Formula Sheets

0. **Metric**

$$g^{\mu\nu} = 0 \ (\mu \neq \nu), \quad g^{00} = -g^{11} = -g^{22} = -g^{33} = 1$$

$$a^\mu = (a^0, \vec{a}), \quad a_\mu = (a^0, -\vec{a})$$

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = (\frac{\partial}{\partial t}, \nabla)$$

$$\partial_\mu a^\mu(x) = \frac{\partial}{\partial t} a^0(x) + \nabla \cdot \vec{a}(x)$$

1. **Fields and equations of motion**

(spin-0) $$\phi(x) = \sum_{\vec{p}} [a_{\vec{p}}\, e_{\vec{p}}(x) + a^\dagger_{\vec{p}}\, e^*_{\vec{p}}(x)], \quad (\partial^2 + m^2)\phi = 0$$

$$\mathcal{L} = \tfrac{1}{2}\left(\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2\right) \quad \text{(neutral)}$$
$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi \quad \text{(charged; with } a^\dagger_{\vec{p}} \to b^\dagger_{\vec{p}} \text{ above)}$$

(spin-1/2) $$\psi(x) = \sum_{\vec{p},s} [a_{\vec{p},s}\, f_{\vec{p},s}(x) + b^\dagger_{\vec{p},s}\, g_{\vec{p},s}(x)], \quad (i\not{\partial} - m)\psi = 0$$

$$\mathcal{L} = \overline{\psi}(i\not{\partial} - m)\psi$$

(spin-1) $$A^\mu(x) = \sum_{\vec{p}\lambda} [a_{\vec{p}\lambda} h^\mu_{\vec{p}\lambda}(x) + a^\dagger_{\vec{p}\lambda} h^{\mu *}_{\vec{p}\lambda}(x)], \quad \partial_\nu F^{\mu\nu} + m^2 A^\mu = 0$$

$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + \tfrac{m^2}{2} A_\mu A^\mu \quad \text{(neutral)}$$
$$\mathcal{L} = -\tfrac{1}{2} F^\dagger_{\mu\nu} F^{\mu\nu} + m^2 A^\dagger_\mu A^\mu \quad \text{(charged; with } a^\dagger_{\vec{p}\lambda} \to b^\dagger_{\vec{p}\lambda} \text{ above)}$$

$$F^{\mu\nu} \equiv \partial^\nu A^\mu - \partial^\mu A^\nu$$

$$\left(e_{\vec{p}}(x) = \frac{e^{-ipx}}{\sqrt{2p^0 V}}, \quad f_{\vec{p},s}(x) = u_{\vec{p},s}\, e_{\vec{p}}(x), \quad g_{\vec{p},s}(x) = v_{\vec{p},s}\, e^*_{\vec{p}}(x), \quad h^\mu_{\vec{p}\lambda}(x) = \epsilon^\mu_{\vec{p}\lambda} e_{\vec{p}}(x) \right)$$

2. **Pauli matrices**

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_i \sigma_j = i\sigma_k \quad (i,j,k \text{ cyclic}), \quad \sigma_i^2 = 1$$

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k\,, \quad \{\sigma_i, \sigma_j\} = 2\delta_{ij}$$

$$(\vec{a}\cdot\vec{\sigma})(\vec{b}\cdot\vec{\sigma}) = \vec{a}\cdot\vec{b} + i\vec{\sigma}\cdot(\vec{a}\times\vec{b})$$

$$e^{i\vec{a}\cdot\vec{\sigma}} = \cos a + i\hat{a}\cdot\vec{\sigma}\sin a, \quad (a = |\vec{a}|,\ \hat{a} = \vec{a}/a)$$

$$\epsilon_{ijk}\epsilon_{i'j'k} = \delta_{ii'}\delta_{jj'} - \delta_{ij'}\delta_{ji'}$$

$$\epsilon_{ijk}\epsilon_{i'jk} = 2\delta_{ii'}\,, \qquad \epsilon_{ijk}\epsilon_{ijk} = 6$$

3. **Gamma matrices**

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \quad \gamma_5 = \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3, \quad \{\gamma^5, \gamma^\mu\} = 0$$

$$\sigma_{\mu\nu} = \frac{i}{2}\left[\gamma_\mu, \gamma_\nu\right]$$

$$\gamma_0^2 = 1, \quad \gamma_i^2 = -1 \ (i = 1, 2, 3), \quad \gamma_5^2 = 1$$

$$\overline{\gamma}_\mu = \gamma_\mu, \quad \overline{\gamma}_5 = -\gamma_5, \quad \gamma_\mu^\dagger = \gamma^\mu, \quad \gamma_5^\dagger = \gamma_5$$

$$\overline{\sigma^{\mu\nu}} = \sigma^{\mu\nu}, \quad \overline{\gamma_5\gamma^\mu} = \gamma_5\gamma^\mu$$

[Dirac representation]

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

[Weyl representation]

$$\gamma^0 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

4. **Dirac spinors**

$$p^\mu = (p^0, \vec{p}), \quad |\vec{s}| = 1, \quad s^\mu = (0, \vec{s}) \text{ (in rest frame)}$$

$$p \cdot s = 0 \text{ (in any frame)}$$

$$(\not{p} - m)u_{\vec{p},s} = 0, \quad (\not{p} + m)v_{\vec{p},s} = 0$$

$$\overline{u}_{\vec{p},s}(\not{p} - m) = 0, \quad \overline{v}_{\vec{p},s}(\not{p} + m) = 0$$

$$\overline{u}_{\vec{p},s_1}\, u_{\vec{p},s_2} = 2m\delta_{s_1s_2} \qquad \overline{v}_{\vec{p},s_1}\, v_{\vec{p},s_2} = -2m\delta_{s_1s_2}$$

$$\overline{u}_{\vec{p},s_1}\, v_{\vec{p},s_2} = 0 \qquad (s_{1,2} = \pm 1)$$

$$u^\dagger_{\vec{p},s_1}\, u_{\vec{p},s_2} = 2E\delta_{s_1s_2} \qquad v^\dagger_{-\vec{p},s_1}\, v_{-\vec{p},s_2} = 2E\delta_{s_1s_2}$$

$$u^\dagger_{\vec{p},s_1}\, v_{-\vec{p},s_2} = 0 \qquad (s_{1,2} = \pm 1)$$

$$u_{\vec{p},s}\,\overline{u}_{\vec{p},s} = (\not{p} + m)\frac{1 + \gamma_5\not{s}}{2}, \quad v_{\vec{p},s}\,\overline{v}_{\vec{p},s} = (\not{p} - m)\frac{1 + \gamma_5\not{s}}{2}$$

$$\sum_s u_{\vec{p},s}\,\overline{u}_{\vec{p},s} = (\not{p} + m), \quad \sum_s v_{\vec{p},s}\,\overline{v}_{\vec{p},s} = (\not{p} - m)$$

[Explicit expressions in the Dirac representation]

$$u_{\vec{p},\pm} = \sqrt{E + m}\begin{pmatrix} 1 \\ \frac{\vec{p}\cdot\vec{\sigma}}{E+m} \end{pmatrix}\chi_\pm, \quad v_{\vec{p},\pm} = \sqrt{E + m}\begin{pmatrix} \frac{\vec{p}\cdot\vec{\sigma}}{E+m} \\ 1 \end{pmatrix}\chi_\mp$$

$$(\vec{s}\cdot\vec{\sigma})\chi_\pm = \pm\chi_\pm, \quad \chi^\dagger_{s_1}\chi_{s_2} = \delta_{s_1s_2} \ (s_{1,2} = \pm)$$

$$\chi_+ = \frac{1}{\sqrt{2(1 + s_z)}}\begin{pmatrix} 1 + s_z \\ s_+ \end{pmatrix}, \quad \chi_- = \frac{1}{\sqrt{2(1 - s_z)}}\begin{pmatrix} s_z - 1 \\ s_+ \end{pmatrix} \quad (s_\pm = s_x \pm i s_y)$$

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (\vec{s} = \hat{z})$$

5. **Spin-1 polarization vectors**

$$\epsilon^\mu_{\vec{0}\lambda} = (0, \hat{e}_{\vec{0}\lambda}) \quad \hat{e}_{\lambda=1,2,3} = \hat{e}_x, \hat{e}_y, \hat{e}_z \quad \text{(in rest frame)}, \qquad \epsilon_{\vec{p}\lambda} \cdot p = 0 \text{ (in any frame)}$$

$$\epsilon^\mu_{\vec{p}1} = (o, \hat{e}_{\vec{p}1}), \quad \epsilon^\mu_{\vec{p}2} = (o, \hat{e}_{\vec{p}2}), \quad \epsilon^\mu_{\vec{p}3} = (\frac{|\vec{p}|}{m}, \frac{p^0}{m}\hat{p}) \quad \text{(in any frame)}$$

$$\epsilon^\mu_{\vec{p}\pm} \equiv \mp \frac{1}{\sqrt{2}}(\epsilon^\mu_{\vec{p}1} \pm i\epsilon^\mu_{\vec{p}2}), \quad \epsilon^\mu_{\vec{p}0} \equiv \epsilon^\mu_{\vec{p}3} \quad \text{(helicity basis)}$$

Polarization sums

$$(m \neq 0) \qquad \sum_\lambda \epsilon^\mu_{\vec{p}\lambda}\epsilon^{\nu *}_{\vec{p}\lambda} = -g^{\mu\nu} + \frac{p^\mu p^\nu}{m^2} \quad (\lambda = 1,2,3 \text{ or } \pm, 0)$$

$$(m = 0) \quad \begin{cases} \sum_\lambda \epsilon^i_{\vec{p}\lambda}\epsilon^{j*}_{\vec{p}\lambda} = \delta_{ij} - \hat{p}_i\hat{p}_j \\ \sum_\lambda \epsilon^\mu_{\vec{p}\lambda}\epsilon^{\nu *}_{\vec{p}\lambda} = -g^{\mu\nu} + \frac{\bar{k}^\mu k^\nu + k^\mu \bar{k}^\nu}{\bar{k}\cdot k} \\ \sum_\lambda \epsilon^\mu_{\vec{p}\lambda}\epsilon^{\nu *}_{\vec{p}\lambda} = -g^{\mu\nu} \quad \text{(QED)} \end{cases} \quad \begin{pmatrix} \lambda = 1,2 \text{ or } \pm\text{. Coulomb gauge: } \epsilon^0_{\vec{p}\lambda} = 0 \\ k = (k^0, \vec{k}), \bar{k} = (k^0, -\vec{k}) \end{pmatrix}$$

6. **Trace theorems and related formuli**

$$\not{a}\not{b} + \not{b}\not{a} = 2a\cdot b \qquad \not{a}\not{a} = a^2$$

$$\text{Tr}\gamma_{i_1}\ldots\gamma_{i_{2n+1}} = 0 \quad (i's = 0,1,2,3: \text{ odd number of } \gamma\text{'s. } \gamma_5 \text{ count as 0.})$$

$$\text{Tr}\gamma_{i_1}\ldots\gamma_{i_{2n}} = \text{Tr}\gamma_{i_{2n}}\ldots\gamma_{i_1} \quad \text{(reverse order)}$$

$$\text{Tr}1 = 4$$

$$\text{Tr}\not{a}\not{b} = 4\,a\cdot b \qquad \text{Tr}\gamma_\mu\gamma_\nu = 4g_{\mu\nu}$$

$$\text{Tr}\not{a}\not{b}\not{c}\not{d} = 4(a\cdot b\,c\cdot d - a\cdot c\,b\cdot d + a\cdot d\,b\cdot c) \qquad \text{Tr}\gamma_\mu\gamma_\nu\gamma_\alpha\gamma_\beta = 4(g_{\mu\nu}g_{\alpha\beta} - g_{\mu\alpha}g_{\nu\beta} + g_{\mu\beta}g_{\nu\alpha})$$

$$\begin{aligned} \text{Tr}\not{a}_1\not{a}_2\ldots\not{a}_{2n} =& \sum_{i=2}^{2n}(-1)^i a_1\cdot a_i \text{Tr}\not{a}_2\ldots\not{a}_{i-1}\not{a}_{i+1}\ldots\not{a}_{2n} \\ =& \; a_1\cdot a_2\text{Tr}\not{a}_3\not{a}_4\ldots\not{a}_{2n} - a_1\cdot a_3\text{Tr}\not{a}_2\not{a}_4\ldots\not{a}_{2n} \\ & + \ldots + a_1\cdot a_{2n}\text{Tr}\not{a}_2\not{a}_3\ldots\not{a}_{2n-1} \end{aligned}$$

$$\text{Tr}\gamma_5 = 0$$

$$\text{Tr}\gamma_5\not{a}\not{b} = 0 \qquad \text{Tr}\gamma_5\gamma_\mu\gamma_\nu = 0$$

$$\text{Tr}\gamma_5\not{a}\not{b}\not{c}\not{d} = 4i\epsilon^{\mu\nu\alpha\beta}a_\mu b_\nu c_\alpha d_\beta \qquad \text{Tr}\gamma_5\gamma_\mu\gamma_\nu\gamma_\alpha\gamma_\beta = 4i\epsilon_{\mu\nu\alpha\beta}$$

$$(\epsilon_{0123} \equiv +1)$$

$$\gamma_\mu\gamma^\mu = 4$$

$$\gamma_\mu\not{a}\gamma^\mu = -2\not{a} \qquad \gamma_\mu\gamma_\alpha\gamma^\mu = -2\gamma_\alpha$$

$$\gamma_\mu\not{a}\not{b}\gamma^\mu = 4a\cdot b \qquad \gamma_\mu\gamma_\alpha\gamma_\beta\gamma^\mu = 4g_{\alpha\beta}$$

$$\gamma_\mu\not{a}\not{b}\not{c}\gamma^\mu = -2\not{c}\not{b}\not{a} \qquad \gamma_\mu\gamma_\alpha\gamma_\beta\gamma_\sigma\gamma^\mu = -2\gamma_\sigma\gamma_\beta\gamma_\alpha$$

$$g_{\mu\nu}g^{\mu\nu} = 4 \qquad \epsilon^{\mu\nu\alpha\beta}\epsilon_{\mu\nu\rho\sigma} = 2(g^\alpha_\sigma g^\beta_\rho - g^\alpha_\rho g^\beta_\sigma)$$

$$\epsilon^{\mu\nu\alpha\beta}\epsilon_{\mu\nu\alpha\sigma} = -6g^{\beta}{}_{\sigma} \qquad \epsilon^{\mu\nu\alpha\beta}\epsilon_{\mu\nu\alpha\beta} = -24$$

7. **Cross sections and decay rates**

$$S_{fi} = \frac{(2\pi)^4\delta^4(\sum_i p_i - \sum_f p_f)}{\sqrt{\prod_i(2p_i^0 V)\prod_j(2p_f^0 V)}}\mathcal{M} \quad (\mathcal{M}\text{: Lorentz-invariant matrix element})$$

$$d\Phi_n = \delta^4(\textstyle\sum_i p_i - \sum_f p_f)\displaystyle\prod_f \frac{d^3 p_f}{(2\pi)^3 2p_f^0} \quad \text{(n-body Lorentz-invariant phase space)}$$

(a) Cross sections (general: $p_1 + p_2 = \sum_f q_f$)

$$d\sigma = \frac{(2\pi)^4}{4\sqrt{(p_1\cdot p_2)^2 - m_1^2 m_2^2}}\overline{|\mathcal{M}|^2}\, d\Phi_n$$

$$\sqrt{(p_1\cdot p_2)^2 - m_1^2 m_2^2} = \begin{cases} m_1|\vec{p}_2| & \text{(1 at rest)} \\ M|\vec{p}| & \text{(c.m. } M\text{: total c.m. energy)} \end{cases}$$

2-body final state ($p_1 + p_2 = p_3 + p_4$: all masses could be different)

$$\frac{d\sigma}{d\Omega} = \frac{\overline{|\mathcal{M}|^2}}{(8\pi M)^2}\frac{|\vec{p}_f|}{|\vec{p}_i|} \qquad \text{(c.m. system)}$$

$$\frac{d\sigma}{dt} = \frac{\overline{|\mathcal{M}|^2}}{16\pi\lambda(s, m_1^2, m_2^2)} \qquad \text{(no azimuth dependece. any frame)}$$

$$\text{where } \begin{cases} s = (p_1 + p_2)^2 \\ t = (p_3 - p_1)^2 \end{cases} \qquad \lambda(x,y,z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx$$

$$\frac{d\sigma}{d\Omega_3} = \frac{|\vec{p}_3|\overline{|\mathcal{M}|^2}}{64\pi^2 m_1|\vec{p}_2|\,[m_1 + E_2(1 - \frac{\beta_2}{\beta_3}\cos\theta_3)]} \qquad \text{(Lab frame; 1 at rest)}$$

(b) **Decay rates (c.m. system)**

$$d\Gamma = \frac{(2\pi)^4}{2M}\overline{|\mathcal{M}|^2}\, d\Phi_n \quad (M\text{: parent mass})$$

$$\frac{d\Gamma}{d\Omega} = \frac{|\vec{p}|}{32\pi^2 M^2}\overline{|\mathcal{M}|^2} \quad \text{(2-body)}$$

$$\Gamma = \frac{|\vec{p}|}{8\pi M^2}\overline{|\mathcal{M}|^2} \quad \text{(2-body, parent not polarized)}$$

$$d\Gamma = \frac{\overline{|\mathcal{M}|^2}}{(2\pi)^3 8M}dE_1 dE_2 = \frac{\overline{|\mathcal{M}|^2}}{(2\pi)^3 32M^3}ds_{23}ds_{31} \quad \text{(3-body, parent not polarized)}$$

8. **Feynman rules (tree level)**

External legs and propagators: (f: fermion, $\bar{f}$: anti-fermion)

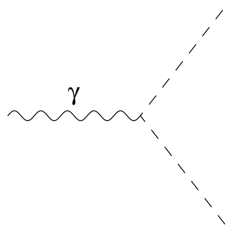
	initial state	final state	propagator
(spin-0)	1	1	$\dfrac{i}{p^2 - m^2 + i\epsilon}$
(spin-1/2)	$u_{\vec{p},s}(f)\ \overline{v}_{\vec{p},s}(\overline{f})$	$\overline{u}_{\vec{p},s}(f)\ v_{\vec{p},s}(\overline{f})$	$\dfrac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon} \equiv \dfrac{i}{\not{p} - m + i\epsilon}$
(spin-1)	$\epsilon^{\mu}_{\vec{p}\lambda}$	$\epsilon^{\mu *}_{\vec{p}\lambda}$	$\dfrac{i(-g^{\mu\nu} + \frac{p^\mu p^\nu}{m^2})}{p^2 - m^2 + i\epsilon}\ (m \neq 0)$ $\quad \dfrac{-ig^{\mu\nu}}{p^2 + i\epsilon}$ (photon)

Vertexes:

($e = 0.303$ ($e^2 = 4\pi\alpha$), $g = 0.652$)

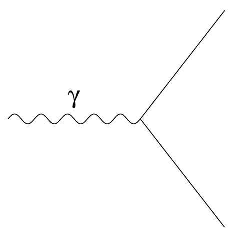
scalar-photon

$-ie(p_1 - p_2)$
($p_{1,2}$: 4-momenta flowing into the vertex))



fermion-photon

$-iq\gamma_\mu$
(q: charge of the fermion. $q = e$ if electron)



fermion-W

$\frac{ig}{2\sqrt{2}}\gamma_\mu(1 - \gamma_5)\, V_{ij}$
(V_{ij}: CKM matrix element; $= 1$ for leptons.)