Chapter 1

Lorentz Group and Lorentz Invariance

In studying Lorentz-invariant wave equations, it is essential that we put our understanding of the Lorentz group on firm ground. We first define the Lorentz transformation as any transformation that keeps the 4-vector inner product invariant, and proceed to classify such transformations according to the determinant of the transformation matrix and the sign of the time component. We then introduce the generators of the Lorentz group by which any Lorentz transformation continuously connected to the identity can be written in an exponential form. The generators of the Lorentz group will later play a critical role in finding the transformation property of the Dirac spinors.

1.1 Lorentz Boost

Throughout this book, we will use a unit system in which the speed of light c is unity. This may be accomplished for example by taking the unit of time to be one second and that of length to be $2.99792458 \times 10^{10}$ cm (this number is *exact*[1]), or taking the unit of length to be 1 cm and that of time to be $(2.99792458 \times 10^{10})^{-1}$ second. How it is accomplished is irrelevant at this point.

Suppose an inertial frame K (space-time coordinates labeled by t, x, y, z) is moving with velocity β in another inertial frame K' (space-time coordinates labeled by t', x', y', z') as shown in Figure 1.1. The 3-component velocity of the origin of K

[1]One cm is *defined* (1983) such that the speed of light in vacuum is $2.99792458 \times 10^{10}$ cm per second, where one second is defined (1967) to be 9192631770 times the oscillation period of the hyper-fine splitting of the Cs^{133} ground state.

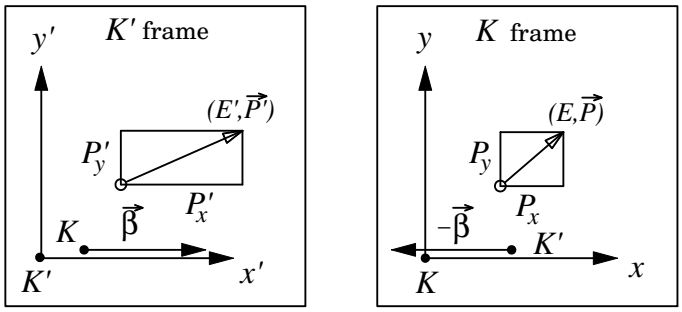


Figure 1.1: The origin of frame K is moving with velocity $\vec{\beta} = (\beta, 0, 0)$ in frame K', and the origin of frame K' is moving with velocity $-\vec{\beta}$ in frame K. The axes x and x' are parallel in both frames, and similarly for y and z axes. A particle has energy momentum $(E, \vec{P})$ in frame K and $(E', \vec{P}')$ in frame K'.

measured in the frame K', $\vec{\beta}'_K$, is taken to be in the $+x'$ direction; namely,

$$\vec{\beta}'_K \text{ (velocity of } K \text{ in } K') = (\beta, 0, 0) \stackrel{\text{def}}{\equiv} \vec{\beta}. \tag{1.1}$$

Assume that, in the frame K', the axes x, y, z are parallel to the axes x', y', z'. Then, the velocity of the origin of K' in K, $\vec{\beta}_{K'}$, is

$$\vec{\beta}_{K'} = -\vec{\beta}'_K = (-\beta, 0, 0) \quad \text{(velocity of } K' \text{ in } K). \tag{1.2}$$

Note that $\vec{\beta}'_K$ ($\vec{\beta}_{K'}$) is measured with respect to the axes of K' (K).

If a particle (or any system) has energy and momentum $(E, \vec{P})$ in the frame K, then the energy and momentum $(E', \vec{P}')$ of the same particle viewed in the frame K' are given by

$$\begin{aligned} E' &= \frac{E + \beta P_x}{\sqrt{1-\beta^2}}, \qquad & P'_y &= P_y, \\ P'_x &= \frac{\beta E + P_x}{\sqrt{1-\beta^2}}, \qquad & P'_z &= P_z, . \end{aligned} \tag{1.3}$$

This can be written in a matrix form as

$$\begin{pmatrix} E' \\ P'_x \end{pmatrix} = \begin{pmatrix} \gamma & \eta \\ \eta & \gamma \end{pmatrix} \begin{pmatrix} E \\ P_x \end{pmatrix}, \qquad \begin{pmatrix} P'_y \\ P'_z \end{pmatrix} = \begin{pmatrix} P_y \\ P_z \end{pmatrix} \tag{1.4}$$

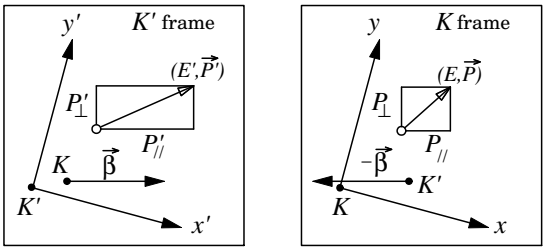


Figure 1.2: Starting from the configuration of Figure 1.1, the same rotation is applied to the axes in each frame. The resulting transformation represents a general Lorentz boost.

with

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}}, \qquad \eta \equiv \beta\gamma = \frac{\beta}{\sqrt{1-\beta^2}}. \tag{1.5}$$

Note that γ and η are related by

$$\boxed{\gamma^2 - \eta^2 = 1}. \tag{1.6}$$

Now start from Figure 1.1 and apply the *same rotation* to the axes of K and K' *within each frame* without changing the motions of the origins of the frames and without touching the paqrticle (Figure 1.2). Suppose the rotation is represented by a 3×3 matrix R. Then, the velocity of K' in K, $\vec{\beta}_{K'}$, and and the velocity of K in K', $\vec{\beta}'_K$, are rotated by the same matrix R,

$$\vec{\beta}'_K \to R\vec{\beta}'_K\,, \qquad \vec{\beta}_{K'} \to R\vec{\beta}_{K'}\,, \tag{1.7}$$

and thus we still have

$$\vec{\beta}'_K = -\vec{\beta}_{K'} \stackrel{\text{def}}{\equiv} \vec{\beta}\,, \tag{1.8}$$

where we have also redefined the vector $\vec{\beta}$ which is well-defined in both K and K' frames in terms of $\vec{\beta}_{K'}$ and $\vec{\beta}'_K$, respectively. The transformation in this case can be obtained by noting that, in (1.4), the component of momentum transverse to $\vec{\beta}$ does not change and that P_x, P'_x are the components of $\vec{P}$, $\vec{P}'$ along $\vec{\beta}$ in each frame. Namely, the tranformation can be written as

$$\boxed{\begin{pmatrix} E' \\ P'_\parallel \end{pmatrix} = \begin{pmatrix} \gamma & \eta \\ \eta & \gamma \end{pmatrix} \begin{pmatrix} E \\ P_\parallel \end{pmatrix}\,, \qquad \vec{P}'_\perp = \vec{P}_\perp}, \tag{1.9}$$

where $_\parallel$ and $_\perp$ denote components parallel and perpecdicular to $\vec{\beta}$, repectively. Note that $\vec{P}'_\perp$ and $\vec{P}_\perp$ are 3-component quantities and the relation $\vec{P}'_\perp = \vec{P}_\perp$ holds component by component because we have applied the same roation R in each frame.

The axes of K viewed in the frame K' are no longer perpendicular to each other since they are contracted in the direction of $\vec{\beta}'_K$. Thus, the axes of K in general are not parallel to the corresponding axes of K' at any time. However, since the same rotation is applied in each frame, and since components transverse to $\vec{\beta}$ are the same in both frames, the corresponding axes of K and K' are exactly parallel when projected onto a plane perpendicular to $\vec{\beta}$ in either frames. The transformation (1.9) is thus correct for the specific relative orientation of two frames as defined here, and such transformation is called a Lorentz boost, which is a special case of *Lorentz transformation* defined later in this chapter for which the relative orientation of the two frames is arbitrary.

1.2 4-vectors and the metric tensor $g_{\mu\nu}$

The quantity $E^2 - \vec{P}^2$ is invariant under the Lorentz boost (1.9); namely, it has the same numerical value in K and K':

$$\begin{aligned} E'^2 - \vec{P}'^2 &= E'^2 - (P'^2_\parallel + \vec{P}'^2_\perp) \\ &= (\gamma E + \eta P_\parallel)^2 - \left[(\eta E + \gamma P_\parallel)^2 + \vec{P}^2_\perp\right] \\ &= \underbrace{(\gamma^2 - \eta^2)}_{1} E^2 + \underbrace{(\eta^2 - \gamma^2)}_{-1} P^2_\parallel - \vec{P}^2_\perp \\ &= E^2 - \vec{P}^2 \,, \end{aligned} \tag{1.10}$$

which is the invariant mass squared m^2 of the system. This invariance applies to any number of particles or any object as long as E and $\vec{P}$ refer to the same object.

The relative minus sign between E^2 and $\vec{P}^2$ above can be treated elegantly as follows. Define a 4-vector P^μ ($\mu = 0, 1, 2, 3$) by

$$P^\mu = (P^0, P^1, P^2, P^3) \stackrel{\text{def}}{\equiv} (E, P_x, P_y, P_z) = (E, \vec{P}) \tag{1.11}$$

called an energy-momentum 4-vector where the index μ is called the Lorentz index (or the space-time index). The $\mu = 0$ component of a 4-vector is often called 'time component', and the $\mu = 1, 2, 3$ components 'space components.'

Define the inner product (or 'dot' product) $A \cdot B$ of two 4-vectors $A^\mu = (A^0, \vec{A})$ and $B^\mu = (B^0, \vec{B})$ by

$$A \cdot B \stackrel{\text{def}}{\equiv} A^0 B^0 - \vec{A} \cdot \vec{B} = A^0 B^0 - A^1 B^1 - A^2 B^2 - A^3 B^3 \,. \tag{1.12}$$

Then, $P^2 \equiv P \cdot P$ is nothing but m^2:

$$P^2 = P^{0\,2} - \vec{P}^2 = E^2 - \vec{P}^2 = m^2 \tag{1.13}$$

which is invariant under Lorentz boost. This inner product $P \cdot P$ is similar to the norm squared $\vec{x}^2$ of an ordinary 3-dimensional vector $\vec{x}$, which is invariant under rotation, except for the minus signs for the space components in the definition of the inner product. In order to handle these minus signs conveniently, we define 'subscripted' components of a 4-vector by

$$A_0 \stackrel{\text{def}}{\equiv} A^0, \quad A_i \stackrel{\text{def}}{\equiv} -A^i \; (i = 1, 2, 3)\,. \tag{1.14}$$

Then the inner product (1.12) can be written as

$$A \cdot B = A_0 B^0 + A_1 B^1 + A_2 B^2 + A_3 B^3 \stackrel{\text{def}}{\equiv} A_\mu B^\mu = A^\mu B_\mu\,, \tag{1.15}$$

where we have used the convention that when a pair of the same index appears in the *same term*, then summation over all possible values of the index ($\mu = 0, 1, 2, 3$ in this case) is implied. In general, we will use Roman letters for space indices (take values 1,2,3) and greek letters for space-time (Lorentz) indices (take values 0,1,2,3). Thus,

$$x^i y^i = \sum_{i=1}^{3} x^i y^i \; (= \vec{x} \cdot \vec{y}), \qquad (A^\mu + B^\mu) C_\mu = \sum_{\mu=0}^{3} (A^\mu + B^\mu) C_\mu\,, \tag{1.16}$$

but no sum over μ or ν in

$$A_\mu B^\nu + C_\mu D^\nu \quad (\mu,\, \nu \text{ not in the same term}). \tag{1.17}$$

When a pair of Lorentz indices is summed over, usually one index is a subscript and the other is a superscript. Such indices are said to be 'contracted'. Also, it is important that there is only one pair of a given index per term. We do not consider implied summations such as $A^\mu B^\mu C_\mu$ to be well-defined. [$(A^\mu + B^\mu) C_\mu$ is well-defined since it is equal to $A^\mu C_\mu + B^\mu C_\mu$.]

Now, define the metric tensor $g_{\mu\nu}$ by

$$g_{00} = 1\,,\; g_{11} = g_{22} = g_{33} = -1\,, \quad g_{\mu\nu} = 0 \; (\mu \neq \nu) \tag{1.18}$$

which is symmetric:

$$g_{\mu\nu} = g_{\nu\mu}\,. \tag{1.19}$$

The corresponding matrix G is defined as

$$\{g_{\mu\nu}\} \stackrel{\text{def}}{\equiv} \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \| \\ \mu \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \quad = \nu \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \end{array} \stackrel{\text{def}}{\equiv} G \tag{1.20}$$

When we form a matrix out of a quantity with two indices, by definition we take the first index to increase downward, and the second to increase to the right.

As defined in (1.14) for a 4-vector, switching an index between superscript and subscript results in a sign change when the index is 1,2, or 3, while the sign is unchanged when the index is zero. We adopt the same rule for the indices of $g_{\mu\nu}$. In fact, from now on, we enforce the same rule for all space-time indices (unless otherwise stated, such as for the Kronecker delta below). Then we have

$$g_{\mu\nu} = g^{\mu\nu}, \quad g_\mu{}^\nu = g^\mu{}_\nu = \delta_{\mu\nu} \tag{1.21}$$

where $\delta_{\mu\nu}$ is the Kronecker's delta ($\delta_{\mu\nu} = 1$ if $\mu = \nu$, 0 otherwise) which we define to have only subscripts. Then, $g_{\mu\nu}$ can be used together with contraction to 'lower' or 'raise' indices:

$$A_\nu = g_{\mu\nu} A^\mu, \quad A^\nu = g^{\mu\nu} A_\mu \tag{1.22}$$

which are equivalent to the rule (1.14).

The inner product of 4-vectors A and B (1.12) can also be written in matrix form as

$$A \cdot B = A^\mu g_{\mu\nu} B^\nu = (A^0\ A^1\ A^2\ A^3) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} B^0 \\ B^1 \\ B^2 \\ B^3 \end{pmatrix} = A^T G B\,. \tag{1.23}$$

When we use 4-vectors in matrix form, they are understood to be column vectors with *superscripts*, while their transpose are row vectors:

$$A \overset{\text{def}}{\equiv} \begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix} = \begin{pmatrix} A^0 \\ A_x \\ A_y \\ A_z \end{pmatrix}, \quad A^T = (A^0, A^1, A^2, A^3) \qquad \text{(in matrix form)}. \tag{1.24}$$

1.3 Lorentz group

The Lorentz boost (1.4) can be written in matrix form as

$$P' = \Lambda P \tag{1.25}$$

with

$$P' = \begin{pmatrix} E' \\ P'_x \\ P'_y \\ P'_z \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \gamma & \eta & 0 & 0 \\ \eta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad P = \begin{pmatrix} E \\ P_x \\ P_y \\ P_z \end{pmatrix}. \tag{1.26}$$

In terms of components, this can be written as

$$\boxed{P'^{\mu} = \Lambda^{\mu}{}_{\nu} P^{\nu}}, \tag{1.27}$$

where we have *defined* the components of the matrix Λ by taking the first index to be superscript and the second to be subscript (still the first index increases downward and the second index increases to the right):

$$\Lambda \overset{\text{def}}{\equiv} \{\Lambda^{\mu}{}_{\nu}\} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \| \\ \mu \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \quad = \nu \\ \begin{pmatrix} \gamma & \eta & 0 & 0 \\ \eta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \tag{1.28}$$

For example, $\Lambda^0{}_1 = \eta$ and thus $\Lambda^{01} = -\eta$, etc. Why do we define Λ in this way? The superscripts and subscript in (1.27) were chosen such that the index ν is contracted and that the index μ on both sides of the equality has consistent position, namely, both are superscript.

We have seen that $P^2 = E^2 - \vec{P}^2$ is invariant under the Lorentz boost given by (1.4) or (1.9). We will now find the necessary and sufficient condition for a 4×4 matrix Λ to leave the inner product of any two 4-vectors invariant. Suppose A^{μ} and B^{μ} transform by the same matrix Λ:

$$A'^{\mu} = \Lambda^{\mu}{}_{\alpha} A^{\alpha}, \quad B'^{\nu} = \Lambda^{\nu}{}_{\beta} B^{\beta}\,. \tag{1.29}$$

Then the inner products $A' \cdot B'$ and $A \cdot B$ can be written using (1.22) as

$$\begin{aligned} A' \cdot B' &= \underbrace{A'_{\nu}}_{g_{\mu\nu}\underbrace{A'^{\mu}}_{\Lambda^{\mu}{}_{\alpha}A^{\alpha}}} \underbrace{B'^{\nu}}_{\Lambda^{\nu}{}_{\beta}B^{\beta}} = (g_{\mu\nu}\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}) A^{\alpha} B^{\beta} \\ A \cdot B &= \underbrace{A_{\beta}}_{g_{\alpha\beta}A^{\alpha}} B^{\beta} = g_{\alpha\beta} A^{\alpha} B^{\beta}\,. \end{aligned} \tag{1.30}$$

In order for $A' \cdot B' = A \cdot B$ to hold for any A and B, the coefficients of $A^{\alpha}B^{\beta}$ should be the same term by term (To see this, set $A^{\nu} = 1$ for $\nu = \alpha$ and 0 for all else, and $B^{\nu} = 1$ for $\nu = \beta$ and 0 for all else.):

$$\boxed{g_{\mu\nu}\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta} = g_{\alpha\beta}}. \tag{1.31}$$

On the other hand, if Λ satisfies this condition, the same derivation above can be traced backward to show that the inner product $A \cdot B$ defined by (1.12) is invariant. Thus, (1.31) is the necessary and sufficient condition.

What does the condition (1.31) tell us about the nature of the matrix Λ? Using (1.22), we have $g_{\mu\nu}\Lambda^{\mu}{}_{\alpha} = \Lambda_{\nu\alpha}$, then the condition becomes

$$\Lambda_{\nu\alpha}\Lambda^{\nu}{}_{\beta} = g_{\alpha\beta} \quad \overset{\text{raise } \alpha \text{ on both sides}}{\longrightarrow} \quad \Lambda_{\nu}{}^{\alpha}\Lambda^{\nu}{}_{\beta} = g^{\alpha}{}_{\beta}\,(= \delta_{\alpha\beta})\,. \tag{1.32}$$

Compare this with the definition of the inverse transformation Λ^{-1}:

$$\Lambda^{-1}\Lambda = I \quad \text{or} \quad (\Lambda^{-1})^{\alpha}{}_{\nu}\Lambda^{\nu}{}_{\beta} = \delta_{\alpha\beta}\,, \tag{1.33}$$

where I is the 4×4 indentity matrix. The indexes of Λ^{-1} are superscript for the first and subscript for the second as before, and the matrix product is formed as usual by summing over the second index of the first matrix and the first index of the second matrix. We see that the inverse matrix of Λ is obtained by

$$\boxed{(\Lambda^{-1})^{\alpha}{}_{\nu} = \Lambda_{\nu}{}^{\alpha}}\,, \tag{1.34}$$

which means that one simply has to change the sign of the components for which only one of the indices is zero (namely, $\Lambda^{0}{}_{i}$ and $\Lambda^{i}{}_{0}$) and then transpose it:

$$\Lambda = \begin{pmatrix} \Lambda^{0}{}_{0} & \Lambda^{0}{}_{1} & \Lambda^{0}{}_{2} & \Lambda^{0}{}_{3} \\ \Lambda^{1}{}_{0} & \Lambda^{1}{}_{1} & \Lambda^{1}{}_{2} & \Lambda^{1}{}_{3} \\ \Lambda^{2}{}_{0} & \Lambda^{2}{}_{1} & \Lambda^{2}{}_{2} & \Lambda^{2}{}_{3} \\ \Lambda^{3}{}_{0} & \Lambda^{3}{}_{1} & \Lambda^{3}{}_{2} & \Lambda^{3}{}_{3} \end{pmatrix}, \quad \longrightarrow \Lambda^{-1} = \begin{pmatrix} \Lambda^{0}{}_{0} & -\Lambda^{1}{}_{0} & -\Lambda^{2}{}_{0} & -\Lambda^{3}{}_{0} \\ -\Lambda^{0}{}_{1} & \Lambda^{1}{}_{1} & \Lambda^{2}{}_{1} & \Lambda^{3}{}_{1} \\ -\Lambda^{0}{}_{2} & \Lambda^{1}{}_{2} & \Lambda^{2}{}_{2} & \Lambda^{3}{}_{2} \\ -\Lambda^{0}{}_{3} & \Lambda^{1}{}_{3} & \Lambda^{2}{}_{3} & \Lambda^{3}{}_{3} \end{pmatrix}. \tag{1.35}$$

Thus, the set of matrices that keep the inner product of 4-vectors invariant is made of matrices that become their own inverse when the signs of components with one time index are flipped and then transposed. As we will see below, such set of matrices forms a group, called the *Lorentz group*, and any such transformation [namely, one that keeps the 4-vector inner product invariant, or equivalently that satisfies the condition (1.31)] is defined as a *Lorentz transformation*.

To show that such set of matrices forms a group, it is convenient to write the condition (1.31) in matrix form. Noting that when written in terms of components, we can change the ordering of product in any way we want, the condition can be written as

$$\Lambda^{\mu}{}_{\alpha}g_{\mu\nu}\Lambda^{\nu}{}_{\beta} = g_{\alpha\beta}, \quad \text{or} \quad \Lambda^{T}G\Lambda = G\,. \tag{1.36}$$

A set forms a group when for any two elements of the set x_1 and x_2, a 'product' x_1x_2 can be defined such that

1. (Closure) The product x_1x_2 also belongs to the set.

2. (Associativity) For any elements x_1, x_2 and x_3, $(x_1x_2)x_3 = x_1(x_2x_3)$.

3. (Identity) There exists an element I in the set that satisfies $Ix = xI = x$ for any element x.

4. (Inverse) For any element x, there exists an element x^{-1} in the set that satisfies $x^{-1}x = xx^{-1} = I$.

In our case at hand, the set is all 4×4 matrices that satisfy $\Lambda^T G \Lambda = G$, and we take the ordinary matrix multiplication as the 'product' which defines the group. The proof is straightforward:

1. Suppose Λ_1 and Λ_2 belong to the set (i.e. $\Lambda_1^T G \Lambda_1 = G$ and $\Lambda_2^T G \Lambda_2 = G$). Then,

$$(\Lambda_1\Lambda_2)^T G (\Lambda_1\Lambda_2) = \Lambda_2^T \underbrace{\Lambda_1^T G \Lambda_1}_{G} \Lambda_2 = \Lambda_2^T G \Lambda_2 = G. \tag{1.37}$$

   Thus, the product $\Lambda_1\Lambda_2$ also belongs to the set.

2. The matrix multiplication is of course associative: $(\Lambda_1\Lambda_2)\Lambda_3 = \Lambda_1(\Lambda_2\Lambda_3)$.

3. The identity matrix I ($I^\mu{}_\nu = \delta_{\mu\nu}$) belongs to the set ($I^T G I = G$), and satisfies $I\Lambda = \Lambda I = \Lambda$ for any element.

4. We have already seen that if a 4×4 matrix Λ satisfies $\Lambda^T G \Lambda = G$, then its inverse exists as given by (1.34). It is instructive, however, to prove it more formally. Taking the determinant of $\Lambda^T G \Lambda = G$,

$$\underbrace{\det \Lambda^T}_{\det \Lambda} \underbrace{\det G}_{-1} \det \Lambda = \underbrace{\det G}_{-1} \quad \rightarrow \quad (\det \Lambda)^2 = 1\,, \tag{1.38}$$

   where we have used the property of determinant

$$\det(MN) = \det M \det N \tag{1.39}$$

   with M and N being square matrices of same rank. Thus, $\det \Lambda \neq 0$ and therefore its inverse Λ^{-1} exists. Also, it belongs to the set: multiplying $\Lambda^T G \Lambda = G$ by $(\Lambda^{-1})^T$ from the left and by Λ^{-1} from the right,

$$\underbrace{\underbrace{(\Lambda^{-1})^T \Lambda^T}_{(\underbrace{\Lambda\Lambda^{-1}}_{I})^T}}_{I} G \underbrace{\Lambda\Lambda^{-1}}_{I} = (\Lambda^{-1})^T G \Lambda^{-1} \;\rightarrow\; (\Lambda^{-1})^T G \Lambda^{-1} = G\,. \tag{1.40}$$

   This completes the proof that Λ's that satisfy (1.36) form a group.

Since the inverse of a Lorentz transformation is also a Lorentz transformation as just proven above, it should satisfy the condition (1.31)

$$g_{\alpha\beta} = g_{\mu\nu}(\Lambda^{-1})^{\mu}{}_{\alpha}(\Lambda^{-1})^{\nu}{}_{\beta} = g_{\mu\nu}\Lambda_{\alpha}{}^{\mu}\Lambda_{\beta}{}^{\nu} \quad \rightarrow \quad \boxed{g_{\mu\nu}\Lambda_{\alpha}{}^{\mu}\Lambda_{\beta}{}^{\nu} = g_{\alpha\beta}}, \tag{1.41}$$

where we have used the inversion rule (1.34). The formulas (1.31), (1.41), and their variations are then summarized as follows: on the left hand side of the form $g\ \Lambda\ \Lambda = g$, an index of g (call it μ) is contracted with an index of a Λ and the other index of g (call it ν) with an index of the other Λ. As long as μ and ν are both first or both second indices on the Λ's, and as long as the rest of the indices are the same (including superscript/subscript) on both sides of the equality, any possible way of indexing gives a correct formula. Similarly, on the left hand side of the form $\Lambda\ \Lambda = g$, an index of a Λ and an index of the other Λ are contracted. As long as the contracted indices are both first or both second indices on Λ's, and as long as the rest of the indices are the same on both sides of the equality, any possible way of indexing gives a correct formula.

A natural question at this point is whether the Lorentz group defined in this way is any larger than the set of Lorentz boosts defined by (1.9). The answer is yes. Clearly, any rotation in the 3-dimensional space keeps $\vec{A}\cdot\vec{B}$ invariant while it does not change the time components A^0 and B^0. Thus, it keeps the 4-vector inner product $A\cdot B = A^0B^0 - \vec{A}\cdot\vec{B}$ invariant, and as a result it belongs to the Lorentz group by definition. On the other hand, the only way the boost (1.9) does not change the time component is to set $\beta = 0$ in which case the transformation is the identity transformation. Thus, any finite rotation in the 3-dimensional space is not a boost while it is a Lorentz transformation.

Furthermore, the time reversal T and the space inversion P defined by

$$T \stackrel{\text{def}}{\equiv} \{T^{\mu}{}_{\nu}\} \stackrel{\text{def}}{\equiv} \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad P \stackrel{\text{def}}{\equiv} \{P^{\mu}{}_{\nu}\} \stackrel{\text{def}}{\equiv} \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \tag{1.42}$$

satisfy

$$T^T G T = G, \quad P^T G P = G\,, \tag{1.43}$$

and thus belong to the Lorentz group. Even though the matrix P has the same numerical form as G, it should be noted that P is a Lorentz transformation but G is not (it is a metric). The difference is also reflected in the fact that the matrix P is defined by the first index being superscript and the second subscript (because it is a Lorentz transformation), while the matrix G is defined by both indices being subscript (or both superscript).

As we will see later, boosts and rotations can be formed by consecutive infinitesimal transformations starting from identity I (they are 'continuously connected' to I), while T and P cannot (they are 'disconnected' from I, or said to be 'discrete' transformations). Any product of boosts, rotation, T, and P belongs to the Lorentz group, and it turns out that they saturate the Lorentz group. Thus, we write symbolically

$$\text{Lorentz group} \;=\; \text{boost} \;+\; \text{rotation} \;+ T + P\,. \tag{1.44}$$

Later, we will see that any Lorentz transformation continuously connected to I is a boost, a rotation, or a combination thereof.

If the origins of the inertial frames K and K' touch at $t = t' = 0$ and $\vec{x} = \vec{x}' = 0$, the coordinate $x^\mu = (t, \vec{x})$ of any event transforms in the same way as P^μ:

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu\,. \tag{1.45}$$

This can be extended to include space-time translation between the two frames:

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \;+ a^\mu\,, \tag{1.46}$$

where a^μ is a constant 4-vector. The transformation of energy-momentum is not affected by the space-time translation, and is still given by $P' = \Lambda P$. Such transformations that include space-time translation also form a group and called the 'inhomogeneous Lorentz group' or the 'Poincaré group'. The group formed by the transformations with $a^\mu = 0$ is sometimes called the homogeneous Lorentz group. Unless otherwise stated, we will deal with the homogeneous Lorentz group.

1.4 Classification of Lorentz transformations

Up to this point, we have not specified that Lorentz transformations are real (namely, all the elements are real). In fact, Lorentz transformations as defined by (1.31) in general can be complex and the complex Lorentz transformations plays an important role in a formal proof of an important symmetry theorem called CPT theorem which states that the laws of physics are invariant under the combination of particle-antiparticle exchange (C), mirror inversion (P), and time reversal (T) under certain natural assumptions. In this book, however, we will assume that Lorentz transformations are real.

As seen in (1.38), all Lorentz transformation satisfy $(\det\Lambda)^2 = 1$, or equivalently, $\det\Lambda = +1$ or -1. We define 'proper' and 'improper' Lorentz transformations as

$$\begin{cases} \det\Lambda \;= +1 \;: & \text{proper} \\ \det\Lambda \;= -1 \;: & \text{improper} \end{cases}\,. \tag{1.47}$$

Since $\det(\Lambda_1\Lambda_2) = \det\Lambda_1 \det\Lambda_2$, the product of two proper transformations or two improper transformations is proper, while the product of a proper transformation and a improper transformation is improper.

Next, look at the $(\alpha, \beta) = (0, 0)$ component of the defining condition $g_{\mu\nu}\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta = g_{\alpha\beta}$:

$$g_{\mu\nu}\Lambda^\mu{}_0\Lambda^\nu{}_0 = g_{00} = 1\,, \quad \rightarrow \quad (\Lambda^0{}_0)^2 - \sum_{i=1}^{3}(\Lambda^i{}_0)^2 = 1 \tag{1.48}$$

or

$$(\Lambda^0{}_0)^2 = 1 + \sum_{i=1}^{3}(\Lambda^i{}_0)^2 \geq 1\,, \tag{1.49}$$

which means $\Lambda^0{}_0 \geq 1$ or $\Lambda^0{}_0 \leq -1$, and this defines the 'orthochronous' and 'non-orthochronous' Lorentz transformations:

$$\begin{cases} \Lambda^0{}_0 \geq 1 : & \text{orthochronous} \\ \Lambda^0{}_0 \leq -1 : & \text{non-orthochronous} \end{cases} . \tag{1.50}$$

It is easy to show that the product of two orthochronous transformations or two non-orthochronous transformations is orthochronous, and the product of an orthochronous transformation and a non-orthochronous transformation is non-orthochronous.

From the definitions (1.42) and $I^\mu{}_\nu = \delta_{\mu\nu}$, we have

$$\begin{gathered} \det I = \det(TP) = +1, \;\; \det T = \det P = -1, \\ I^0{}_0 = P^0{}_0 = +1, \qquad T^0{}_0 = (TP)^0{}_0 = -1 \end{gathered} \tag{1.51}$$

Thus, the identity I is proper and orthochronous, P is improper and orthochronous, T is improper and non-orthochronous, and TP is proper and non-orthochronous. Accordingly, we can multiply any proper and orthochronous transformations by each of these to form four sets of transformations of given properness and orthochronousness as shown in Table 1.1. Any Lorentz transformation is proper or improper (i.e. $\det\Lambda = \pm 1$) and orthochronous or non-orthochronous (i.e. $|\Lambda^0{}_0|^2 \geq 1$). Since any improper transformation can be made proper by multiplying P and any nonorthochronous transformation can be made orthochronous by multiplying T, the four forms of transformations in Table 1.1 saturate the Lorentz group. For example, if Λ is improper and orthochronous, then $P\Lambda \stackrel{\text{def}}{=} \Lambda^{(\text{po})}$ is proper and orthochronous, and Λ can be written as $\Lambda = P\Lambda^{(\text{po})}$.

It is straightforward to show that the set of proper transformations and the set of orthochronous transformations separately form a group, and that proper and orthochronous transformations by themselves form a group. Also, the set of proper and orthochronous transformations and the set of improper and non-orthochronous transformations together form a group.

	$\Lambda^0{}_0 \geq 1$ orthochronous	$\Lambda^0{}_0 \leq -1$ non-orthochronous
$\det\Lambda=+1$ proper	$\Lambda^{(\mathrm{po})}$	$TP\Lambda^{(\mathrm{po})}$
$\det\Lambda=-1$ improper	$P\Lambda^{(\mathrm{po})}$	$T\Lambda^{(\mathrm{po})}$

Table 1.1: Classification of the Lorentz group. $\Lambda^{(\mathrm{po})}$ is any proper and orthochronous Lorentz transformation.

Exercise 1.1 *Classification of Lorentz transformations.*
(*a*) *Suppose* $\Lambda = AB$ *where* $\Lambda, A,$ *and* B *are Lorentz transformations. Prove that* Λ *is orthochronous if* A *and* B *are both orthochronous or both non-orthochronous, and that* Λ *is non-orthochronous if one of* A *and* B *is orthochronous and the other is non-orthochronous.*
[*hint: Write* $\Lambda^0{}_0 = A^0{}_0 B^0{}_0 + \vec{a}\cdot\vec{b}$ *with* $\vec{a} \equiv (A^0{}_1, A^0{}_2, A^0{}_3)$ *and* $\vec{b} \equiv (B^1{}_0, B^2{}_0, B^3{}_0)$. *One can use* $|\vec{a}\cdot\vec{b}| \leq |\vec{a}||\vec{b}|$ *and* $\vec{a}^2 = {A^0{}_0}^2 - 1, \vec{b}^2 = {B^0{}_0}^2 - 1$ (*derive them*) *to show* $|\vec{a}\cdot\vec{b}| \leq |A^0{}_0 B^0{}_0|$.]
(*b*) *Show that the following sets of Lorentz transformations each form a group:*

1. *proper transformations*
2. *orthochronous transformations*
3. *proper and orthochronous transformations*
4. *proper and orthochronous transformations plus improper and non-orthochronous transformations*

As mentioned earlier (and as will be shown later) boosts and rotations are continuously connected to the identity. Are they then proper and orthochronous? To show that this is the case, it suffices to prove that an infinitesimal transformation can change $\det\Lambda$ and $\Lambda^0{}_0$ only infinitesimally, since then multiplying an infinitesimal transformation cannot jump across the gap between $\det\Lambda = +1$ and $\det\Lambda = -1$ or the gap between $\Lambda^0{}_0 \geq 1$ and $\Lambda^0{}_0 \leq -1$.

An infinitesimal transformation is a transformation that is very close to the identity I and any such transformation λ can be written as

$$\lambda = I + dH \tag{1.52}$$

where d is a small number and H is a 4×4 matrix of order unity meaning the maximum of the absolute values of its elements is about 1. To be specific, we could define it such

that $\max_{\alpha,\beta} |H^\alpha{}_\beta| = 1$ and $d \geq 0$, which uniquely defines the decomposition above. We want to show that for any Lorentz transformation Λ, multiplying $I + dH$ changes the determinant or the $(0,0)$ component only infinitesimally; namely, the change in the determinant vanish as we take d to zero.

The determinant of a $n \times n$ matrix A is defined by

$$\det A \stackrel{\text{def}}{\equiv} \sum_{\text{permutations}} s_{i_1,i_2,\ldots i_n} A_{i_1 1} A_{i_2 2} \ldots A_{i_n n} \tag{1.53}$$

where the sum is taken over $(i_1, i_2, \ldots i_n)$ which is any permutation of $(1, 2, \ldots n)$, and $s_{i_1,i_2,\ldots i_n}$ is $1(-1)$ if $(i_1, i_2, \ldots i_n)$ is an even(odd) permutation. When applied to 4×4 Lorentz transformations, this can be written as

$$\det A \stackrel{\text{def}}{\equiv} \epsilon_{\alpha\beta\gamma\delta} A^\alpha{}_0 A^\beta{}_1 A^\gamma{}_2 A^\delta{}_3 \,, \tag{1.54}$$

where the implicit sum is over $\alpha, \beta, \gamma, \delta = 0, 1, 2, 3$ and $\epsilon_{\alpha\beta\gamma\delta}$ is the totally anti-symmetric 4-th rank tensor defined by

$$\epsilon_{\alpha\beta\gamma\delta} \stackrel{\text{def}}{\equiv} \begin{cases} = \left\{ \begin{array}{c} +1 \\ -1 \end{array} \right\} & \text{if } (\alpha\beta\gamma\delta) \text{ is an } \left\{ \begin{array}{c} \text{even} \\ \text{odd} \end{array} \right\} \text{ permutation of } (0,1,2,3) \\ = 0 & \text{if any of } \alpha\beta\gamma\delta \text{ are equal} \end{cases} \tag{1.55}$$

The standard superscript/subscript rule applies to the indices of $\epsilon_{\alpha\beta\gamma\delta}$; namely, $\epsilon_{0123} = -\epsilon^{0123} = 1$, etc. Then, it is easy to show that

$$\det(I + dH) = 1 + d\,\mathrm{Tr}H + (\text{higher orders in } d) \,, \tag{1.56}$$

where the 'trace' of a matrix A is defined as the sum of the diagonal elements:

$$\mathrm{Tr}A \stackrel{\text{def}}{\equiv} \sum_{\alpha=0}^{3} A^\alpha{}_\alpha \,. \tag{1.57}$$

Exercise 1.2 *Determinant and trace.*
Determinant of a $n \times n$ matrix is defined by

$$\det A \stackrel{\text{def}}{\equiv} s_{i_1 i_2,\ldots i_n} A_{i_1 1} A_{i_2 2} \ldots A_{i_n n}$$

where sum over $i_1, i_2 \ldots i_n$ is implied (each taking values 1 through n) and $s(i_1, i_2 \ldots i_n)$ is the totally asymmetric n-th rank tensor:

$$s_{i_1 i_2 \ldots i_n} \equiv \begin{cases} +1(-1) & \textit{if } (i_1, i_2 \ldots i_n) \textit{ is an even (odd) permutation of } (1, 2, \ldots n). \\ 0 & \textit{if any of } i_1, i_2 \ldots i_n \textit{ are equal.} \end{cases}$$

Show that to first order of a small number d, the determinant of a matrix that is infinitesimally close to the identity matrix I is given by

$$\det(I + dH) = 1 + d\mathrm{Tr}H + (\textit{higher orders in } d)\,,$$

where H is a certain matrix whose size is of order 1, and the trace (Tr) *of a matrix is defined by*

$$\mathrm{Tr}H \equiv \sum_{i=1}^{n} H_{ii}\,.$$

Since all diagonal elements of H are of order unity or smaller, (1.56) tells us that $\det\lambda \to 1$ as we take $d \to 0$. In fact, the infinitesimal transformation λ is a Lorentz transformation, so we know that $\det\lambda = \pm 1$. Thus, we see that the determinant of an infinitesimal transformation is strictly $+1$. It then follows from $\det(\lambda\Lambda) = \det\lambda \det\Lambda$ that multiplying an infinitesimal transformation λ to any transformation Λ does not change the determinant of the transformation.

The $(0,0)$ component of $\lambda\Lambda$ is

$$(\lambda\Lambda)^0{}_0 = [(I + dH)\Lambda]^0{}_0 = [\Lambda + dH\Lambda]^0{}_0 = \Lambda^0{}_0 + d\,(H\Lambda)^0{}_0\,. \tag{1.58}$$

Since $(H\Lambda)^0{}_0$ is a finite number for a finite Λ, the change in the $(0,0)$ component tends to zero as we take $d \to 0$. Thus, no matter how many infinitesimal transformations are multiplied to Λ, the $(0,0)$ component cannot jump across the gap between $+1$ and -1.

Thus, continuously connected Lorentz transformations have the same 'properness' and 'orthochronousness.' Therefore, boosts and rotations, which are continuously connected to the identity, are proper and orthochronous.

Do Lorentz boosts form a group?
A natural question is whether Lorentz boosts form a group by themselves. The answer is no, and this is because two consecutive boosts in different directions turn out to be a boost *plus* a rotation as we will see when we study the generators of the Lorentz group. Thus, boosts and rotations have to be combined to form a group. On the other hand, rotations form a group by themselves.

1.5 Tensors

Suppose A^μ and B^μ are 4-vectors. Each is a set of 4 numbers that transform under a Lorentz transformation Λ as

$$A'^\mu = \Lambda^\mu{}_\alpha A^\alpha, \quad B'^\nu = \Lambda^\nu{}_\beta B^\beta\,. \tag{1.59}$$

Then, the set of 16 numbers $A^\mu B^\nu$ ($\mu, \nu = 0, 1, 2, 3$) transforms as

$$A'^\mu B'^\nu = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta A^\alpha B^\beta \,. \tag{1.60}$$

Anything that has 2 Lorentz indices, which is a set of 16 numbers, and transforms as

$$(\quad)'^{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta (\quad)^{\alpha\beta} \tag{1.61}$$

is called a second rank tensor (or simply a 'tensor'). It may be real, complex, or even a set of operators. Similarly, a quantity that has 3 indices and transforms as

$$(\quad)'^{\mu\nu\sigma} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \Lambda^\sigma{}_\gamma (\quad)^{\alpha\beta\gamma} \tag{1.62}$$

is called a third-rank tensor, and so on. A 4-vector (or simply a 'vector') is a first-rank tensor. A Lorentz-invariant quantity, sometimes called a 'scalar', has no Lorentz index, and thus it is a zero-th rank tensor:

$$(\quad)' = (\quad) \qquad \text{(scalar)} \,. \tag{1.63}$$

Contracted indices do not count in deciding the rank of a tensor. For example,

$$A^\mu B_\mu \;:\; \text{(scalar)}, \quad A_\mu T^{\mu\nu} \;:\; \text{(vector)}, \quad F^{\mu\nu} G_{\mu\sigma} \;:\; \text{(tensor)}, \quad \text{etc.} \tag{1.64}$$

The metric $g_{\mu\nu}$ has two Lorentz indices and thus can be considered a second-rank tensor (thus, the metric *tensor*), then it should transform as

$$g'^{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta g^{\alpha\beta} = g^{\mu\nu} \tag{1.65}$$

where the second equality is due to (1.31). Namely, the metric tensor is invariant under Lorentz transformations.

In order for some equation to be *Lorentz-invariant*, the Lorentz indices have to be the same on both sides of the equality, including the superscript/subscript distinction. By 'Lorentz-invariant', we mean that if an equation holds in one frame, then it holds in any other frame after all the quantities that appear in the equation are evaluated in the new frame. In the literature, such equations are sometimes called *Lorentz covariant*: both sides of the equality change values but the form stays the same. For example, if an equation $A^{\mu\nu} = B^{\mu\nu}$ (which is actually a set of 16 equations) holds in a frame, then it also holds in any other frame:

$$A'^{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta A^{\alpha\beta} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta B^{\alpha\beta} = B'^{\mu\nu} \,. \tag{1.66}$$

Thus, equations such as

$$\begin{aligned} m^2 &= P^\mu P_\mu \,, \\ P^\mu &= A^\mu + B^\mu \,, \\ F^{\mu\nu} &= A^\mu B^\nu \end{aligned} \tag{1.67}$$

are all Lorentz-invariant, assuming of course that the quantities transform in the well-defined ways as described above.

1.6 Fields (classical) and space-time derivatives

A field is a quantity that is a function of space-time point $x^\mu = (t, \vec{x})$ (or 'event'). A scalar quantity that is a function of space time is called a scalar field, a vector quantity that is a function of space time is called a vector field, etc. The rank of a field and the Lorentz transformation properties (scalar, vector, tensor, etc.) are defined in the same way as before, provided that the quantities are evaluated at the same event point before and after a Lorentz transformation; namely,

$$\boxed{\begin{array}{rrl} \text{Scalar field :} & \phi'(x') & = \phi(x) \\ \text{Vector field :} & A'^\mu(x') & = \Lambda^\mu{}_\alpha A^\alpha \\ \text{Tensor field :} & T'^{\mu\nu}(x') & = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta T^{\alpha\beta}(x) \end{array}} \tag{1.68}$$

where x' and x are related by

$$x'^\mu = \Lambda^\mu{}_\alpha x^\alpha \,. \tag{1.69}$$

For example, a vector field associates a set of 4 numbers $A^\mu(x)$ to an event point x, say when an ant sneezes. In another frame, there are a set of 4 numbers $A'^\mu(x')$ associated with the *same event* x', namely, when the ant sneezes in *that* frame, and they are related to the 4 numbers $A^\mu(x)$ in the original frame by the matrix Λ. The functional shape of a primed field is in general different from that of the corresponding unprimed field. Namely, if one plots $\phi(x)$ as a function of x and $\phi'(x')$ as a function of x', they will look different.

When a quantity is a function of x, we naturally encounter space-time derivatives of such quantity. Then a question arises as to how they transform under a Lorentz transformation. Take a scalar field $f(x)$, and form a set of 4 numbers (fields) by taking space-time derivatives:

$$\begin{aligned} \frac{\partial f}{\partial x^\mu}(x) &= \left(\frac{\partial f}{\partial x^0}(x), \frac{\partial f}{\partial x^1}(x), \frac{\partial f}{\partial x^2}(x), \frac{\partial f}{\partial x^3}(x) \right) \\ &= \left(\frac{\partial f}{\partial t}(x), \frac{\partial f}{\partial x}(x), \frac{\partial f}{\partial y}(x), \frac{\partial f}{\partial z}(x) \right) . \end{aligned} \tag{1.70}$$

Then pick two space-time points x_1 and x_2 which are close in space and in time. The argument below is based on the observation that the difference between the values of the scalar field at the two event points is the same in any frame. Since $f(x)$ is a scalar field, the values at a given event is the same before and after a Lorentz transformation:

$$f'(x'_1) = f(x_1), \qquad f'(x'_2) = f(x_2) \,, \tag{1.71}$$

which gives

$$f'(x'_1) - f'(x'_2) = f(x_1) - f(x_2) \,. \tag{1.72}$$

Since x_1 and x_2 are close, this can be written as

$$dx'^{\mu}\frac{\partial f'}{\partial x'^{\mu}}(x'_1) = dx^{\mu}\frac{\partial f}{\partial x^{\mu}}(x_1)\,, \tag{1.73}$$

where summation over μ is implied, and

$$dx'^{\mu} \stackrel{\text{def}}{\equiv} x'^{\mu}_1 - x'^{\mu}_2\,, \quad dx^{\mu} \stackrel{\text{def}}{\equiv} x^{\mu}_1 - x^{\mu}_2\,. \tag{1.74}$$

which tells us that the quantity $dx^{\mu}(\partial f/\partial x^{\mu})$ is Lorentz-invariant. Since $dx^{\mu} = x^{\mu}_1 - x^{\mu}_2$ is a superscripted 4-vector, it follows that $\partial f/\partial x^{\mu}$ should transform as a subscripted 4-vector (which transforms as $A'_{\mu} = \Lambda_{\mu}{}^{\alpha}A_{\alpha}$):

$$\frac{\partial f'}{\partial x'^{\mu}}(x') = \Lambda_{\mu}{}^{\alpha}\frac{\partial f}{\partial x^{\alpha}}(x)\,. \tag{1.75}$$

In fact, together with $dx'^{\mu} = \Lambda^{\mu}{}_{\beta}dx^{\beta}$, we have

$$\begin{aligned} dx'^{\mu}\frac{\partial f'}{\partial x'^{\mu}}(x') &= \left(\Lambda^{\mu}{}_{\beta}dx^{\beta}\right)\left(\Lambda_{\mu}{}^{\alpha}\frac{\partial f}{\partial x^{\alpha}}(x)\right) \\ &= \underbrace{\Lambda_{\mu}{}^{\alpha}\Lambda^{\mu}{}_{\beta}}_{g^{\alpha}{}_{\beta} \quad \text{by (1.32)}}dx^{\beta}\frac{\partial f}{\partial x^{\alpha}}(x) \\ &= dx^{\alpha}\frac{\partial f}{\partial x^{\alpha}}(x)\,, \end{aligned} \tag{1.76}$$

showing that it is indeed Lorentz-invariant.

Thus, the index μ in the differential operator $\partial/\partial x^{\mu}$ acts as a subscript even though it is a superscript on x. To make this point clear, $\partial/\partial x^{\mu}$ is often written using a subscript as

$$\partial_{\mu} \stackrel{\text{def}}{\equiv} \frac{\partial}{\partial x^{\mu}} = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \left(\frac{\partial}{\partial t}, \vec{\nabla}\right)\,. \tag{1.77}$$

Once ∂_{μ} is defined, the standard subscript/superscript rule applies; namely, $\partial^{\mu} = \partial/\partial x_{\mu}$, etc. Symbolically, the operator ∂^{μ} then transforms as a superscripted 4-vector:

$$\partial'^{\mu} = \Lambda^{\mu}{}_{\nu}\partial^{\nu}\,, \tag{1.78}$$

with $\partial'^{\mu} \equiv \partial/\partial x'_{\mu}$.

1.7 Generators of the Lorentz group

In this section, we will focus on the proper and orthochronous Lorentz group. Other elements of the Lorentz group can be obtained by multiplying T, P, and TP to the

elements of this group. The goal is to show that any element Λ that is continuously connected to the identity can be written as[2]

$$\Lambda = e^{\xi_i K_i + \theta_i L_i}, \quad (i = 1, 2, 3) \tag{1.79}$$

where ξ_i and θ_i are real numbers and K_i and L_i are 4×4 matrices. Such group whose elements can be parametrized by a set of continuous real numbers (in our case they are ξ_i and θ_i) is called a *Lie group*. The operators K_i and L_i are called the *generators* of the Lie group.

Any element of the proper and orthochronous Lorentz group is continuously connected to the identity. Actually we have not proven this, but we will at least show that all boosts, rotations and combinations thereof are continuously connected to the identity (and vice versa).

1.7.1 Infinitesimal transformations

Let's start by looking at a Lorentz transformation which is infinitesimally close to the identity:

$$\Lambda^{\mu}{}_{\nu} = g^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu} \tag{1.80}$$

where $\omega^{\mu}{}_{\nu}$ is a set of small (real) numbers. Inserting this to the defining condition (1.31) or equivalently $\Lambda_{\nu\alpha}\Lambda^{\nu}{}_{\beta} = g_{\alpha\beta}$ (1.32), we get

$$\begin{aligned} g_{\alpha\beta} &= \Lambda_{\nu\alpha}\Lambda^{\nu}{}_{\beta} \\ &= (g_{\nu\alpha} + \omega_{\nu\alpha})(g^{\nu}{}_{\beta} + \omega^{\nu}{}_{\beta}) \\ &= g_{\nu\alpha}g^{\nu}{}_{\beta} + \omega_{\nu\alpha}g^{\nu}{}_{\beta} + g_{\nu\alpha}\omega^{\nu}{}_{\beta} + \omega_{\nu\alpha}\omega^{\nu}{}_{\beta} \\ &= g_{\alpha\beta} + \omega_{\beta\alpha} + \omega_{\alpha\beta} + \omega_{\nu\alpha}\omega^{\nu}{}_{\beta}\,. \end{aligned} \tag{1.81}$$

Keeping terms to the first order in ω, we then obtain

$$\omega_{\beta\alpha} = -\omega_{\alpha\beta}\,. \tag{1.82}$$

Namely, $\omega_{\alpha\beta}$ is anti-symmetric (which is true when the indices are both subscript or both superscript; in fact, $\omega^{\alpha}{}_{\beta}$ is *not* anti-symmetric under $\alpha \leftrightarrow \beta$), and thus it has 6 independent parameters:

$$\{\omega_{\alpha\beta}\} = \begin{array}{c} \\ \alpha \\ \downarrow \end{array} \begin{array}{c} \beta \longrightarrow \\ \begin{pmatrix} 0 & \boxed{\omega_{01}} & \boxed{\omega_{02}} & \boxed{\omega_{03}} \\ -\omega_{01} & 0 & \boxed{\omega_{12}} & \boxed{\omega_{13}} \\ -\omega_{02} & -\omega_{12} & 0 & \boxed{\omega_{23}} \\ -\omega_{03} & -\omega_{13} & -\omega_{23} & 0 \end{pmatrix} \end{array} \tag{1.83}$$

[2]In the literature, it is often defined as $\exp i(\xi_i K_i + \theta_i L_i)$, which would make the operators hermitian if the transformation were unitary (e.g. representations of the Lorentz group in the Hilbert space). The Lorentz transformation matrices in space-time are in general not unitary, and for now, we will define without the 'i' so that the expressions become simpler.

This can be conveniently parametrized using 6 anti-symmetric matrices as

$$\begin{aligned}\{\omega_{\alpha\beta}\} &= \omega_{01}\{(M^{01})_{\alpha\beta}\} + \omega_{02}\{(M^{02})_{\alpha\beta}\} + \omega_{03}\{(M^{03})_{\alpha\beta}\} \\ &+ \omega_{23}\{(M^{23})_{\alpha\beta}\} + \omega_{13}\{(M^{13})_{\alpha\beta}\} + \omega_{12}\{(M^{12})_{\alpha\beta}\} \\ &= \sum_{\mu<\nu} \omega_{\mu\nu}\{(M^{\mu\nu})_{\alpha\beta}\}\end{aligned} \tag{1.84}$$

with

$$\begin{aligned}\{(M^{01})_{\alpha\beta}\} &= \left(\begin{array}{c|ccc} 0 & 1 & 0 & 0 \\ \hline -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), \quad \{(M^{23})_{\alpha\beta}\} = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{array}\right), \\ \{(M^{02})_{\alpha\beta}\} &= \left(\begin{array}{c|ccc} 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), \quad \{(M^{13})_{\alpha\beta}\} = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right), \\ \{(M^{03})_{\alpha\beta}\} &= \left(\begin{array}{c|ccc} 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{array}\right), \quad \{(M^{12})_{\alpha\beta}\} = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)\end{aligned} \tag{1.85}$$

Note that for a given pair of μ and ν, $\{(M^{\mu\nu})_{\alpha\beta}\}$ is a 4×4 matrix, while $\omega^{\mu\nu}$ is a real number. The elements $(M^{\mu\nu})_{\alpha\beta}$ can be written in a concise form as follows: first, we note that in the upper right half of each matrix (i.e. for $\alpha < \beta$), the element with $(\alpha,\beta) = (\mu,\nu)$ is 1 and all else are zero, which can be written as $g^{\mu}{}_{\alpha}\, g^{\nu}{}_{\beta}$. For the lower half, all we have to do is to flip α and β and add a minus sign. Combining the two halves, we get

$$(M^{\mu\nu})_{\alpha\beta} = g^{\mu}{}_{\alpha}\, g^{\nu}{}_{\beta} - g^{\mu}{}_{\beta}\, g^{\nu}{}_{\alpha}\,. \tag{1.86}$$

This is defined only for $\mu < \nu$ so far. For $\mu > \nu$, we will use this same expression (1.86) as the definition; then, $(M^{\mu\nu})_{\alpha\beta}$ is anti-symmetric with respect to $(\mu \leftrightarrow \nu)$:

$$(M^{\mu\nu})_{\alpha\beta} = -(M^{\nu\mu})_{\alpha\beta}\,, \tag{1.87}$$

which also means $(M^{\mu\nu})_{\alpha\beta} = 0$ if $\mu = \nu$. Together with $\omega_{\mu\nu} = -\omega_{\nu\mu}$, (1.84) becomes

$$\omega_{\alpha\beta} = \sum_{\mu<\nu} \omega_{\mu\nu}(M^{\mu\nu})_{\alpha\beta} = \sum_{\mu>\nu} \omega_{\mu\nu}(M^{\mu\nu})_{\alpha\beta} = \frac{1}{2}\omega_{\mu\nu}(M^{\mu\nu})_{\alpha\beta}\,, \tag{1.88}$$

where in the last expression, sum over all values of μ and ν is implied. The infinitesimal transformation (1.80) can then be written as

$$\Lambda^{\alpha}{}_{\beta} = g^{\alpha}{}_{\beta} + \frac{1}{2}\omega_{\mu\nu}(M^{\mu\nu})^{\alpha}{}_{\beta}\,, \tag{1.89}$$

or in matrix form,

$$\Lambda = I + \frac{1}{2}\omega_{\mu\nu}M^{\mu\nu}\,. \tag{1.90}$$

where the first indices of $M^{\mu\nu}$, which is a 4×4 matrix for given μ and ν, is taken to be superscript and the second subscript; namely, in the same way as Lorentz transformation. Namely, when no explicit indexes for elements are given, the 4×4 matrix $M^{\mu\nu}$ is defined as

$$M^{\mu\nu} \stackrel{\rm def}{\equiv} \{(M^{\mu\nu})^{\alpha}{}_{\beta}\}\,. \tag{1.91}$$

It is convenient to divide the six matrices to two groups as

$$K_i \stackrel{\rm def}{\equiv} M^{0i}, \qquad L_i \stackrel{\rm def}{\equiv} M^{jk} \quad (i,j,k:\text{cyclic})\,. \tag{1.92}$$

We always use subscripts for K_i and L_i since only possible values are $i=1,2,3$. The elements of the matrices K_i's and L_i's are defined by taking the first Lorentz index to be superscript and the second subscript as is the case for $M^{\mu\nu}$:

$$K_i \stackrel{\rm def}{\equiv} \{(K_i)^{\alpha}{}_{\beta}\}\,, \quad L_i \stackrel{\rm def}{\equiv} \{(L_i)^{\alpha}{}_{\beta}\}\,. \tag{1.93}$$

Later, we will see that K's generate boosts and L's generate rotations. Explicitly, they can be obtained by raising the index α in (1.85) (note also the the minus sign in $L_2 = -M^{13}$):

$$K_1 = \left(\begin{array}{c|ccc} 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), K_2 = \left(\begin{array}{c|ccc} 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), K_3 = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right) \tag{1.94}$$

$$L_1 = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right), L_2 = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right), L_3 = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \tag{1.95}$$

By inspection, we see that the elements of K's and L's can be written as

$$\begin{array}{l} (K_i)^{j}{}_{k} = 0\,, \qquad (K_i)^{0}{}_{\mu} = (K_i)^{\mu}{}_{0} = g^{i}{}_{\mu}\,, \\ (L_i)^{j}{}_{k} = -\epsilon_{ijk}\,, \quad (L_i)^{0}{}_{\mu} = (L_i)^{\mu}{}_{0} = 0\,, \end{array} \qquad (i,j,k=1,2,3;\ \mu=0,1,2,3) \tag{1.96}$$

where ϵ_{ijk} is a totally anti-symmetric quantity defined for $i,j,k=1,2,3$:

$$\epsilon_{ijk} \stackrel{\rm def}{\equiv} \begin{cases} = \left\{\begin{array}{c} +1 \\ -1 \end{array}\right\} & \text{if } (i,j,k) \text{ is an } \left\{\begin{array}{c} \text{even} \\ \text{odd} \end{array}\right\} \text{ permutation of } (1,2,3), \\ = 0 & \text{if any of } i,j,k \text{ are equal.} \end{cases} \tag{1.97}$$

An explicit calculation shows that K's and L's satisfy the following commutation relations:

$$\begin{aligned} [K_i, K_j] &= -\epsilon_{ijk} L_k \\ [L_i\,, L_j\,] &= \epsilon_{ijk} L_k \\ [L_i\,, K_j] &= \epsilon_{ijk} K_k\,, \end{aligned} \tag{1.98}$$

where sum over $k = 1, 2, 3$ is implied, and the *commutator* of two operators A, B is defined as

$$[A, B] \stackrel{\text{def}}{\equiv} AB - BA\,. \tag{1.99}$$

Note that the relation $[K_i, K_j] = -\epsilon_{ijk} L_k$ can also be written as $[K_i, K_j] = -L_k$ (i, j, k: cyclic), etc.

Exercise 1.3 *Verify the commutation relations (1.98). You may numerically verify them, or you may try proving generally by using the general formulas (1.86) and (1.96) for the elements of the matrixes.*

Exercise 1.4 *Boost in a general direction.*
Start from the formula for boost (1.9) where $P_\parallel$ is the component of $\vec{P}$ parallel to $\vec{\beta}$, and $\vec{P}_\perp$ is the component perpendicular to $\vec{\beta}$; namely,

$$P_\parallel = \vec{P} \cdot \vec{n}, \qquad \text{and} \quad \vec{P}_\perp = \vec{P} - P_\parallel \vec{n}$$

with $\vec{n} = \vec{\beta}/\beta$ (and similarly for $\vec{P}'$). Note that $\vec{\beta}$ is well-defined in the primed frame also by the particular relative orientation of the two frames chosen.
(a) Show that the corresponding Lorentz transformation matrix is given by

$$\Lambda = \begin{pmatrix} \gamma & \gamma\beta_x & \gamma\beta_y & \gamma\beta_z \\ \gamma\beta_x & 1+\rho\beta_x^2 & \rho\beta_x\beta_y & \rho\beta_x\beta_z \\ \gamma\beta_y & \rho\beta_x\beta_y & 1+\rho\beta_y^2 & \rho\beta_y\beta_z \\ \gamma\beta_z & \rho\beta_x\beta_z & \rho\beta_y\beta_z & 1+\rho\beta_z^2 \end{pmatrix}, \quad \text{with} \quad \rho \equiv \frac{\gamma - 1}{\beta^2}.$$

(b) Show that when β is small, the Lorentz transformation matrix for a boost is given to the first order in β by

$$\Lambda = 1 + \beta_i K_i\,. \quad (\textit{summed over } i = 1, 2, 3)$$

(c) In the explicit expression of Λ given above, one notes that the top row $[\Lambda^0{}_\mu\ (\mu = 0, 1, 2, 3)]$ and the left-most column $[\Lambda^\mu{}_0\ (\mu = 0, 1, 2, 3)]$ are nothing but the velocity 4-vector $\eta^\mu = (\gamma, \vec{\beta}\gamma)$. Let's see how it works for general Lorentz transformations (proper and orthochronous). Suppose the relative orientation of the two frames K and K' is not given by $\vec{\beta}'_K = -\vec{\beta}_{K'}$, where $\vec{\beta}'_K$ is the velocity of the origin of K

measured in K', and $\vec{\beta}_{K'}$ is the velocity of the origin of K' measured in K. Let Λ be the corresponding Lorentz transformation. Express ${\Lambda^0}_\mu$ and ${\Lambda^\mu}_0$ in terms of $\vec{\beta}'_K$ and $\vec{\beta}_{K'}$. (hint: Place a mass m at the origin of K and view it from K', and place a mass at the origin of K' and view it from K. Their 4-momenta are known in terms of $\vec{\beta}'_K$ and $\vec{\beta}_{K'}$ which can also be written using Λ.)

1.7.2 Finite transformations

Now we will show that any finite (namely, not infinitesimal) rotation can be written as $e^{\theta_i L_i}$, and any finite boost can be written as $e^{\xi_i K_i}$, where θ_i and ξ_i $(i = 1, 2, 3)$ are some finite real numbers. First, however, let us review some relavant mathematics:

Matrix exponentiations

The exponential of a $m \times m$ matrix A is also a $m \times m$ matrix defined by

$$\boxed{e^A \stackrel{\text{def}}{\equiv} \lim_{n\to\infty} \left(I + \frac{A}{n}\right)^n}, \tag{1.100}$$

which can be expanded on the right hand side as

$$e^A = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{n(n-1)\dots(n-k+1)}{k\,!} \frac{A^k}{n^k}\,. \tag{1.101}$$

Since the sum is a rapidly converging series, one can sum only the terms with $k \ll n$ for which $n(n-1)\dots(n-k+1) \approx n^k$. It then leads to

$$\boxed{e^A = \sum_{k=0}^{\infty} \frac{A^k}{k\,!}}, \tag{1.102}$$

which can also be regarded as a definition of e^A.

Using the definition (1.100) or (1.102), we see that

$$\boxed{\left(e^A\right)^\dagger = e^{A^\dagger}}, \tag{1.103}$$

where the hermitian conjugate of a matrix A is defined by $(A^\dagger)_{ij} \equiv A^*_{ji}$. The determinant of e^A can be written using (1.100) as

$$\begin{aligned} \det e^A &= \lim_{n\to\infty} \left[\det\left(I + \frac{A}{n}\right)\right]^n \\ &= \lim_{n\to\infty} \left(1 + \frac{\mathrm{Tr}A}{n} + \dots + \frac{c_k}{n^k} + \dots\right)^n, \end{aligned} \tag{1.104}$$

where we have used (1.56). This does not depend on c_k ($k > 1$) since the derivative with respect to c_k vanishes in the limit $n \to \infty$ as can be readily verified. Thus, c_k ($k > 1$) can be set to zero and we have

$$\boxed{\det e^A = e^{\mathrm{Tr}A}}\,. \tag{1.105}$$

The derivative of e^{xA} (x is a number, while A is a constant matrix) with respect to x can be obtained using (1.102),

$$\frac{d}{dx}e^{xA} = \sum_{k=1}^{\infty}\frac{(k\,x^{k-1})A^k}{k\,!} = A\sum_{k=1}^{\infty}\frac{x^{k-1}A^{k-1}}{(k-1)\,!}\,; \tag{1.106}$$

thus,

$$\boxed{\frac{d}{dx}e^{xA} = Ae^{xA}}\,. \tag{1.107}$$

There is an important theorem that expresses a product of two exponentials in terms of single exponential, called the *Campbell-Baker-Hausdorff (CBH) theorem* (presented here without proof):

$$\boxed{e^Ae^B = e^{A+B+\frac{1}{2}[A,B]+\cdots}}\,, \tag{1.108}$$

where '$\cdots$' denotes the higher-order commutators of A and B such as $[A,[A,B]]$, $[A,[[A,B],B]]$ etc. with *known coefficients*. Note that the innermost commutator is always $[A,B]$ since otherwise it is zero ($[A,A] = [B,B] = 0$), and thus if $[A,B]$ is a commuting quantity (a c-number), then '$\cdots$' is zero. Applying (1.108) to $B = -A$, we get

$$e^Ae^{-A} = e^{A-A} = I\,, \tag{1.109}$$

or

$$\boxed{(e^A)^{-1} = e^{-A}}\,. \tag{1.110}$$

∎

Rotation

An infinitesimal rotation around the z-axis by $\delta\theta$ [Figure 1.3(a)] can be written as

$$\begin{pmatrix}x'\\y'\end{pmatrix} = \begin{pmatrix}x-\delta\theta\,y\\y+\delta\theta\,x\end{pmatrix} = \begin{pmatrix}1 & -\delta\theta\\ \delta\theta & 1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = (I+\delta\theta L_z)\begin{pmatrix}x\\y\end{pmatrix}\,, \tag{1.111}$$

with

$$L_z = \begin{array}{c} \\ x\\ y\end{array}\begin{array}{c}\begin{array}{cc} x & y\end{array}\\ \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\end{array}\,. \tag{1.112}$$

Then, a rotation by a finite angle θ is constructed as n consecutive rotations by θ/n each and taking the limit $n \to \infty$. Using (1.111), it can be written as

$$\begin{aligned} \begin{pmatrix} x' \\ y' \end{pmatrix} &= \lim_{n\to\infty} \left(I + \frac{\theta}{n} L_z \right)^n \begin{pmatrix} x \\ y \end{pmatrix} \\ &= e^{\theta L_z} \begin{pmatrix} x \\ y \end{pmatrix}, \end{aligned} \tag{1.113}$$

where we have used the definition (1.100).

Now let us find out the exlicit form of $e^{\theta L_z}$. From the explicit expression of L_z (1.112), we have $L_z^2 = -I$, $L_z^3 = -L_z$, $L_z^4 = I$, etc. In general,

$$L_z^{4n} = I, \quad L_z^{4n+1} = L_z, \quad L_z^{4n+2} = -I, \quad L_z^{4n+3} = -L_z, \tag{1.114}$$

where n is an integer. Using the second definition of e^A (1.102), the rotation matrix $e^{\theta L_z}$ can then be written in terms of the trigonometirc functions as

$$e^{\theta L_z} = I + \theta L_z + \frac{\theta^2}{2!} \underbrace{L_z^2}_{-I} + \frac{\theta^3}{3!} \underbrace{L_z^3}_{-L_z} + \ldots \tag{1.115}$$

$$= \underbrace{\left(1 - \frac{\theta^2}{2!} + \ldots\right)}_{\cos\theta} I + \underbrace{\left(\theta - \frac{\theta^3}{3!} + \ldots\right)}_{\sin\theta} L_z \tag{1.116}$$

$$= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \tag{1.117}$$

which is probably a more familiar form of a rotation around the z-axis by an angle θ.

Similarly, rotations around x and y axes are generated by L_x and L_y as obtained by cyclic permutations of (x, y, z) in the derivation above. Switching to numerical

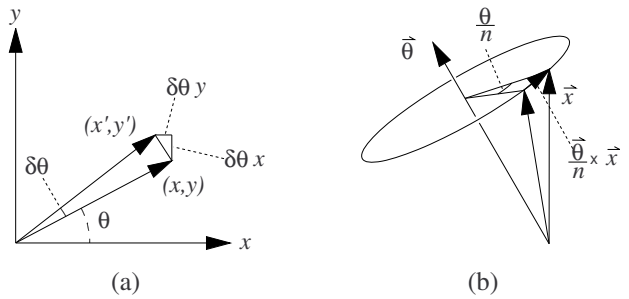


Figure 1.3: Infinitesimal rotation around the z-axis by an angle $\delta\theta$ (a), and around a general direction $\vec{\theta}$ by an angle θ/n (b).

indices $[(L_x, L_y, L_z) \equiv (L_1, L_2, L_3)]$,

$$L_1 = \begin{matrix} \\ 2 \\ 3 \end{matrix} \begin{matrix} \begin{matrix} 2 & 3 \end{matrix} \\ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \end{matrix}, \quad L_2 = \begin{matrix} \\ 3 \\ 1 \end{matrix} \begin{matrix} \begin{matrix} 3 & 1 \end{matrix} \\ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \end{matrix}, \quad L_3 = \begin{matrix} \\ 1 \\ 2 \end{matrix} \begin{matrix} \begin{matrix} 1 & 2 \end{matrix} \\ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \end{matrix}. \tag{1.118}$$

Are these identical to the definition (1.95) which was given in 4×4 matrix form, or equivalently (1.96)? Since $\delta\theta L_i$ is the *change* of coordinates by the rotation, the elements of a 4×4 matrix corresponding to unchanged coordinates should be zero. We then see that the L's given above are indeed identical to (1.95); namely, in 2×2 form or in 4×4 form, $(L_i)^j{}_k = -1$ and $(L_i)^k{}_j = 1$ and all other elements are zero where (ijk) are cyclic.

A general rotation would then be given by

$$e^{\theta_i L_i} = e^{\vec{\theta}\cdot\vec{L}}, \tag{1.119}$$

where

$$\vec{\theta} \stackrel{\text{def}}{\equiv} (\theta_1, \theta_2, \theta_3), \quad \vec{L} \stackrel{\text{def}}{\equiv} (L_1, L_2, L_3). \tag{1.120}$$

As we will see below, this is a rotation around the direction $\vec{\theta}$ by an angle $\theta \equiv |\vec{\theta}|$. To see this, first we write $e^{\theta_i L_i}$ using the definition (1.100):

$$e^{\theta_i L_i} = \lim_{n\to\infty} \left(I + \frac{\theta_i L_i}{n} \right)^n, \tag{1.121}$$

which shows that it is a series of small rotations each given by $I + \theta_i L_i/n$. The action of such an infinitesimal transformation [Figure 1.3(b)] on $\vec{x}$ is (noting that $I^j{}_0 = (L_i)^j{}_0 = 0$)

$$\begin{aligned} x'^j &= \left(I + \frac{\theta_i L_i}{n} \right)^j{}_\mu x^\mu = \left(I + \frac{\theta_i L_i}{n} \right)^j{}_k x^k \\ &= g^j{}_k x^k + \frac{1}{n}\theta_i \underbrace{(L_i)^j{}_k}_{-\epsilon_{ijk} \text{ by (1.96)}} x^k \\ &= x^j - \frac{1}{n}\epsilon_{ijk}\,\theta_i x^k \\ &= x^j + \frac{1}{n}(\vec{\theta}\times\vec{x})^j \end{aligned} \tag{1.122}$$

where we have used the definition of the three-dimensional cross product

$$(\vec{a}\times\vec{b})_i = \epsilon_{ijk}\, a_j b_k\,. \tag{1.123}$$

Thus, $I+\theta_i L_i/n$ is nothing but a small rotation around $\vec{\theta}$ by an angle θ/n (Figure 1.3). Then n such rotations applied successively will result in a rotation by an angle θ around the same axis $\vec{\theta}$.

Boosts

A boost in x direction by a velocity β is given by (1.26):

$$\Lambda = \begin{array}{c} \\ t \\ x \end{array} \begin{array}{c} \begin{array}{cc} t & x \end{array} \\ \begin{pmatrix} \gamma & \eta \\ \eta & \gamma \end{pmatrix} \end{array}, \qquad \left(\gamma = \frac{1}{\sqrt{1-\beta^2}}, \quad \eta = \beta\gamma\right). \tag{1.124}$$

When β is small (let's call it δ), $\gamma \approx 1$ and $\eta \approx \delta$ to the first order in δ; then, the infinitesimal boost can be written as

$$\Lambda = \begin{pmatrix} 1 & \delta \\ \delta & 1 \end{pmatrix} = I + \delta K_x\,, \tag{1.125}$$

with

$$K_x = \begin{array}{c} \\ t \\ x \end{array} \begin{array}{c} \begin{array}{cc} t & x \end{array} \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{array}. \tag{1.126}$$

Suppose we apply n such boosts consecutively, where we take n to infinity while $n\delta$ is fixed to a certain value ξ:

$$n\delta = \xi\,. \tag{1.127}$$

Then the resulting transformation is

$$\Lambda = \lim_{n\to\infty}\left(I + \frac{\xi}{n}K_x\right)^n = e^{\xi K_x}\,, \tag{1.128}$$

where we have used the definition (1.100). Is ξ the velocity of this boost? The answer is no, even though it is a function of the velocity. Let's expand the exponential above by the second definition (1.100) and use $K_x^2 = I$:

$$\begin{aligned} \Lambda &= e^{\xi K_x} \qquad (1.129)\\ &= I + \xi K_x + \frac{\xi^2}{2\,!}\underbrace{K_x^2}_{I} + \frac{\xi^3}{3\,!}\underbrace{K_x^3}_{K_x} + \ldots \\ &= \underbrace{\left(1 + \frac{\xi^2}{2\,!} + \ldots\right)}_{\cosh\xi} I + \underbrace{\left(\xi + \frac{\xi^3}{3\,!} + \ldots\right)}_{\sinh\xi} K_x \\ &= \begin{pmatrix} \cosh\xi & \sinh\xi \\ \sinh\xi & \cosh\xi \end{pmatrix}. \qquad (1.130) \end{aligned}$$

Comparing with (1.125), we see that this is a boost of a velocity β given by

$$\gamma = \cosh\xi, \qquad \eta = \sinh\xi \tag{1.131}$$

or

$$\beta = \frac{\eta}{\gamma} = \tanh\xi\,. \tag{1.132}$$

Note that the relation $\gamma^2 - \eta^2 = 1$ (1.6) is automatically satisfied since $\cosh\xi^2 - \sinh\xi^2 = 1$.

Thus, n consecutive boosts by a velocity ξ/n each did not result in a boost of a velocity ξ; rather, it was a boost of a velocity $\beta = \tanh\xi$. This breakdown of the simple addition rule of velocity is well known: the relativistic rule of velocity addition states that two consecutive boosts, by β_1 and by β_2, do not result in a boost of $\beta_1+\beta_2$, but in a boost of a velocity β_0 given by

$$\beta_0 = \frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2}\,. \tag{1.133}$$

Due to the identity $\tanh(\xi_1+\xi_2) = (\tanh\xi_1 + \tanh\xi_1)/(1 + \tanh\xi_1\tanh\xi_2)$, however, it becomes additive when velocities are transformed by $\beta_i = \tanh\xi_i$ $(i = 0, 1, 2)$; namely, $\xi_0 = \xi_1 + \xi_2$ holds.

The matrix $K_x(\equiv K_1)$ given in (1.126) is identical to the 4×4 form given in (1.94) when all other elements that correspond to unchanged coordinates are set to zero. The boosts along y and z directions are obtained by simply replacing x with y or z in the derivation above. Thus, we see that K_2 and K_3 given in (1.94) indeed generate boosts in y and z directions, respectively.

A boost in a general direction would then be given by

$$\Lambda = e^{\xi_i K_i}\,, \tag{1.134}$$

where $\vec{\xi} \equiv (\xi_1, \xi_2, \xi_3)$ are the parameters of the boost. In order to see what kind of transformation this represents, let's write it as a series of infinitesimal transformations using (1.100):

$$e^{\xi_i K_i} = \lim_{n\to\infty}\left(I + \frac{\xi_i}{n}K_i\right)^n\,. \tag{1.135}$$

From the explicit forms of K_i (1.94), we can write the infinitesimal transformation as

$$I + \frac{\xi_i}{n}K_i = I + \frac{1}{n}\left(\begin{array}{c|ccc} 0 & \xi_1 & \xi_2 & \xi_3 \\ \hline \xi_1 & & & \\ \xi_2 & & 0 & \\ \xi_3 & & & \end{array}\right)\,. \tag{1.136}$$

On the other hand, a boost in a general direction by a small velocity $\vec{\delta}$ is given by (1.9) with $\gamma \approx 1$, $\eta \approx \delta$ and $\delta \equiv |\vec{\delta}|$:

$$\begin{pmatrix} E' \\ P'_\parallel \end{pmatrix} = \begin{pmatrix} 1 & \delta \\ \delta & 1 \end{pmatrix} \begin{pmatrix} E \\ P_\parallel \end{pmatrix}, \qquad \vec{P}'_\perp = \vec{P}_\perp, \tag{1.137}$$

or

$$\left\{ \begin{array}{lcll} E' &=& E + \delta P_\parallel & \to E' = E + \vec{\delta}\cdot\vec{P} \\ P'_\parallel &=& P_\parallel + \delta E & \to \vec{P}' = \vec{P} + E\,\vec{\delta} \\ \vec{P}'_\perp &=& \vec{P}_\perp & \end{array} \right. \tag{1.138}$$

where we have used $\vec{P}^{(\prime)} = P_\parallel^{(\prime)}\hat{\delta} + \vec{P}_\perp^{(\prime)}$ ($\hat{\delta} \equiv \vec{\delta}/\delta$). This can be written in 4×4 matrix form as

$$\begin{pmatrix} E' \\ P'_x \\ P'_y \\ P'_z \end{pmatrix} = \left[I + \left(\begin{array}{c|ccc} 0 & \delta_1 & \delta_2 & \delta_3 \\ \hline \delta_1 & & & \\ \delta_2 & & 0 & \\ \delta_3 & & & \end{array} \right) \right] \begin{pmatrix} E \\ P_x \\ P_y \\ P_z \end{pmatrix}. \tag{1.139}$$

Comparing this with (1.136), we can identify that $I + \xi_i K_i/n$ as a boost in $\vec{\xi}$ direction by a velocity parameter ξ/n ($\xi \equiv |\vec{\xi}|$). Then n consecutive such boosts will result in a boost in the same direction. Since the rule of addition of velocity (1.133) is valid in any direction as long as the boosts are in the same direction, the n boosts by velocity ξ/n each will result in a single boost of velocity $\beta = \tanh\xi$ as before. Thus, $e^{\xi_i K_i}$ represents a boost in $\vec{\xi}$ direction by a velocity $\beta = \tanh\xi$.

Boost + rotation

First, we show that a rotation followed by a rotation is a rotation, but a boost followed by a boost is in general not a boost. Consider a rotation $e^{\theta_i L_i}$ followed by another rotation $e^{\phi_i L_i}$ where $\vec{\theta}$ and $\vec{\phi}$ are arbitrary vectors. Using the CBH theorem (1.108), we can write the product of the two transformations as

$$e^{\phi_i L_i} e^{\theta_j L_j} = e^{\phi_i L_i + \theta_j L_j + \frac{1}{2}[\phi_i L_i, \theta_j L_j] + \cdots}, \tag{1.140}$$

where '$\cdots$' represents terms with higher-order commutators such as $[\phi_i L_i, [\phi_j L_j, \theta_k L_k]]$ etc. Now we can use the commutation relations (1.98) to remove all commutators in the exponent on the right hand side. The result will be a linear combination of L's with well-defined coefficients (call them α_i) since the coefficients in '$\cdots$' in the CBH theorem are known. Here, there will be no K's appearing in the linear combination because of the commutation relation $[L_i, L_j] = \epsilon_{ijk} L_k$. Thus, the product is written as

$$e^{\phi_i L_i} e^{\theta_j L_j} = e^{\alpha_i L_i} \quad (\vec{\alpha} : \text{a function of } \vec{\phi}, \vec{\theta}), \tag{1.141}$$

which is just another rotation.

Next, consider a boost $e^{\xi_i K_i}$ followed by another boost $e^{\xi'_i K_i}$:

$$e^{\xi'_i K_i} e^{\xi_j K_j} = e^{\xi'_i K_i + \xi_j K_j + \frac{1}{2}[\xi'_i K_i, \xi_j K_j] + \cdots}. \tag{1.142}$$

Again, the brackets can be removed by the commutation relations (1.98) reducing the exponent to a linear combination of K's and L's. This time, there will be L's appearing through the relation $[K_i, K_j] = -\epsilon_{ijk} L_k$ which are in general not cancelled among different terms. Thus, a boost followed by a boost is not in general another boost; rather, it is a combination of boost and rotation:

$$e^{\xi'_i K_i} e^{\xi_j K_j} = e^{\alpha_i K_i + \beta_i L_i} \quad (\vec{\alpha}, \vec{\beta} : \text{functions of } \vec{\xi}, \vec{\xi}'). \tag{1.143}$$

It is easy to show, however, that if two boosts are in the same direction, then the product is also a boost.

Any combinations of boosts and rotations can then be written as

$$\boxed{\Lambda = e^{\xi_i K_i + \theta_i L_i} = e^{\frac{1}{2} a_{\mu\nu} M^{\mu\nu}}}, \tag{1.144}$$

where we have defined the anti-symmetric tensor $a_{\mu\nu}$ by

$$a_{0i} \overset{\text{def}}{\equiv} \xi_i\,, \quad a_{ij} \overset{\text{def}}{\equiv} \theta_k \ (i, j, k : \text{cyclic}), \quad a_{\mu\nu} = -a_{\nu\mu}\,, \tag{1.145}$$

and the factor $1/2$ arises since terms with $\mu > \nu$ as well as $\mu < \nu$ are included in the sum. The expression of an infinitesimal transformation (1.90) is nothing but this expression in the limit of small $a_{\mu\nu}$. Since we now know that any product of such transformations can also be written as (1.144) by the CBH theorem, we see that the set of Lorentz transformations connected to the identity is saturated by boosts and rotations.

We have seen that the generators K's and L's and their commutation relations (called the *Lie algebra*) play critical roles in understanding the Lorentz group. In fact, generators and their commutation relations completely determine the structure of the Lie group, as described briefly below.

Structure constants

When the commutators of generators of a Lie group are expressed as linear combinations of the generators themselves, the coefficients of the linear expressions are called the *structure constants* of the Lie group. For example, the coeffients $\pm\epsilon_{ijk}$ in (1.98) are the structure constants of the Lorentz group. We will now show that the structure constants completely define the structure of a Lie group. To see this, we have to define what we mean by 'same structure'. Two sets $\mathcal{F}(\ni f)$ and $\mathcal{G}(\ni g)$ are

said to have the same structure if there is a mapping between $\mathcal{F}$ and $\mathcal{G}$ such that if $f_1, f_2 \in \mathcal{F}$ and $g_1, g_2 \in \mathcal{G}$ are mapped to each other:

$$f_1 \leftrightarrow g_1, \quad f_2 \leftrightarrow g_2 \tag{1.146}$$

then, the products $f_1 f_2$ and $g_1 g_1$ are also mapped to each other by the same mapping:

$$f_1 f_2 \leftrightarrow g_1 g_2 \,; \tag{1.147}$$

namely, the mapping *preserves* the product rule.

Suppose the sets $\mathcal{F}$ and $\mathcal{G}$ are Lie groups with the same number of generators F_i and G_i and that they have the same set of structure constants c_{ijk}

$$[F_i, F_j] = c_{ijk} F_k, \quad [G_i, G_j] = c_{ijk} G_k \,. \tag{1.148}$$

Since $\mathcal{F}$ and $\mathcal{G}$ are Lie groups, any element of $\mathcal{F}$ and $\mathcal{G}$ can be expressed in exponential form using the corresponding generators and a set of real parameters. We define the mapping between $\mathcal{F}$ and $\mathcal{G}$ by the same set of the real parameters:

$$f = e^{\alpha_i F_i} \leftrightarrow g = e^{\alpha_i G_i} \,. \tag{1.149}$$

If $f_1 \leftrightarrow g_1$ and $f_2 \leftrightarrow g_2$, then they can be written as

$$f_1 = e^{\alpha_i F_i} \;\leftrightarrow\; g_1 = e^{\alpha_i G_i} \tag{1.150}$$

$$f_2 = e^{\beta_i F_i} \;\leftrightarrow\; g_2 = e^{\beta_i G_i}, \tag{1.151}$$

where α_i and β_i are certain sets of real parameters. Then the question is whether the products $f_1 f_2$ and $g_1 g_2$ are mapped to each other by the same mapping. The products $f_1 f_2$ and $g_1 g_2$ can be written using the CBH theorem as

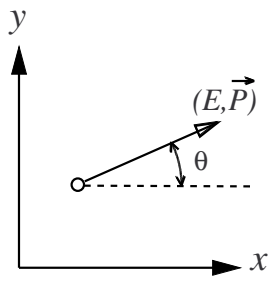
$$f_1 f_2 = \; e^{\alpha_i F_i} e^{\beta_j F_j} = e^{\alpha_i F_i + \beta_j F_j + \frac{1}{2}[\alpha_i F_i, \beta_j F_j] + \cdots} \; = e^{\phi_i F_i}, \tag{1.152}$$

$$g_1 g_2 = \; e^{\alpha_i G_i} e^{\beta_j G_j} = e^{\alpha_i G_i + \beta_j G_j + \frac{1}{2}[\alpha_i G_i, \beta_j G_j] + \cdots} \; = e^{\gamma_i G_i}, \tag{1.153}$$

The numbers ϕ_i and γ_i are obtained by removing the commutators using the commutation relations (1.148), and thus completely determined by α_i, β_i and c_{ijk}; namely, $\phi_i = \gamma_i$, and thus $f_1 f_2$ and $g_1 g_2$ are mapped to each other by (1.149). Thus, if two Lie groups have the same set of structure constants, then they have the same structure. ∎

Problems

1.1 *Boost and invariant mass.*
A particle with energy E and momentum P is moving in the x-y plane at an angle θ anti clockwise from the x axis.



(a) Boost the particle in $+x$ direction with velocity β (or look at this particle from a Lorentz frame moving in $-x$ direction with velocity β), and write down the resulting 4-momentum in terms of E, $p \equiv |\vec{P}|$, θ and β.
(b) Calculate explicitly the invariant mass of the boosted 4-momentum and verify that it is the same as in the original frame; i.e. $E^2 - P^2$.
(c) Express the $\tan\theta'$ in terms of β, θ and β^0, where θ' is the angle of the direction of the particle with respect to the x axis after the boost and β^0 is the velocity of the particle in the original frame.

1.2 *Two-body decay.*
Consider the decay of a particle of mass M to two particles of masses m_1 and m_2.
(a) Show that the momentum of the daughter particles in the rest frame of M is given by

$$p = \frac{\sqrt{\lambda(M^2, m_1^2, m_2^2)}}{2M}$$

with

$$\lambda(x, y, z) \equiv x^2 + y^2 + z^2 - 2xy - 2yz - 2zx\,.$$

(b) Suppose M is moving with velocity β in the lab frame, and it decays to two particles uniformly in 4π steradian in its own rest frame (no spin polarization). What is the maximum and minimum energies of daughter particle 1 in the lab frame? Find the energy distribution $f(E_1)dE_1$ of daughter particle 1 in the lab frame. Normalize $f(E_1)$ such that

$$\int_0^\infty f(E_1)dE_1 = 1\,.$$

(hint: The uniform decay means that $\cos\theta$ is distributed uniformly from -1 to 1, where θ is the polar angle of the particle with respect to some axis.)

1.3 *Show that a finite rotation by $\vec{\xi}$ and a finite boost by $\vec{\theta}$ commute when $\vec{\xi}$ and $\vec{\theta}$ are in a same direction, and that a finite boost by $\vec{\xi}$ and another boost by $\vec{\xi}'$ commute if $\vec{\xi}$ and $\vec{\xi}'$ are in the same direction:*

$$\left[e^{\xi_i K_i}, e^{\theta_j L_j}\right] = 0 \quad \text{if } \vec{\theta} = c\,\vec{\xi},$$

$$\left[e^{\xi_i K_i}, e^{\xi'_j K_j}\right] = 0 \quad \text{if } \vec{\xi}' = c\,\vec{\xi},$$

where summations over i, j are implicit and c is a constant. (hint: use the CBH theorem.)

1.4 *Generators of $SU(n)$ group.*
The Lie group formed by a set of complex $n \times n$ matrices which are unitary:

$$U^\dagger U = I$$

and whose determinants are unity:

$$\det U = 1$$

is called the special unitary group in n dimensions or $SU(n)$. Show that the generators G_k of a $SU(n)$ group are traceless hermitian matrixes and that there are $n^2 - 1$ of those:

$$\{G_k\ (k = 1, \ldots n^2 - 1)\}, \quad \text{with} \quad G_k^\dagger = G_k\,, \quad \mathrm{Tr} G_k = 0\,.$$

Here, any element of the group is written in terms of the generators as

$$U = e^{i\,\gamma_k G_k}\,,$$

where γ_k is a real number and the index k is summed over.

Chapter 2

Lorentz-invariant Wave Equations

In this chapter we introduce Lorentz-invariant wave equations which are defined as follows: if a wave function satisfies a certain equation in one frame, then when the wave function is transformed to another frame (i.e. 'viewed' in that frame), it satisfies the same form of equation in terms of the coordinates in the latter frame. We then attempt to construct such equation by applying the standard correspondence between energy-momentum and the differential operators $(i\partial_0, -i\vec{\nabla})$ to the relativistic relation $E^2 - \vec{P}^2 = m^2$, which leads us to the Klein-Gordon equation. The probability current is constructed from the wave function such that it is conserved. We will then see that if we adhere to the standard interpretations of energy and probability, it has solutions with negative energy and negative probability. We start this chapter by examining what is meant by energy and probability.

2.1 Energy and momentum

In non-relativistic quantum mechanics, the differential operators that correspond to energy and momentum are given by

$$E: \; i\frac{\partial}{\partial t}\,, \qquad \vec{P}: \; -i\vec{\nabla}\,. \tag{2.1}$$

Thus, in general, a wave function which is an eigenfunction of given values of energy-momentum $(E, \vec{P})$ has the plane-wave form

$$\phi(t, \vec{x}) = C(E, \vec{P}) e^{-i(Et - \vec{P}\cdot\vec{x})}, \tag{2.2}$$

where the constant coefficient $C(E, \vec{P})$ in general can have multiple components [which will make $\phi(t, \vec{x})$ multiple-component] and can depend on $(E, \vec{P})$. It clearly gives the desired eigenvalues for energy-momentum:

$$\left(i\frac{\partial}{\partial t}\right)\phi = E\phi\,, \qquad \left(-i\vec{\nabla}\right)\phi = \vec{P}\phi\,. \tag{2.3}$$

Namely, the wave length at a fixed time gives the momentum and the oscillation frequency at a fixed position gives the energy. According to the definition (2.1), the sense of the phase rotation with respect to time gives the sign of energy:

$$\phi \sim e^{-i|E|t} : \text{positive energy}, \quad \phi \sim e^{+i|E|t} : \text{negative energy}\,. \tag{2.4}$$

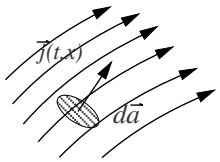
2.2 Conserved current and probability

The concept of conserved current is simple and natural. When there is anything that 'exists' and 'moves around' in space and if it is not created or destroyed, then there is a conserved current. The electric charge and the classical density of gas flow are some examples. In particular we expect that the probability density of a wave fucntion to have such property at least when there is no interactions. Thus, our strategy is to look for a conserved quantity for a given wave equation, and attempt to interpret it as the probability current.

Let's formulate the concept of conserved current more precisely. First, assume that there is some density of some 'material' $\rho(t,\vec{x})$ measured in a given frame which we call 'the laboratory frame' without loss of generaility. Then, one can form a 3-vector 'flux' defined by

$$\vec{j}(t,\vec{x}) \stackrel{\text{def}}{\equiv} \rho(t,\vec{x})\vec{\beta}(t,\vec{x}) \tag{2.5}$$

where $\beta(t,\vec{x})$ is the velocity of this 'material' at $(t,\vec{x})$. Then the quantity per unit time that flows across a small area $d\vec{a}$, which is fixed in the laboratory frame, is given by



$$\vec{j}\cdot d\vec{a} \qquad \text{(flow across } d\vec{a}\text{)} \tag{2.6}$$

No net creation nor destruction of the material means that, if we fix a volume V in space, the change in the total amount of the material in the volume, $\int_V \rho dv$, is entirely accounted for by how much is flowing in across the boundary surface A of the volume. Namely,

$$\frac{d}{dt}\int_V \rho(t,\vec{x})dv = -\int_A \vec{j}\cdot d\vec{a} \tag{2.7}$$

or moving the time derivative to the inside of the integral and using the Gauss's theorem (which is correct for any $\vec{j}$, conserved or not) on the right hand side,

$$\int_V \frac{d}{dt}\rho(t,\vec{x})dv = -\int_V \vec{\nabla}\cdot\vec{j}dv\,. \tag{2.8}$$

Since this should hold for any volume V, the integrands should be equal point by point:

$$\frac{d\rho}{dt} + \vec{\nabla} \cdot \vec{j} = 0\,. \tag{2.9}$$

Note that ρ could in general have negative values such as in the case of the electric charge.

If we define a 4-component quantity $j^\mu \equiv (\rho, \vec{j})$, then (2.9) above can be written as $\partial_\mu j^\mu = 0$, which looks like Lorentz-invariant. This, however, is true only if j^μ transforms as a Lorentz 4-vector. As we will see now, this is the case for the electric charge current or for any flow of some material such as gas flow.

Suppose the charge is carried by some medium, such as gas of ions, then pick a space-time point x and let $\rho_0(x)$ be the charge density in the rest frame of the medium and $\vec{\beta}(x)$ be the velocity of the medium at that point. Then the charge density ρ in the frame in question is larger than ρ_0 by the factor $\gamma = 1/\sqrt{1-\beta^2}$ due to Lorentz contraction

$$\rho = \rho_0 \gamma\,. \tag{2.10}$$

Since $\vec{j} = \rho\vec{\beta}$, j^μ can be written as

$$j^\mu = (\rho, \vec{j}) = (\rho_0\gamma, \rho_0\gamma\vec{\beta}) = \rho_0(\gamma, \vec{\eta}) = \rho_0\eta^\mu\,, \tag{2.11}$$

where we have defined the '4-velocity' η^μ by

$$\eta^\mu = (\eta^0, \vec{\eta}) \stackrel{\text{def}}{\equiv} (\gamma, \gamma\vec{\beta})\,. \tag{2.12}$$

On the other hand, the 4-momentum of a particle with mass m can be written as

$$P^\mu = (m\gamma, m\gamma\vec{\beta}) = m\eta^\mu \tag{2.13}$$

which means that the 4-velocity η^μ is a Lorentz 4-vector, and therefore so is j^μ. When the charge is carried by more than one different media, unique rest frame of the media where ρ^0 is defined does not exist. The total j^μ, however, is the sum of j^μ for each medium. Since j^μ for each medium is a 4-vector, the sum is also a 4-vector. Thus, $\partial_\mu j^\mu = 0$ is a Lorentz-invariant equation; namely, if it is true in one frame, then it is true in any frame. If charge is conserved in one frame, it is conserved in any other frame.

2.3 The Schrödinger equation

Let's briefly review how the Schrödinger equation was introduced and how the probability density was defined. In the classical mechanics, the energy of a particle in the absence of interactions is given by

$$E = \frac{\vec{P}^2}{2m}\,. \tag{2.14}$$

The Schrödinger equation is then obtained by replacing E by $i\partial/\partial t$ and $\vec{P}$ by $-i\vec{\nabla}$ in the above and let them act on a complex scalar function $\phi(x)$ with $x^\mu = (t, \vec{x})$:

$$i\frac{\partial}{\partial t}\phi(x) = -\frac{1}{2m}\nabla^2\phi(x)\,. \quad \text{(Schrödinger equation)} \tag{2.15}$$

When applied to the plane-wave form (2.2), the operator $i\partial/\partial t$ 'pulls down' a factor E from the exponent, and the operator $-i\vec{\nabla}$ 'pulls down' a factor $\vec{P}$. Thus, if we substitute the plane-wave form in the above, we recover the relation between energy and momentum. The solution of the Schrödinger equation with given energy and momentum is then

$$\phi(x) = Ne^{-i(Et-\vec{P}\cdot\vec{x})}, \quad \text{with} \quad E = \frac{\vec{P}^2}{2m}, \tag{2.16}$$

where N is a normalization constant. Namely, the plane-wave form becomes a solution of the Schrödinger equation when the constants E and $\vec{P}$ in the exponent are related by $E = \vec{P}^2/2m$.

Next, a conserved quantity, which is interpreted as the probability density, can be constructed as follows: Multiplying ϕ^* to (2.15) from the left, we get

$$i\phi^*\Big(\frac{\partial}{\partial t}\phi\Big) = -\frac{1}{2m}\phi^*(\nabla^2\phi) \tag{2.17}$$

Taking the complex conjugate of (2.15) and multiplying ϕ from the right, we get

$$-i\Big(\frac{\partial}{\partial t}\phi^*\Big)\phi = -\frac{1}{2m}(\nabla^2\phi^*)\phi\,. \tag{2.18}$$

Subtracting (2.18) from (2.17),

$$\underbrace{i\left[\phi^*\Big(\frac{\partial}{\partial t}\phi\Big) + \Big(\frac{\partial}{\partial t}\phi^*\Big)\phi\right]}_{\displaystyle\frac{\partial}{\partial t}(\phi^*\phi)} = -\frac{1}{2m}\underbrace{\left[\phi^*(\nabla^2\phi) - (\nabla^2\phi^*)\phi\right]}_{\displaystyle\vec{\nabla}\cdot\left[\phi^*(\vec{\nabla}\phi) - (\vec{\nabla}\phi^*)\phi\right]}, \tag{2.19}$$

where we have used on the right hand side $\vec{\nabla}\cdot(\phi\vec{A}) = \vec{\nabla}\phi\cdot\vec{A} + \phi\vec{\nabla}\cdot\vec{A}$:

$$\begin{aligned}\vec{\nabla}\cdot\left[\phi^*(\vec{\nabla}\phi) - (\vec{\nabla}\phi^*)\phi\right] &= \phi^*(\nabla^2\phi) + \vec{\nabla}\phi^*\cdot\vec{\nabla}\phi - \vec{\nabla}\phi^*\cdot\vec{\nabla}\phi - (\nabla^2\phi^*)\phi \\ &= \phi^*(\nabla^2\phi) - (\nabla^2\phi^*)\phi\,. \end{aligned} \tag{2.20}$$

Then, (2.19) can be written as

$$\frac{\partial}{\partial t}\rho = -\vec{\nabla}\cdot\vec{j} \tag{2.21}$$

with

$$\rho = \phi^*\phi, \qquad \vec{j} = -\frac{i}{2m}\Big[\phi^*(\vec{\nabla}\phi) - (\vec{\nabla}\phi^*)\phi\Big]\,. \tag{2.22}$$

Thus, assuming that ϕ satisfies the Schrödinger equation, the quantities $(\rho, \vec{j})$ defined this way satisfy the same equation of conservation (2.9) introduced for a flow of some classical material. There, the density ρ and the current $\vec{j}$ were related by $\vec{j} = \rho\vec{\beta}$. What is $\vec{\beta}$ in the case of the Schrödinger equation? Using the plane-wave solution (2.16) in the definition of ρ and $\vec{j}$ above, we get

$$\rho = |N|^2, \quad \vec{j} = -\frac{i}{2m}|N|^2\Big[(i\vec{P}) - (-i\vec{P})\Big] = |N^2|\frac{\vec{P}}{m}\,; \tag{2.23}$$

thus,

$$\vec{j} = \frac{\vec{P}}{m}\,\rho\,. \tag{2.24}$$

Since $\vec{\beta} = \vec{P}/m$ in classical mechanics, this indeed corresponds to $\vec{j} = \rho\vec{\beta}$. Note also that the density $\rho = |N|^2$ is always positive. Thus, the conserved current defined for the Schrödinger equation as (2.22) can be interpreted as the probability current.

The Schrödinger equation, however, is not Lorentz-invariant; namely, if we use the 4-vector nature of the operator ∂_μ and the scalar nature of the field ϕ [see (1.68) and the discussion that immediately follows it]:

$$\partial'_\mu = \Lambda_\mu{}^\nu\partial_\nu, \quad \phi'(x') = \phi(x) \quad (x'^\mu = \Lambda^\mu{}_\nu x^\nu), \tag{2.25}$$

then, the Schrödinger equation in one frame does not lead to the same form of equation in the Lorentz-transformed frame:

$$i\frac{\partial}{\partial t}\phi(x) = -\frac{1}{2m}\nabla^2\phi(x) \quad \not\to \quad i\frac{\partial}{\partial t'}\phi'(x') = -\frac{1}{2m}\nabla'^2\phi'(x')\,; \tag{2.26}$$

where $\vec{\nabla}'$ denotes the derivatives with respect to the primed coordinates. Rather, it ends up in a terrible mess. Similarly, under the same transformation given by (2.25), the 4-component quantity $j^\mu = (\rho, \vec{j})$, where ρ and $\vec{j}$ are defined as (2.22), does not transform as a 4-vector:

$$j'^\mu(x') \neq \Lambda^\mu{}_\nu j^\nu(x), \tag{2.27}$$

which can easily be seen by noting that ρ as defined here is a scalar quantity: using $\phi'(x') = \phi(x)$,

$$\rho'(x') \equiv \phi'^*(x')\phi'(x') = \phi^*(x)\phi(x) = \rho(x)\,. \tag{2.28}$$

Thus, $j^0 = \rho$ cannot be the time component of a 4-vector which has to change its value under general Lorentz transformation.

2.4 Klein-Gordon equation

In order to look for a Lorentz-invariant wave equation, we start from the relativistic energy-momentum relation for a particle

$$E^2 = \vec{P}^2 + m^2 \,, \tag{2.29}$$

and apply the substitution (2.1) as in the case of the Schrödinger equation to get

$$-\frac{\partial^2}{\partial t^2}\phi(x) = (-\nabla^2 + m^2)\phi(x) \,, \tag{2.30}$$

or

$$\boxed{(\partial_\mu \partial^\mu + m^2)\phi(x) = 0 \,,} \tag{2.31}$$

where

$$\partial_\mu \partial^\mu = \partial_0 \partial^0 + \partial_1 \partial^1 + \partial_2 \partial^2 + \partial_3 \partial^3 = \frac{\partial^2}{\partial t^2} - \nabla^2 \,. \tag{2.32}$$

This equation (2.31) is called the Klein-Gordon equation. The operator $\partial_\mu \partial^\mu$ is sometimes written in different ways:

$$\partial_\mu \partial^\mu \equiv \partial^2 \equiv \Box \tag{2.33}$$

and called the D'Alembertian operator.

Now, the equation (2.31) looks Lorentz-invariant, and indeed it is. Using the transformation (2.25),

$$\begin{aligned} \partial'_\mu \partial'^\mu \phi'(x') &= (\Lambda_\mu{}^\alpha \partial_\alpha)(\Lambda^\mu{}_\beta \partial^\beta)\phi(x) \\ &= \underbrace{\Lambda_\mu{}^\alpha \Lambda^\mu{}_\beta}_{g^\alpha{}_\beta} \partial_\alpha \partial^\beta \phi(x) = \partial_\alpha \partial^\alpha \phi(x) \,. \end{aligned} \tag{2.34}$$

Thus, if a wave function $\phi(x)$ satisfies the Klein-Gordon equation in one frame, then the same form of equation is satified by the transformed wave function $\phi'(x')$ provided that $\phi(x)$ transforms as a scalar field [namely, $\phi'(x') = \phi(x)$]:

$$(\partial^2 + m^2)\phi(x) = 0 \quad \rightarrow \quad (\partial'^2 + m^2)\phi'(x') = 0 \,. \tag{2.35}$$

Thus, the Klein-Gordon equation is Lorentz-invariant.

Next, let's examine the sign of the energy for solutions of the Klein-Gordon equation. To do so, we will again try the plane-wave form (2.2), this time expressed using the energy-momentum 4-vector:

$$\phi(x) = N e^{-i(Et - \vec{P}\cdot\vec{x})} = N e^{-ip\cdot x}, \qquad [p^\mu \equiv (E, \vec{P})] \tag{2.36}$$

Noting that

$$\partial_\mu(p\cdot x) = \frac{\partial}{\partial x^\mu}(p_\nu x^\nu) = p_\nu \underbrace{\frac{\partial x^\nu}{\partial x^\mu}}_{\delta_{\mu\nu}} = p_\mu\,, \tag{2.37}$$

$$\rightarrow \quad \partial_\mu e^{-ip\cdot x} = -ip_\mu e^{-ip\cdot x}\,, \tag{2.38}$$

we get

$$(\partial_\mu\partial^\mu + m^2)\phi = (\underbrace{(-ip_\mu)(-ip^\mu)}_{-p^2} + m^2)\phi = 0 \quad \rightarrow \quad p^2 = m^2\,, \tag{2.39}$$

which is nothing but the relativistic energy-momentum relation (2.29). Namely, a plane-wave (2.36) is a solution of the Klein-Gordon equation as long as the constants E and $\vec{P}$ satisfy the relation $E^2 = \vec{P}^2 + m^2$.

In contrast to the case of the Schrödinger equation where the condition $E = \vec{P}^2/2m$ required that E be positive, in this case E can be positive or negative. One way to get around the negative energies may be simply not to use the negative-energy solutions. However, when interactions are included in the theory, it turns out that the theory will *predict* that the positive energy states will eventually fall into the negative energy states (by emitting photons, for example).

Leaving the negative-energy problem as it is, let's turn to the conserved current of the Klein-Gordon theory in order to study the sign of the probability. Multiplying the Klein-Gordon equation (2.31) with ϕ^* on the left, and multiplying the complex conjugate of (2.31) with ϕ on the right, we get

$$\phi^*(\partial_\mu\partial^\mu\phi) + m^2\phi^*\phi = 0 \tag{2.40}$$

$$(\partial_\mu\partial^\mu\phi^*)\phi + m^2\phi^*\phi = 0\,. \tag{2.41}$$

Taking the difference of the two, we have

$$\begin{aligned} 0 &= \phi^*(\partial_\mu\partial^\mu\phi) - (\partial_\mu\partial^\mu\phi^*)\phi \\ &= \Big[\phi^*(\partial_\mu\partial^\mu\phi) + \partial_\mu\phi^*\partial^\mu\phi\Big] - \Big[\partial_\mu\phi^*\partial^\mu\phi + (\partial_\mu\partial^\mu\phi^*)\phi\Big] \\ &= \partial_\mu\Big[\phi^*(\partial^\mu\phi) - (\partial^\mu\phi^*)\phi\Big]\,. \end{aligned} \tag{2.42}$$

Thus, we have a conserved current

$$\partial_\mu j^\mu = 0\,, \tag{2.43}$$

where the current j^μ is defined by

$$j^\mu = i\,[\phi^*(\partial^\mu\phi) - (\partial^\mu\phi^*)\phi]\,, \tag{2.44}$$

which can be symbolically written as

$$\boxed{j^\mu = i\,\phi^* \overset{\leftrightarrow}{\partial^\mu} \phi}\,, \tag{2.45}$$

The symbol $\overset{\leftrightarrow}{\partial^\mu}$ used above is defined by

$$a \overset{\leftrightarrow}{\partial^\mu} b \overset{\text{def}}{\equiv} a(\partial^\mu b) - (\partial^\mu a) b\,; \tag{2.46}$$

namely, it operates on everything to the right and then everything to the left with a minus sign. Note that the second term in (2.44) is the complex conjugate of the first term. The factor 'i' in (2.44) is added to make j^μ a real quantity.

Using the plane-wave form (2.36) in (2.44),

$$j^\mu = i|N|^2[(-ip^\mu) - (ip^\mu)] = 2|N|^2 p^\mu\,. \tag{2.47}$$

The time component is then supposed to be the probability density:

$$j^0 = 2|N|^2 E \tag{2.48}$$

which can be both positive or negative since E can be positive or negative as discussed earlier.

Thus, in the Klein-Gordon theory, the problem of negative energy and that of negative probability are related. For the probability current, a natural question is whether one can construct a conserved current whose time component is always positive. People have tried, and were not successful. The true resolution of these problems will be accomplished in the framework of the quantum field theory. There, the probability current defined above will be reinterpreted as the charge current which can naturally be negative, and the energy becomes the eigenvalue of the Hamiltonian operator, which turns out to be always positive. Then, what happens to the probability density in the quantum field theory? In a nut shell, the concept of the probability that a particle is found at a given position loses its usefulness since particle and anti-particle can be pair-created out of vacuum for a short period of time, which is a result of the multi-particle nature of the quantum field theory as opposed to the single-particle nature of the quantum mechanical wave function which describes a state of a only one particle. For now, we will stay within the framework of the quantum mechanical wave function, and move on to the wave equation for a spin 1/2 particle - the Dirac equation.

Problems

2.1 *Conserved current.*
(*a*) *Suppose two different scaler fields* $\phi_1(x)$ *and* $\phi_2(x)$ *satisfy the Schrödinger equation:*

$$i\frac{\partial}{\partial t}\phi_i = -\frac{1}{2m}\nabla^2\phi_i \quad (i=1,2)\,.$$

Show that the current $j^\mu = (j^0, \vec{j})$ *given by*

$$j^0 = \phi_1^*\phi_2\,, \quad \vec{j} = -\frac{i}{2m}\left(\phi_1^*(\vec{\nabla}\phi_2) - (\vec{\nabla}\phi_1^*)\phi_2\right)$$

is conserved; namely, $\partial_\mu j^\mu = 0$*. Is this a Lorentz invariant equation? In other words, if it happens to be conserved in one inertial frame, does the same form hold in any other frame?*
(*b*) *Suppose two different scaler fields* $\phi_1(x)$ *and* $\phi_2(x)$ *satisfy the Klein-Gordon equation:*

$$(\partial^2 + m^2)\phi_i = 0 \quad (i=1,2)$$

Show that the current defined by

$$j^\mu = i\left[\phi_1^*(\partial^\mu\phi_2) - (\partial^\mu\phi_1^*)\phi_2\right]$$

is conserved; i.e. $\partial_\mu j^\mu = 0$*, then show that this equation of conservation is Lorentz invariant. (comment: What is this quantity that looks like the probability current but formed with different fields? It is a measure of overlap of the two fields. Such quantity is needed when another field is created out of the overlap of the two. And such quantity can be formed in a Lorentz invariant way as seen above. In a sense, the probability current is considered to be an overlap of a field with itself.)*

2.2 *Conserved currents with potential.*
The Klein-Gordon equation with the electromagnetic 4-potential

$$A^\mu(x) = (\Phi(x), \vec{A}(x)) \quad \text{(real)} \tag{2.49}$$

can be obtained by the so-called minimal substitution

$$i\partial^\mu \to i\partial^\mu - eA^\mu \tag{2.50}$$

where e *is the electric charge. Namely, the Klein-Gordon equation* $(\partial^2 + m^2)\phi = 0$ *becomes*

$$\left[(\partial_\mu + ieA_\mu)(\partial^\mu + ieA^\mu) + m^2\right]\phi = 0 \tag{2.51}$$

Here, the differential operator ∂_μ *operates on everything to its right up to* $\phi(x)$ *in each term when the expression is expanded. Show that the current defined by*

$$j^\mu = i\left[\phi_1^*(\partial^\mu\phi_2) - (\partial^\mu\phi_1^*)\phi_2\right] - 2eA^\mu\phi_1^*\phi_2 \tag{2.52}$$

is conserved, i.e. $\partial_\mu j^\mu = 0$*, where* ϕ_1 *and* ϕ_2 *both satisfy the modified Klein-Gordon equation (2.51).*

Chapter 3

The Dirac Theory

In this chapter, we will study the Dirac equation which describes spin-1/2 particles such as quarks and leptons. We start from the observation that the apparent reason why the Klein-Gordon equation gave solutions with negative energies and negative probabilities was that the energy appeared as E^2 in the relation $E^2 = \vec{P}^2 + m^2$, allowing negative as well as positive value of E. Thus, we will proceed by trying to construct a wave equation which is linear in the time derivative, hoping that it will give us solutions with only positive energies and positive probabilities.

It is not clear exactly what kind of thought process was followed by Dirac when he set out to construct a relativistic theory of electron in late 1920's, even though it seems that he was particularly disturbed by the negative probability. As we will see, he actually succeeded in solving the problem of negative probability, but not that of negative energy. In fact, even if one constructs a wave equation that is linear in time derivative, there is no guarantee that the energy will always be positive. When one applies a differential equation linear in $i\partial_0$ to a plane-wave form, one obtains a relation between E and $\vec{P}$ in the form $E = f(\vec{P}, m)$ as opposed to $E^2 = f(\vec{P}, m)$. The function $f(\vec{P}, m)$, however, may in general be positive or negative, which is actually what happens in the case of the Dirac equation. Dirac partially 'solved' the negative-energy problem by the so-called hole theory in which the vacuum is assumed to be the state where all the negative energy-states are filled up thereby preventing positive energy states to fall into the negative-energy states by the exclusion principle. However, such a scenario cannot work for particles with integer spin for which the exclusion principle does not apply. As in the case of the Klein-Gordon theory, the problem of negative energy in the Dirac theory will be satisfactorily solved in the framework of the quantum field theory.

In retrospect, the importance of the Dirac equation does not have much to do with negative energy or negative probability. Its importance lies in the fact that, in trying to solve these 'problems', Dirac stumbled upon a quantity called a spinor which describes a particle with spin 1/2, and led to a correct description of the magnetic

moment of the electron. Here, we will use the linearity in the time derivative simply as a convenient guide to introduce the Dirac equation. In this section and later, we will generically refer to the particle represented by the Dirac equation as an 'electron', but the discussions apply to any point-like spin-1/2 particle which is not antiparticle of itself. [1]

3.1 The Dirac equation

We now search for a wave equation that is linear in time derivative and consistent with the relativistic energy-momentum relation $E^2 = \vec{P}^2 + m^2$. Let's start from the following Schrödinger form of equation:

$$i\frac{\partial}{\partial t}\psi = H\psi\,, \tag{3.1}$$

where ψ is some wave function representing the state of the electron. The operator H is assumed to be some function of the momentum operator $-i\vec{\nabla}$ and the mass m, presumably representing the energy of the electron. Since we are now interested in a free electron, we assume that H does not depend on time. We then want this to give the relativistic relation $E^2 = \vec{P}^2 + m^2$ when acting upon a plane wave solution $\psi \propto e^{-ip\cdot x}$. This can be accomplished if (3.1) somehow leads to

$$\left(i\frac{\partial}{\partial t}\right)^2\psi = [(-i\vec{\nabla})^2 + m^2]\psi\,. \tag{3.2}$$

We first take the time derivative $i\partial/\partial t$ of (3.1) to get

$$\begin{aligned}\left(i\frac{\partial}{\partial t}\right)^2\psi \;&=\; H\,\underbrace{i\frac{\partial}{\partial t}\psi}_{H\psi} = H^2\psi \\ &\text{(which we would like to become)} \\ &=\; [(-i\vec{\nabla})^2 + m^2]\psi\,,\end{aligned} \tag{3.3}$$

or

$$H^2 = (-i\vec{\nabla})^2 + m^2\,. \tag{3.4}$$

Note that the equation (3.2) is nothing but the Klein-Gordon equation. Yes, if a wave function satisfies the Dirac equation, then it will satisfy the Klein-Gordon equation. In fact, any relativistic wave function would satisfy the Klein-Gordon equation as long as it is consistent with $E^2 = \vec{P}^2 + m^2$.

[1] A spin-1/2 particle that is antiparticle of itself is called a Majorana particle. So far, no such particles have been found in nature.

Since H^2 should become a quadratic function of $-i\vec{\nabla}$ and m, it seems reasonable (though not mandatory) to look for H which is linear in $-i\vec{\nabla}$ and m:

$$H = \vec{\alpha} \cdot (-i\vec{\nabla}) + \beta m \,, \tag{3.5}$$

where $\vec{\alpha} \equiv (\alpha_1, \alpha_2, \alpha_3)$ and β are some *constants* which turn out to be matrices. Keeping track of the ordering of the (matrix) products of α_i and β while noting that m is just a real number, the condition (3.4) then becomes

$$\begin{aligned}
H^2 &= [\alpha_i(-i\nabla_i) + \beta m]\,[\alpha_j(-i\nabla_j) + \beta m] \\
&= \sum_{i,j} \alpha_i\alpha_j(-i\nabla_i)(-i\nabla_j) \quad \rightarrow \left(\begin{array}{l} \sum_i \alpha_i^2(-i\nabla_i)^2 \\ +\sum_{i>j}(\alpha_i\alpha_j + \alpha_j\alpha_i)(-i\nabla_i)(-i\nabla_j) \end{array} \right. \\
&\quad \left. \begin{array}{l} +\sum_i \alpha_i\beta(-i\nabla_i)m \\ +\sum_j \beta\alpha_j(-i\nabla_j)m \\ +\beta^2 m^2 \end{array} \right\} \quad \rightarrow \sum_i(\alpha_i\beta + \beta\alpha_i)(-i\nabla_i)m \\
&\quad \text{(which should be equal to)} \\
&= (-i\vec{\nabla})^2 + m^2 \,,
\end{aligned} \tag{3.6}$$

which is satisfied if

$$\alpha_i^2 = \beta^2 = 1, \qquad \left(\begin{array}{l} \alpha_i\alpha_j + \alpha_j\alpha_i = 0 \;\; (i \neq j) \\ \alpha_i\beta + \beta\alpha_i = 0 \end{array} \right), \quad (i, j = 1, 2, 3)\,. \tag{3.7}$$

Using the anticommutator symbol,

$$\{A, B\} \stackrel{\text{def}}{\equiv} AB + BA \,, \tag{3.8}$$

this condition can be written as

$$\boxed{\begin{array}{ll} \{\alpha_i\,, \alpha_j\} = 0 \;\; (i \neq j), & \\ \{\alpha_i\,, \beta\} = 0, & \alpha_i^2 = \beta^2 = 1, \quad (i, j = 1, 2, 3) \end{array}}\,. \tag{3.9}$$

Note that this condition is symmetric among the four quantities $\alpha_1, \alpha_2, \alpha_3$ and β; namely, the square of each is unity and each anticommutes with another. They cannot be ordinary numbers since they do not commute. They can be matrices, however, and we will now show that they are traceless hermitian matrixes of rank greater than or equal to four:

1. *α_i and β are hermitian.* Since H represents the energy, we want it to have real eigenvalues; thus, H is hermitian. Since the operator $-i\vec{\nabla}$ acts as a hermitian operator, then α_i and β should also be hermitian (for a proof, see Exercise 3.4):

$$\alpha_i^\dagger = \alpha_i, \qquad \beta^\dagger = \beta\,. \tag{3.10}$$

2. *The traces of α_i and β are zero.* From (3.9), we have

$$\alpha_i\beta = -\beta\alpha_i \quad \overset{\times\beta \text{ from right}}{\longrightarrow} \quad \alpha_i \underbrace{\beta^2}_{1} = -\beta\alpha_i\beta\,. \tag{3.11}$$

   Taking trace of both sides,

$$\mathrm{Tr}\alpha_i = -\underbrace{\mathrm{Tr}(\beta\alpha_i\beta)}_{\mathrm{Tr}(\beta^2\alpha_i)} = -\mathrm{Tr}\alpha_i\,, \quad \to \quad \mathrm{Tr}\alpha_i = 0\,, \tag{3.12}$$

   where we have used

$$\mathrm{Tr}(AB) = \mathrm{Tr}(BA)\,, \tag{3.13}$$

   where A and B are two arbitrary square matrices of same rank. Similarly, multiplying α_i instead of β in (3.11), we get $\mathrm{Tr}\beta = 0$.

3. *The eigenvalues of α_i and β are ± 1.* Suppose u is an eigenvector of β with an eigenvalue c; namely, $\beta u = cu$. Applying β from the left again, we get

$$\beta^2 u = c\,\beta u \quad \to \quad u = c^2 u \quad \to \quad c^2 = 1. \tag{3.14}$$

   Thus, the eigenvalues of β should be ± 1. Similarly, the eigenvalues of α_i should also be ± 1.

4. *The rank of α_i and β is even.* Since β is hermitian, it can be diagonalized by some matrix S:

$$S\beta S^{-1} = \begin{pmatrix} c_1 & & \\ & \ddots & \\ & & c_n \end{pmatrix}, \tag{3.15}$$

   where c_i are the eigenvalues of β. Taking the trace, and using (3.13),

$$\mathrm{Tr}(S\beta S^{-1}) = \mathrm{Tr}(S^{-1}S\beta) = \mathrm{Tr}\beta = \sum_i c_i\,. \tag{3.16}$$

   Since $\mathrm{Tr}\beta = 0$ and $c_i = \pm 1$, n should be even. The proof is the same for α_i.

5. *α_i and β are linearly independent.* Suppose β can be written as a linear combination of α_i: $\beta = b_i\alpha_i$ where b_i are some c-numbers. Then, using $\{\beta\,,\alpha_j\} = 0$ (3.9),

$$0 = \{\beta\,,\alpha_j\} = \{b_i\alpha_i\,,\alpha_j\} = b_i\{\alpha_i\,,\alpha_j\} = 2b_j\,. \tag{3.17}$$

where we have used the linearity of the anticommutator

$$\{cA, B\} = \{A, cB\} = c\{A, B\} \quad (c : c\text{-number};\ A, B : \text{matrices}). \tag{3.18}$$

We thus have $b_j = 0$ $(j = 1, 2, 3)$; namely, β cannot be written as a linear combination of α_i's. Similarly, α_i cannot be written as a linear combination of the rest.

An $n \times n$ complex matrix A has $2n^2$ degrees of freedom, and out of which n^2 are taken away by the hermitian condition $A^\dagger = A$.[2] Together with the traceless requirement which takes away one degree of freedom, we are left with $2n^2 - n^2 - 1 = n^2 - 1$ degrees of freedom. Thus, n should be larger than 2 to have at least four independent such matrices ($\vec{\alpha}$ and β). Since n should be even, α_i and β are then independent, traceless, hermitian anticommuting matrices of rank 4 or more.

Now we will explicitly construct the 4×4 matrices α_i and β. A well-known set of linearly independent anticommuting matrices is the Pauli matrices σ_i $(i = 1, 2, 3)$ given by

$$\boxed{\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},} \tag{3.19}$$

which are also hermitian and traceless

$$\sigma_i^\dagger = \sigma_i, \qquad \mathrm{Tr}\sigma_i = 0 \quad (i = 1, 2, 3) \tag{3.20}$$

and satisfy

$$\sigma_i^2 = 1 \quad (i = 1, 2, 3)\,, \qquad \sigma_i\sigma_j = -\sigma_j\sigma_i = i\sigma_k \quad (i, j, k : \text{cyclic}), \tag{3.21}$$

or equivalently,

$$\boxed{\begin{array}{rcll} \{\sigma_i, \sigma_j\} & = & 2\,\delta_{ij} & \\ [\sigma_i, \sigma_j] & = & 2\,i\,\epsilon_{ijk}\sigma_k & (i, j, k = 1, 2, 3) \end{array}}. \tag{3.22}$$

In (3.21), '1' is actually a 2×2 identity matrix, and in (3.22) the identity matrix on the right hand side is omitted. Hereafter, the $n \times n$ identity matrix is often written as the number '1' for simplicity.

[2]The hermiticity condition of an $n \times n$ matrix A is given by $A^*_{ij} = A_{ji}$. It contains n equations for the diagonal elements $A^*_{ii} = A_{ii}$ each of which removes one degree of freedom, and $n(n-1)/2$ equations of the form $A^*_{ij} = A_{ji}$ $(i \neq j)$ for off-diagonal elements each of which removes two degrees of freedom. Thus, the total degrees of freedom removed is $n + 2 \times n(n-1)/2 = n^2$.

Exercise 3.1 *Explicitly verify the relations (3.22).*

Exercise 3.2 *Use the properties of the Pauli matrixes (3.22) to prove*

$$(\vec{a}\cdot\vec{\sigma})(\vec{b}\cdot\vec{\sigma}) = \vec{a}\cdot\vec{b} + i\vec{\sigma}\cdot(\vec{a}\times\vec{b}) \tag{3.23}$$

and

$$e^{i\vec{a}\cdot\vec{\sigma}} = \cos a + i\hat{a}\cdot\vec{\sigma}\sin a \tag{3.24}$$

where $\vec{a}$ and $\vec{b}$ are 3-component vectors, and

$$a = |a|, \quad \hat{a} = \vec{a}/a, \quad \vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)\,. \tag{3.25}$$

One important feature of the Pauli matrices is that $\sigma_i/2$ act as the spin-1/2 angular momentum operators. Indeed, from (3.22), we see that $\sigma_i/2$ satisfy the commutation relation of angular momentum $[J_i\,, J_j] = i\,\epsilon_{ijk}J_k$:

$$\left[\frac{\sigma_i}{2}, \frac{\sigma_j}{2}\right] = i\,\epsilon_{ijk}\frac{\sigma_k}{2}\,. \tag{3.26}$$

The Pauli matrices are independent, hermitian, traceless, anticommuting, and thier squares are unity. The problem of course is that there are only three of them while we need four. It is, however, a good place to start. So we try the following:

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \tag{3.27}$$

where I is the 2×2 identity matrix. These are clearly hermitian and traceless. It is also straightforward to see that they satisfy the relations (3.9). For example,

$$\begin{aligned}
\{\alpha_i\,, \alpha_j\} &= \alpha_i\alpha_j + \alpha_j\alpha_i \\
&= \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \\
&= \begin{pmatrix} \sigma_i\sigma_j & 0 \\ 0 & \sigma_i\sigma_j \end{pmatrix} + \begin{pmatrix} \sigma_j\sigma_i & 0 \\ 0 & \sigma_j\sigma_i \end{pmatrix} \\
&= \begin{pmatrix} \{\sigma_i, \sigma_j\} & 0 \\ 0 & \{\sigma_i, \sigma_j\} \end{pmatrix} = 2\delta_{ij}\,, \quad \text{etc.}
\end{aligned} \tag{3.28}$$

Thus, we now have an equation which is linear in time derivative and consistent with the relativistic relation $E^2 = \vec{P}^2 + m^2$. Namely, combining (3.1) and (3.5),

$$i\frac{\partial}{\partial t}\psi(x) = [\vec{\alpha}\cdot(-i\vec{\nabla}) + \beta m]\psi(x)\,, \tag{3.29}$$

where α_i and β are 4×4 matrices defined by the relations (3.9) and explicitly given by (3.27). Since α_i and β are 4×4 matrices, the wave function ψ has to be a 4-component quantity (called a *Dirac spinor*, a *4-component spinor*, or simply a *spinor*):

$$\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{pmatrix}, \qquad \psi_n \ (n = 1, 2, 3, 4) : \text{complex} \tag{3.30}$$

The indexes of the four components have *nothing to do* with the space-time. The multiple components were introduced to satisfy the required relations among α_i and β. We will see later that they represent the degrees of freedom corresponding to spin up and down as well as those corresponding to electron and its antiparticle positron. We will call this four-dimensional space the 'spinor space'.

The equation (3.29) can be written in a way that treats the time and space component more symmetrically. This can be done by multiplying β from the left:

$$i\beta\frac{\partial}{\partial t}\psi = [\beta\alpha_i(-i\nabla_i) + m]\psi\,, \tag{3.31}$$

and defining four 'gamma' matrices out of α_i and β:

$$\gamma^0 \equiv \beta, \quad \gamma^i \equiv \beta\alpha_i \quad (i = 1, 2, 3)\,, \tag{3.32}$$

where we will adopt the standard superscript/subscript rule for the index of the gamma matrices. Then, together with $\partial_0 = \partial/\partial t, \nabla_i = \partial_i$, it can be written as

$$\Big[i(\gamma^0\partial_0 + \gamma^i\partial_i) - m\Big]\psi = 0 \tag{3.33}$$

or

$$\boxed{(i\gamma^\mu\partial_\mu - m)\psi = 0}\,. \tag{3.34}$$

This equation is called the *Dirac equation*. It is often written as

$$(i\not{\partial} - m)\psi = 0\,. \tag{3.35}$$

where we have defined the notation

$$\not{a} \stackrel{\text{def}}{\equiv} \gamma^\mu a_\mu \tag{3.36}$$

with a_μ being any 4-component quantity.

From the properties (3.9) of α_i and β, it can be easily shown that the four matrices introduced in (3.32) satisfy the following anticommutation relations of utmost importance:

$$\boxed{\{\gamma^\mu, \gamma^\nu\} = 2\,g^{\mu\nu}\,,} \tag{3.37}$$

or equivalently, $\gamma^{0\,2} = 1$, $\gamma^{i\,2} = -1$, and all four matrices anticommute among themselves. This relation (3.37) is sometimes called the Clifford algebra, and is entirely equivalent to (3.9).

Exercise 3.3 *Gamma matrices.*
Use the definition of the γ matrices $\gamma^0 = \beta$, $\gamma^i = \beta\alpha_i \quad (i = 1, 2, 3)$, and the relations among β and α_i (3.9) to verify the anticommutation relations (3.37).

The matrix γ^0 is hermitian while γ^i are anti-hermitian (i.e. its hermitian conjugate is the negative of itself): since α_i and β are hermitian,

$$\begin{aligned} \gamma^{0\dagger} &= \gamma^0, \qquad (\text{since } \gamma^0 = \beta) \\ \gamma^{i\dagger} &= (\beta\alpha_i)^\dagger = \alpha_i^\dagger \beta^\dagger = \alpha_i \beta = -\beta\alpha_i = -\gamma^i \,. \end{aligned} \tag{3.38}$$

The following, however, is true for all μ:

$$\gamma^0 \gamma^{\mu\dagger} \gamma^0 = \gamma^\mu \,. \tag{3.39}$$

Using the explicit representation (3.27) of α_i and β, the gamma matrices can be written as

$$\boxed{\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}}. \tag{3.40}$$

This is not the only explicit expression of the 4×4 matrices that satisfy (3.37). This particular representation is called the Dirac representation, and is the standard one we will use in this book. The anticommutation relation $\{\gamma^\mu, \gamma^\nu\} = 2\, g^{\mu\nu}$ and the hermiticity relation $\gamma^0 \gamma^{\mu\dagger} \gamma^0 = \gamma^\mu$, however, are independent of representation since they are direct consequences of the anticommutation relation (3.9) and hermiticity of α_i and β.

Just to make sure that we know what is going on, let's completely expand the Dirac equation $(i\partial\!\!\!/ - m)\psi = 0$, or $i\gamma^\mu \partial_\mu \psi = m\psi$, using the Dirac representation:

$$\begin{aligned} i \Bigg[& \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}}_{\gamma^0} \partial_0 + \underbrace{\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}}_{\gamma^1} \partial_1 + \underbrace{\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}}_{\gamma^2} \partial_2 \\ & + \underbrace{\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}}_{\gamma^3} \partial_3 \Bigg] \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = m \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}, \end{aligned} \tag{3.41}$$

which is a set of four equations given by

$$i\left[\partial_0 \psi_1 + (\partial_1 - i\partial_2)\psi_4 + \partial_3 \psi_3\right] \;=\; m\psi_1 \,,$$

$$\begin{aligned} i\,[\partial_0\psi_2 + (\partial_1 + i\partial_2)\psi_3 - \partial_3\psi_4] &= m\psi_2\,, \\ i\,[-\partial_0\psi_3 - (\partial_1 - i\partial_2)\psi_2 - \partial_3\psi_1] &= m\psi_3\,, \\ i\,[-\partial_0\psi_4 - (\partial_1 + i\partial_2)\psi_1 + \partial_3\psi_2] &= m\psi_4\,. \end{aligned} \tag{3.42}$$

This looks complicated, but what matters is the structure of the gamma matrices, and it is almost all contained in the relation $\{\gamma^\mu, \gamma^\nu\} = 2\,g^{\mu\nu}$. It is actually as simple as indicated by the concise expression $(i\partial\!\!\!/ - m)\psi = 0$.

Exercise 3.4 *Hermiticity of $H = \vec{\alpha}\cdot(-i\vec{\nabla}) + m\beta$.*
Suppose a 4-component spinor ψ is an eigenvector of H with an eigenvalue c:

$$H\psi(x) = c\psi(x), \qquad \psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{pmatrix} \tag{3.43}$$

with

$$H = \alpha_i(-i\nabla_i) + m\beta\,, \tag{3.44}$$

or more explicitly,

$$-i\alpha_i(\nabla_i\psi) + m\beta\psi = c\psi\,. \tag{3.45}$$

The parameter m is real, and α_i, β are constant hermitian matrixes:

$$\alpha_i^\dagger = \alpha_i, \qquad \beta^\dagger = \beta\,. \tag{3.46}$$

Show that the eigenvalue c is real. Assume that the wave function $\psi(x)$ vanishes at infinite distance from the origin. (hint: Multiply the row vector $\psi^\dagger$ to (3.45) from the left and integrate over d^3x, Then take hermitian conjugate of (3.45), multiply ψ from the right, and integrate over d^3x. Compare the two using a partial integration.)

3.2 Conserved current

Now, let's see if we can construct a conserved current out of solutions of the Dirac equation. If we can somehow find a conserved current, then it is naturally interpreted as the probability current. The basic procedure is similar to the cases of the Schrödinger equation or the Klein-Gordon equation. Multiplying the hermitian conjugate of a spinor ψ

$$\psi^\dagger \stackrel{\text{def}}{\equiv} (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*) \qquad \text{(a row vector)} \tag{3.47}$$

to the Dirac equation $i\gamma^\mu\partial_\mu\psi = m\psi$ on the left,

$$i\,\psi^\dagger\gamma^\mu\partial_\mu\psi = m\psi^\dagger\psi\,, \tag{3.48}$$

which has the form

$$\overbrace{(\cdot\ \cdot\ \cdot\ \cdot)}^{i\psi^\dagger}\overbrace{\begin{pmatrix}\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot\end{pmatrix}}^{\gamma^\mu}\overbrace{\begin{pmatrix}\cdot\\ \cdot\\ \cdot\\ \cdot\end{pmatrix}}^{\partial_\mu\psi} = \overbrace{(\cdot\ \cdot\ \cdot\ \cdot)}^{m\psi^\dagger}\overbrace{\begin{pmatrix}\cdot\\ \cdot\\ \cdot\\ \cdot\end{pmatrix}}^{\psi}. \tag{3.49}$$

Next, we take the hermitian conjugate of the Dirac equation $i\gamma^\mu\partial_\mu\psi = m\psi$ to obtain

$$-i\,(\partial_\mu\psi)^\dagger\gamma^{\mu\dagger} = m\,\psi^\dagger\,. \tag{3.50}$$

Multiplying ψ from the right,

$$-i\,(\partial_\mu\psi)^\dagger\gamma^{\mu\dagger}\psi = m\,\psi^\dagger\psi\,. \tag{3.51}$$

Subtracting (3.51) from (3.48),

$$i\,[\psi^\dagger\gamma^\mu(\partial_\mu\psi) + (\partial_\mu\psi)^\dagger\underbrace{\gamma^{\mu\dagger}}_{\gamma^\mu?}\psi] = 0 \tag{3.52}$$

Now, only if $\gamma^{\mu\dagger} = \gamma^\mu$ were true, then this would become $i\,\partial_\mu(\psi^\dagger\gamma^\mu\psi) = 0$, and thus we would identify $\psi^\dagger\gamma^\mu\psi$ as the conserved current. However, γ^i $(i = 1, 2, 3)$ are not hermitian as we have seen in (3.38). Instead, what we have is $\gamma^0\gamma^{\mu\dagger}\gamma^0 = \gamma^\mu$ (3.39). As we will see below, we can take advantage of this relation if we define a new kind of inner product of two spinors ψ_1 and ψ_2 by

$$\psi_1^\dagger\gamma^0\psi_2 \overset{\text{def}}{\equiv} \bar{\psi}_1\psi_2\,, \tag{3.53}$$

where we have defined a new kind of adjoint by

$$\boxed{\bar{\psi} \overset{\text{def}}{\equiv} \psi^\dagger\gamma^0}, \tag{3.54}$$

which we will call 'spinor adjoint'. Similarly, we define the spinor adjoint of a 4×4 matrix by

$$\boxed{\bar{M} \overset{\text{def}}{\equiv} \gamma^0 M^\dagger\gamma^0}. \tag{3.55}$$

Using this definition, the relation $\gamma^0\gamma^{\mu\dagger}\gamma^0 = \gamma^\mu$ can be written as

$$\boxed{\overline{\gamma^\mu} = \gamma^\mu}, \tag{3.56}$$

which is equivalent to $\gamma^{0\dagger} = \gamma^0$ and $\gamma^{i\dagger} = -\gamma^i$. Before moving on, let's become familiar with operations of the spinor adjoints.

Spinor adjoints

For two spinors a, b, and 4×4 matrices $M_i\,(i = 1, \ldots, n)$ the quantity $\bar{b}\, M_1 \ldots M_n a$ is a complex number, and its complex conjugate is

$$\begin{aligned}(\bar{b}\, M_1 \ldots M_n a)^* &= (b^\dagger \gamma^0 M_1 \ldots M_n a)^* \\ &= a^\dagger M_n^\dagger \ldots M_1^\dagger \underbrace{\gamma^{0\dagger}}_{\gamma^0} b \\ &\text{(inserting } \gamma^{0\,2} = 1 \text{ almost everywhere)} \\ &= \underbrace{a^\dagger \gamma^0}_{\bar{a}} \underbrace{\gamma^0 M_n^\dagger \gamma^0}_{\bar{M}_n} \gamma^0 \ldots \gamma^0 \underbrace{\gamma^0 M_1^\dagger \gamma^0}_{\bar{M}_1} b \end{aligned} \tag{3.57}$$

Thus, we have

$$\boxed{(\bar{b}\, M_1 \ldots M_n a)^* = \bar{a}\, \bar{M}_n \ldots \bar{M}_1 b}\,. \tag{3.58}$$

Similarly, the adjoint of a column vector $M_1 \ldots M_n a$ is by the definition (3.54),

$$\begin{aligned}\overline{M_1 \ldots M_n a} &\equiv (M_1 \ldots M_n a)^\dagger \gamma^0 \\ &= a^\dagger M_n^\dagger \ldots M_1^\dagger \gamma^0 \\ &= a^\dagger \gamma^{0\,2} M_n^\dagger \gamma^{0\,2} \ldots \gamma^{0\,2} M_1^\dagger \gamma^0 \\ &= \bar{a}\, \bar{M}_n \ldots \bar{M}_1\,. \end{aligned} \tag{3.59}$$

Namely,

$$\boxed{\overline{M_1 \ldots M_n a} = \bar{a} \bar{M}_n \ldots \bar{M}_1}\,. \tag{3.60}$$

The general rule is thus similar to the case of hermitian conjugate; namely, one simply takes spinor adjoint of each, or take it away if it already has the adjoint symbol, and reverse the order. This applies also to products of matrices. Furthermore, a complex number becomes its complex conjugate when the spinor adjoint is taken:

$$\boxed{\overline{\eta a} = \eta^* \bar{a} \quad (\eta : \text{complex number})} \tag{3.61}$$

In passing, we note that (3.60) and (3.61) are valid when each component of the spinor a is an operator which is relevant when we quantize the fields later. The relation (3.62) is also valid when a and b are operators; only difference being that the product $\bar{b}\, M_1 \ldots M_n a$ is now an operator and the complex conjugation needs to be replaced with hermitian conjugation:

$$\boxed{(\bar{b}\, M_1 \ldots M_n a)^\dagger = \bar{a}\, \bar{M}_n \ldots \bar{M}_1 b \qquad (a, b\text{: operators})}\,. \tag{3.62}$$

Now we will repeat the failed attempt to construct a conserved current, this time replacing the hermitian conjugation with the spinor adjoint. Multiplying $\bar{\psi}$ to the Dirac equation $i\gamma^\mu\partial_\mu\psi = m\psi$ from the left,

$$i\,\bar{\psi}\gamma^\mu\partial_\mu\psi = m\,\bar{\psi}\psi\,. \tag{3.63}$$

Noting that $\overline{\partial_\mu\psi} = \partial_\mu\bar{\psi}$, we take the spinor adjoint of the Dirac equation,

$$-i\,(\partial_\mu\bar{\psi})\,\underbrace{\overline{\gamma^\mu}}_{\gamma^\mu} = m\,\bar{\psi}\,, \tag{3.64}$$

where we have used $\overline{\gamma^\mu} = \gamma^\mu$ which is the critical step, then multiply ψ from the right to get

$$-i\,(\partial_\mu\bar{\psi})\gamma^\mu\psi = m\,\bar{\psi}\psi\,. \tag{3.65}$$

Subtracting (3.65) from (3.63), we then have a conserved current:

$$i\,[\bar{\psi}\gamma^\mu(\partial_\mu\psi) + (\partial_\mu\bar{\psi})\gamma^\mu\psi] = i\,\partial_\mu(\bar{\psi}\gamma^\mu\psi) = 0 \tag{3.66}$$

or

$$\boxed{\partial_\mu j^\mu = 0, \quad \text{with} \quad j^\mu \equiv \bar{\psi}\gamma^\mu\psi}\,. \tag{3.67}$$

Is this a Lorentz-invariant equation? Actually, we do not know how j^μ transforms under Lorentz transformation since we do not know how the spinor ψ transforms; it is not any one of the quantities we know so far: scalar, vector, tensor, etc. The spinor ψ is a 4-component quantity, but it cannot be a Lorentz 4-vector since the indexes have nothing to do with the space-time. Since we do not know how ψ transforms under Lorentz transformation, we do not even know if the Dirac equation is Lorentz-invariant or not at this point. Shortly, we will find how a spinor transforms under a Lorentz transformation by requiring that the Dirac equation becomes Lorentz-invariant. For now, however, let's examine the time component of the conserved current we just found, which is supposed to be the probability density: using $\bar{\psi} \equiv \psi^\dagger\gamma^0$,

$$\begin{aligned} j^0 &= \bar{\psi}\gamma^0\psi = (\psi^\dagger\underbrace{\gamma^0)\gamma^0}_{1}\psi = \psi^\dagger\psi \\ &= |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 + |\psi_4|^2 \geq 0\,. \end{aligned} \tag{3.68}$$

Thus, the probability density is always positive, and Dirac seems to have fixed the problem of negative probability. As mentioned earlier, however, this quantity will be

reinterpreted as the charge current in the framework of the quantum field theory, and will acquire both positive and negative values.

Incidentally, (3.64) gives the equation that has to be satisfied by the adjoint spinor $\bar{\psi}$ in order for the original wave function ψ to be a solution of the Dirac equation:

$$-i\,(\partial_\mu \bar{\psi})\gamma^\mu = m\,\bar{\psi} \tag{3.69}$$

which is often written as

$$\bar{\psi}\,(i\gamma^\mu \overleftarrow{\partial}_\mu + m) = 0\,, \tag{3.70}$$

or

$$\bar{\psi}\,(i \overleftarrow{\not\partial} + m) = 0\,, \tag{3.71}$$

where the symbol $\overleftarrow{\partial}_\mu$ is defined to operate on everything to its left:

$$a \overleftarrow{\partial}_\mu b \stackrel{\text{def}}{\equiv} (\partial_\mu a) b\,. \tag{3.72}$$

This funny notation is a result of a compromise between the desire to put γ^μ and ∂_μ next to each other so that it can be combined as $\overleftarrow{\not\partial}$ and the necessity to put the matrix γ^μ to the right of $\bar{\psi}$ which is a row vector.

3.3 Lorentz invariance of the Dirac equation

We have seen that the Klein-Gordon equation is Lorentz-invariant, which meant the following: if a function of space-time $\phi(x)$ satisfies the equation $(\partial^2 + m^2)\phi(x) = 0$, then a new function defined by

$$\phi'(x') = \phi(x), \quad x' = \Lambda x \tag{3.73}$$

satisfies $(\partial'^2 + m^2)\phi'(x') = 0$. Note that the function $\phi'(x')$ is uniquely defined once the original function $\phi(x)$ and the Lorentz transformation Λ are given. In particular, a critical condition in defining $\phi'(x')$ uniquely was the definition of scalar field $\phi'(x') = \phi(x)$, which means that the value at an event point x in the original frame is the same in the transformed frame if measured at the same event point (now given by the coordinate $x' = \Lambda x$ in that frame). The value of ϕ' at a given event point is completely defined by the value of ϕ at the same event point, and does not depend on the values of any other event points. Also, note that the functional shape of ϕ' is in general different from that of ϕ; namely, if we give the same argument to ϕ' and ϕ, then in general $\phi'(x) \neq \phi(x)$.

Similarly, the electromagnetic wave equation

$$\partial^2 A^\mu(x) = j^\mu(x) \quad (\mu = 0, 1, 2, 3)\,, \tag{3.74}$$

where the electromagnetic 4-potential A^μ and the charge current j^μ are both Lorentz 4-vectors, is Lorentz-invariant. [3] Namely, if $A^\mu(x)$ and $j^\mu(x)$ satisfy the above equation, then a new set of fields defined by the vector field condition

$$A'^\mu(x') = \Lambda^\mu{}_\alpha A^\alpha(x), \quad j'^\mu(x') = \Lambda^\mu{}_\beta\, j^\beta(x), \quad x' = \Lambda x\,, \tag{3.75}$$

satisfy $\partial'^2 A'^\mu(x') = j'^\mu(x')$ as can be readily verified. The function $A^\mu(x)$ [or $A'^\mu(x')$] assigns a set of four numbers to each event point, and the four values in the transformed frame $A'^\mu(x')$ at an event point are completely determined by the four values in the original frame at the same event point, they are simply mixed up by the matrix Λ. Note that the new values $A'^\mu(x')$ do not depend on the values of $A^\mu(x)$ at other event points.

Then, how does a Dirac spinor field $\psi(x)$ transform? We do not know at this point, except that as in the case of scalar and vector fields, the four values $\psi'(x')$ in a transformed frame will depend only on the four values in the original frame $\psi(x)$ associated with the same event point, presumably mixed up by some 4×4 matrix S:

$$\boxed{\psi'(x') = S(\Lambda)\psi(x)\,, \quad x' = \Lambda x}\,. \tag{3.76}$$

Also, just like the mixing matrix (Λ) for the vector field depended only on the Lorentz transformation and did not depend on event point x, we expect that S also depends only on the Lorentz transformation Λ.

Our strategy is to derive the condition for S that makes the Dirac equation Lorentz-invariant, then explicitly construct such a matrix, out of which we will obtain explicit solutions of the Dirac equation. We will then find that the solutions contain the spin-1/2 structure and the particle-antiparticle degrees of freedom.

Thus, we require that if $\psi'(x')$ satisfy $(i\partial\!\!\!/' - m)\psi'(x') = 0$ then it leads to $(i\partial\!\!\!/ - m)\psi(x) = 0$ when $\psi'(x')$ and $\psi(x)$ are related by (3.76). Specifically, we require that the numerical values of γ^μ will be the same in any frame; namely, for the Dirac representation, the form of the four equations is given by (3.42) and is the same in any frame. Using $\psi'(x') = S\psi(x)$, $(i\partial\!\!\!/' - m)\psi'(x') = 0$ becomes

$$\begin{aligned}
0 &= (i\gamma^\mu \underbrace{\partial'_\mu}_{\Lambda_\mu{}^\alpha\partial_\alpha} - m)\,\underbrace{\psi'(x')}_{S\psi(x)} \\
&= i\,\gamma^\mu \underbrace{\Lambda_\mu{}^\alpha \partial_\alpha S\psi(x)}_{S\Lambda_\mu{}^\alpha\partial_\alpha\psi(x)} - mS\psi(x) \\
(\times S^{-1}) \quad 0 &= i\,S^{-1}\gamma^\mu S\Lambda_\mu{}^\alpha\partial_\alpha\psi(x) - m\psi(x)\,,
\end{aligned} \tag{3.77}$$

[3] If one uses a gauge which is not Lorentz-invariant, such as the Coulomb gauge, then the correct statement is that the equation is Lorentz-invariant up to gauge transformation. This complication comes about due to the masslessness of photon. We will discuss this point in detail later.

where in the second line, S can come out of ∂_α since S is a constant matrix, and S can move past ${\Lambda_\mu}^\alpha$ because ${\Lambda_\mu}^\alpha$ is just a number for given α and μ. Note that we have consistently suppressed the spinor indexes while all space-time indexes have been explicitly written out. The necessary condition for this to become $i\,\gamma^\alpha\partial_\alpha\psi(x) - m\psi(x) = 0$ for any $\psi(x)$ is then

$$S^{-1}\gamma^\mu S{\Lambda_\mu}^\alpha = \gamma^\alpha \tag{3.78}$$

or multiplying ${\Lambda^\nu}_\alpha$ and summing over α,

$$S^{-1}\gamma^\mu S\underbrace{{\Lambda_\mu}^\alpha{\Lambda^\nu}_\alpha}_{{g_\mu}^\nu} = {\Lambda^\nu}_\alpha\gamma^\alpha\,; \tag{3.79}$$

namely,

$$\boxed{S^{-1}\gamma^\nu S = {\Lambda^\nu}_\alpha\gamma^\alpha}\,. \tag{3.80}$$

By tracing back the derivation, one sees that this is also a sufficient condition; namely, if a spinor field transforms under a Lorentz transformation Λ by $\psi'(x') = S\psi(x)$ where S is a matrix that satisfies $S^{-1}\gamma^\nu S = {\Lambda^\nu}_\alpha\gamma^\alpha$, then the Dirac equation becomes Lorentz-invariant. Note that we did not restrict ourselves to proper and orthochronous Lorentz transformations; thus, the condition (3.80) applies to any Lorentz transformations including T and P.

The left hand side of (3.80) indicates a certain transformation of γ^μ, and the right hand side is exactly like the transformation of a vector. Does that mean that the gamma matrices that appear in the Dirac equation transform as a vector? The answer is no. In fact, the condition (3.80) was obtained by requiring that the gamma matrices in the Dirac equation *do not change* under a Lorentz transformation. It will turn out, however, that one can use (3.80) to construct quantities that transform as a vector. One example is the conserved current $j^\mu = \bar\psi\gamma^\mu\psi$ as will be discussed in detail shortly.

Let's move on to actually constructing $S(\Lambda)$. Our starting point is that the mapping between S and Λ preserves the product rule, which can be seen as follows. Suppose $S(\Lambda_1)$, $S(\Lambda_2)$, and $S(\Lambda_1\Lambda_2)$ correspond to Λ_1, Λ_2, and $\Lambda_1\Lambda_2$, respectively:

$$\begin{aligned} S(\Lambda_1) &\leftrightarrow \Lambda_1\,, \\ S(\Lambda_2) &\leftrightarrow \Lambda_2\,, \\ S(\Lambda_1\Lambda_2) &\leftrightarrow \Lambda_1\Lambda_2\,. \end{aligned} \tag{3.81}$$

Under the Lorentz transformation Λ_1, a spinor ψ will be transformed to $S(\Lambda_1)\psi$. If we perform an additional transformation Λ_2, which makes the total transformation $\Lambda_2\Lambda_1$, then this spinor will transform to $S(\Lambda_2)[S(\Lambda_1)\psi] = [S(\Lambda_2)S(\Lambda_1)]\psi$. Since this should hold for any spinor ψ, we have $S(\Lambda_2\Lambda_1) = S(\Lambda_2)S(\Lambda_1)$, or

$$S(\Lambda_2)S(\Lambda_1) \leftrightarrow \Lambda_2\Lambda_1\,, \tag{3.82}$$

which means that S's and Λ's have exactly the same group structure [see (1.146) through (1.153)]; in fact, S is said to be the *representation of the Lorentz group* in the spinor-space. If we restrict ourselves to Lorentz transformations that are continuously connected to the identity, then S and Λ can be written in exponential forms using generators, and the generators for S and those for Λ should have the same structure constants. The problem then reduces to finding the set of generators for S that satisfy the same commutation relations as $M^{\mu\nu}$ (or K_i and L_i) *and* are consistent with $S^{-1}\gamma^\nu S = \Lambda^\nu{}_\alpha \gamma^\alpha$.

Suppose $B^{\mu\nu}$ be the generators for S corresponding to the generators $M^{\mu\nu}$ for Λ. Namely, the mapping between S and Λ is given by the same real parameters $a_{\mu\nu}$:

$$S = e^{\frac{1}{2}a_{\mu\nu}B^{\mu\nu}} \;\leftrightarrow\; \Lambda = e^{\frac{1}{2}a_{\mu\nu}M^{\mu\nu}} \,. \tag{3.83}$$

Let's first find the condition for $B^{\mu\nu}$ that satisfy $S^{-1}\gamma^\nu S = \Lambda^\nu{}_\alpha\gamma^\alpha$. Assume that $a_{\mu\nu}$ are small and expand both sides of $S^{-1}\gamma^\mu S = \Lambda^\mu{}_\nu\gamma^\nu$: using $(e^A)^{-1} = e^{-A}$ (1.109),

$$\begin{aligned} S^{-1}\gamma^\mu S &= e^{-\frac{1}{2}a_{\alpha\beta}B^{\alpha\beta}}\gamma^\mu e^{\frac{1}{2}a_{\alpha'\beta'}B^{\alpha'\beta'}} \\ &= \Big(1 - \frac{1}{2}a_{\alpha\beta}B^{\alpha\beta}\Big)\gamma^\mu\Big(1 + \frac{1}{2}a_{\alpha\beta}B^{\alpha\beta}\Big) + \cdots \\ &= \gamma^\mu + \frac{1}{2}\underbrace{\Big(\gamma^\mu a_{\alpha\beta}B^{\alpha\beta} - a_{\alpha\beta}B^{\alpha\beta}\gamma^\mu\Big)}_{a_{\alpha\beta}[\gamma^\mu, B^{\alpha\beta}]} + \cdots \,, \end{aligned} \tag{3.84}$$

and the right hand side is

$$\begin{aligned} \Lambda^\mu{}_\nu\gamma^\nu &= \Big(e^{\frac{1}{2}a_{\alpha\beta}M^{\alpha\beta}}\Big)^\mu{}_\nu\gamma^\nu \\ &= \Big(1 + \frac{1}{2}a_{\alpha\beta}M^{\alpha\beta}\Big)^\mu{}_\nu\gamma^\nu + \cdots \\ &= \Big[g^\mu{}_\nu + \frac{1}{2}a_{\alpha\beta}(M^{\alpha\beta})^\mu{}_\nu\Big]\gamma^\nu + \cdots \\ &= \gamma^\mu + \frac{1}{2}a_{\alpha\beta}(M^{\alpha\beta})^\mu{}_\nu\gamma^\nu + \cdots \,. \end{aligned} \tag{3.85}$$

Requiring that (3.84) is equal to (3.85) for any $a_{\alpha\beta}$, we obtain

$$[\gamma^\mu, B^{\alpha\beta}] = (M^{\alpha\beta})^\mu{}_\nu\gamma^\nu \,, \tag{3.86}$$

which is the condition that the generators of S ($B^{\mu\nu}$'s) have to satisfy in order for the Dirac equation to be Lorentz-invariant under the (infinitesimal) transformation $\psi'(x') = S\psi(x)$. Note that γ^μ and $B^{\mu\nu}$ are 4×4 matrices (for given μ,ν) that operate in the spinor space while $(M^{\alpha\beta})^\mu{}_\nu$ is just a number. Again, we are suppressing the spinor indexes while writing out space-time indexes.

We will now show that the solution to (3.86) is given by

$$B^{\alpha\beta} = \frac{1}{4}\,[\gamma^\alpha, \gamma^\beta] \tag{3.87}$$

or equivalently (using $\{\gamma^\alpha, \gamma^\beta\} = 2g^{\alpha\beta}$),

$$B^{\alpha\beta} = \begin{cases} \dfrac{1}{2}\gamma^\alpha\gamma^\beta & (\alpha \neq \beta) \\ 0 & (\alpha = \beta) \end{cases} . \tag{3.88}$$

For $\alpha = \beta$, the condition (3.86) is clearly satisfied since in that case $M^{\alpha\beta} = 0$ and $B^{\alpha\beta} = 0$. For $\alpha \neq \beta$, we have to work out the commutator $[\gamma^\mu, \gamma^\alpha\gamma^\beta]$. Using the identity

$$[A, BC] = \overbrace{ABC - BCA}^{+BAC - BAC} = \{A, B\}C - B\{A, C\}\,, \tag{3.89}$$

we have

$$\begin{aligned} [\gamma^\mu, \gamma^\alpha\gamma^\beta] &= \underbrace{\{\gamma^\mu, \gamma^\alpha\}}_{2g^{\mu\alpha}}\gamma^\beta - \gamma^\alpha\,\underbrace{\{\gamma^\mu, \gamma^\beta\}}_{2g^{\mu\beta}} \\ &= 2\,g^{\alpha\mu}\gamma^\beta - 2\,g^{\beta\mu}\gamma^\alpha \\ &= 2(\underbrace{g^{\alpha\mu}g^\beta{}_\nu - g^{\beta\mu}g^\alpha{}_\nu}_{(M^{\alpha\beta})^\mu{}_\nu})\gamma^\nu\,, \end{aligned} \tag{3.90}$$

where we have used the explicit expression $(M^{\alpha\beta})_{\mu\nu} = g^\alpha{}_\mu\, g^\beta{}_\nu - g^\beta{}_\mu\, g^\alpha{}_\nu$ (1.86). Dividing by 2 on both sides, we have

$$\Big[\gamma^\mu, \frac{1}{2}\,\gamma^\alpha\gamma^\beta\Big] = (M^{\alpha\beta})^\mu{}_\nu\gamma^\nu \qquad (\alpha \neq \beta)\,. \tag{3.91}$$

Namely, the condition (3.86) is satisfied for $B^{\mu\nu}$ given by (3.88) [or (3.87)] for all α and β.

We have shown that for an *infinitesimal Lorentz transformation* given by parameters $a_{\alpha\beta}$, the matrix $S = 1 + (a_{\alpha\beta}/2)B^{\alpha\beta}$ satisfies $S^{-1}\gamma^\nu S = \Lambda^\nu{}_\alpha\gamma^\alpha$ to the first order in $a_{\alpha\beta}$ if $B^{\alpha\beta}$ is given by $B^{\alpha\beta} = [\gamma^\alpha, \gamma^\beta]/4$. Does it hold for a finite Lorentz transformation? Namely, if Λ and S are finite transformations mapped as (3.83) with the same set of parameters $a_{\alpha\beta}$, then do they satisfy $S^{-1}\gamma^\nu S = \Lambda^\nu{}_\alpha\gamma^\alpha$? This can be easily proven by dividing Λ and S into n consecutive small transformations:

$$\Lambda = \lambda^n = \left(1 + \frac{1}{2}\frac{a_{\alpha\beta}}{n}M^{\alpha\beta}\right)^n, \qquad S = s^n = \left(1 + \frac{1}{2}\frac{a_{\alpha\beta}}{n}B^{\alpha\beta}\right)^n . \tag{3.92}$$

Since s and λ are infinitesimal transformations mapped by the same set of parameters, we have already proven that

$$s^{-1}\gamma^\mu s = \lambda^\mu{}_\nu \gamma^\nu \,. \tag{3.93}$$

Multiplying s^{-1} from the left and s from the right, we get

$$(s^{-1})^2\gamma^\mu s^2 = \lambda^\mu{}_\nu \underbrace{s^{-1}\gamma^\nu s}_{\lambda^\nu{}_\alpha \gamma^\alpha} = (\lambda^\mu{}_\nu \lambda^\nu{}_\alpha)\gamma^\alpha = (\lambda^2)^\mu{}_\alpha \gamma^\alpha \,. \tag{3.94}$$

Repeating the process n times, we obtain

$$\begin{aligned} \underbrace{(s^{-1})^n}_{(s^n)^{-1}} \gamma^\mu s^n &= \underbrace{(\lambda^n)^\mu{}_\nu}_{\Lambda^\mu{}_\nu} \gamma^\nu \\ \rightarrow \quad S^{-1}\gamma^\mu S &= \Lambda^\mu{}_\nu \gamma^\nu \,. \end{aligned} \tag{3.95}$$

Thus, we have shown that for any proper and orthochronous Lorentz transformation Λ given by

$$\Lambda = e^{\frac{1}{2}a_{\alpha\beta}M^{\alpha\beta}} \,, \tag{3.96}$$

one can construct a transformation in the spinor space by

$$\boxed{S = e^{\frac{1}{2}a_{\alpha\beta}B^{\alpha\beta}} \quad \text{with} \quad B^{\alpha\beta} = \frac{1}{4}\left[\gamma^\alpha, \gamma^\beta\right]}, \tag{3.97}$$

such that $S^{-1}\gamma^\mu S = \Lambda^\mu{}_\nu \gamma^\nu$ holds, or equivalently, the Dirac equation becomes invariant under the transformation $\psi'(x') = S\psi(x)$. It is straightforward to show that the generators $B^{\alpha\beta}$ satisfy the same commutation relations as those of $M^{\alpha\beta}$, and the proof is left as an exercise.

Exercise 3.5 *Representation of Lorentz group in the spinor space.*
Define the generators of boost and rotation in the 4-component spinor space by

$$\text{boost}: \quad K_i \leftrightarrow B^b_i \equiv B^{0i} \,, \quad \text{rotation}: \quad L_i \leftrightarrow B^r_i \equiv B^{jk} \;\; (ijk : \text{cyclic}) \,. \tag{3.98}$$

Show explicitly that the B^b's and B^r's satisfy the same commutation relations as satisfied by the K's and L's; namely

$$\begin{aligned} [B^b_i \,, B^b_j\,] &= -\epsilon_{ijk} B^r_k \\ [B^r_i \,, B^r_j\,] &= \epsilon_{ijk} B^r_k \\ [B^r_i \,, B^b_j\,] &= \epsilon_{ijk} B^b_k \,, \end{aligned} \tag{3.99}$$

(*It is also possible to prove generally. In any case, all you need is $B^{\alpha\beta} = 1/2\gamma^\alpha\gamma^\beta$ ($\alpha \neq \beta$) and the relation $\{\gamma^\mu, \gamma^\nu\} = 2\,g^{\mu\nu}$; do not use any explicit representation of the γ matrixes*).

An important property of S is that its spinor adjoint is the inverse:

$$\boxed{\bar{S} = S^{-1}, \quad \text{or} \quad \bar{S}S = 1}, \tag{3.100}$$

which can be seen by noting that for $\alpha \neq \beta$,

$$\overline{B^{\alpha\beta}} = \frac{1}{2}\overline{\gamma^\alpha\gamma^\beta} = \frac{1}{2}\overline{\gamma^\beta}\,\overline{\gamma^\alpha} = \frac{1}{2}\gamma^\beta\gamma^\alpha = -\frac{1}{2}\gamma^\alpha\gamma^\beta = -B^{\alpha\beta} \tag{3.101}$$

(for $\alpha = \beta$, $\overline{B^{\alpha\beta}} = -B^{\alpha\beta}$ holds trivially since $B^{\alpha\beta} = 0$) and thus,

$$\begin{aligned} \bar{S} &= \overline{e^{\frac{1}{2}a_{\alpha\beta}B^{\alpha\beta}}} = \sum_{k=0}^{\infty} \frac{1}{k\,!}(\tfrac{1}{2}a_{\alpha\beta}\underbrace{\overline{B^{\alpha\beta}}}_{-B^{\alpha\beta}})^k \\ &= e^{-\frac{1}{2}a_{\alpha\beta}B^{\alpha\beta}} = S^{-1}\,, \end{aligned} \tag{3.102}$$

where we have used $\overline{M^k} = (\bar{M})^k$ and $e^{-A} = (e^A)^{-1}$.

Using (3.100), we can easily see that the inner product $\bar{a}b$ of two spinors a and b is invariant under Lorentz transformation in the spinor space:

$$a' = Sa\,, \quad b' = Sb\,, \tag{3.103}$$

$$\bar{a}'b' = \underbrace{\overline{Sa}}_{\bar{a}\bar{S}}Sb = \bar{a}\underbrace{\bar{S}S}_{1}b = \bar{a}b\,. \tag{3.104}$$

The relation $\bar{S} = S^{-1}$ or $\gamma^0 S^\dagger \gamma^0 = S^{-1}$ can also be written as (by multiplying γ^0 from the left and S from the right)

$$S^\dagger\gamma^0 S = \gamma^0\,. \tag{3.105}$$

Note the parallel between the space-time and the spinor space: in the space-time, the Lorentz transformation of a 4-vector A is given by $A' = \Lambda A$ which keeps the inner product $A \cdot B = A^T G B$ (1.23) invariant, and the metric G is unchanged under the transformation: $\Lambda^T G \Lambda = G$. In the spinor space, the Lorentz transformation of a spinor a is given by $a' = Sa$ which keeps the inner product $\bar{a}b = a^\dagger\gamma^0 b$ invariant, and the 'metric' γ^0 is invariant under the transformation: $S^\dagger\gamma^0 S = \gamma^0$. Table 3.1 summarizes the parallelism.

Now it is trivial to show that the conserved current we derived earlier $j^\mu(x)$ (3.67) is indeed a 4-vector: using $\psi'(x') = S\psi(x)$,

$$\begin{aligned} j'^\mu(x') &= \bar{\psi}'(x')\gamma^\mu\psi'(x') \\ &= \bar{\psi}(x)\underbrace{\bar{S}\,\gamma^\mu\,S}\psi(x) \\ &\qquad S^{-1}\gamma^\mu S = \Lambda^\mu{}_\nu\gamma^\nu \text{ by (3.80)} \\ &= \Lambda^\mu{}_\nu\bar{\psi}(x)\gamma^\nu\psi(x) \\ \rightarrow\ j'^\mu(x') &= \Lambda^\mu{}_\nu j^\nu(x)\,. \end{aligned} \tag{3.106}$$

space	Lorentz transformation	inner product	metric invariance
space-time	$A' = \Lambda A$	$A \cdot B \equiv A^T G B$	$\Lambda^T G \Lambda = G$
spinor	$a' = Sa$	$\bar{a}b \equiv a^\dagger \gamma^0 b$	$S^\dagger \gamma^0 S = \gamma^0$

Table 3.1: Correspondence between the space-time and the spinor space. A and B are 4-vectors and a and b are 4-component spinors.

Note that the current j^μ is a 4-vector even if $\psi(x)$ does not satisfy the Dirac equation. If $\psi(x)$ does satisfy the Dirac equation, then j^μ is conserved: $\partial_\mu j^\mu = 0$, and this, we now know, is a Lorentz-invariant statement; namely, if the current is conserved in one frame, then it is conserved in any other frame. Similarly, a current consisting of two spinor fields $\psi_1(x)$ and $\psi_2(x)$

$$j^\mu_{12}(x) = \bar{\psi}_1(x)\gamma^\mu\psi_2(x) \tag{3.107}$$

is also a 4-vector; namely,

$$j'^\mu_{12}(x') = \Lambda^\mu{}_\nu j^\nu_{12}(x)\,, \tag{3.108}$$

which can be easily shown to be conserved if ψ_1 and ψ_2 are solutions of the Dirac equation (with the same mass).

Thus, we have constructed a scalar quantity and a vector quantity out of two spinors ψ_1 and ψ_2. These are not the only quantities one can form out of two spinors that transforms in a well-defined manner under Lorentz transformation. Such quantities are called the bilinear covariants which we will study systematically in the next section.

3.4 Bilinear covariants

Let's consider a complex number of the form

$$f = \bar{a}\Gamma b \tag{3.109}$$

where Γ is a constant 4×4 complex matrix, and a and b are spinors; namely they transform under a Lorentz transformation Λ as

$$a' = S(\Lambda)a\,, \quad b' = S(\Lambda)b\,. \tag{3.110}$$

Or a and b may be spinor *fields*:

$$a'(x') = S(\Lambda)a(x)\,, \quad b'(x') = S(\Lambda)b(x) \qquad (x' = \Lambda x)\,. \tag{3.111}$$

The following discussion applies to both cases.

A complex 4×4 matrix has 16 *complex* elements, and thus any such matrix can be written as a linear combination of 16 independent matrices with *complex* coefficients. As we will see below, the 16 independent matrices can be chosen such that the quantities

$$\bar{a}\Gamma_i b \quad (i = 1, \ldots, 16) \tag{3.112}$$

are grouped into five sets each of which transform in a well-defined manner under *proper and orthochronous transformations* (scalar, vector and tensor) and the *space inversion* P (how it changes sign). Such quantities are called the bilinear covariants. To study them, let's first find the spinor representation of the space inversion P.

Space inversion in the spinor space

Recall that the condition for the invariance of the Dirac equation, $S^{-1}\gamma^\mu S = \Lambda^\mu{}_\nu \gamma^\nu$, holds for $\Lambda = P$ also, where P is the space inversion:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{3.113}$$

which is also called the *parity transformation*. Namely, $S_P \equiv S(P)$ should satisfy

$$S_P^{-1}\gamma^0 S_P = P^0{}_\nu \gamma^\nu = \gamma^0 \quad \text{or} \quad \gamma^0 S_P = S_P \gamma^0 \tag{3.114}$$

$$S_P^{-1}\gamma^i S_P = P^i{}_\nu \gamma^\nu = -\gamma^i \quad \text{or} \quad \gamma^i S_P = -S_P \gamma^i \,. \tag{3.115}$$

Namely, S_P commutes with γ^0 and anticommutes with γ i. This is accomplished by taking

$$S_P = \eta\gamma^0 \,, \tag{3.116}$$

where η is an arbitrary complex constant. We then require that S_P also satisfies the property $\bar{S}S = 1$ (3.100):

$$\bar{S}_P S_P = \overline{\eta\gamma^0}\,\eta\gamma^0 = \eta^*\eta\,\gamma^{0^2} = |\eta|^2 = 1 \,. \tag{3.117}$$

We will now arbitrarily choose η to be $+1$:

$$\boxed{S_P = \gamma^0 \qquad (\eta = +1)} \,. \tag{3.118}$$

Here we are requiring that the Dirac equation be invariant under space inversion, but should it be? Experiments tell us that electromagnetic and strong interactions are invariant under space inversion, but weak interaction is not. Since there are some interactions that are invariant under space inversion, the free field part should better be invariant also. In fact, the rule of the game is to build in as many symmetries as

possible into the theory as long as they do not contradict with experimental facts; such theory will have more predictive power than otherwise. Later, we will deal with the question of discrete symmetries in more detail in the context of quantized field. Now, in the Dirac representation the space inversion in the spinor space amounts to changing the sign of the bottom two components: noting that $x' = Px$,

$$\psi'(x') = S_P\psi(x) = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{pmatrix} = \begin{pmatrix} \psi_1(P^{-1}x') \\ \psi_2(P^{-1}x') \\ -\psi_3(P^{-1}x') \\ -\psi_4(P^{-1}x') \end{pmatrix} . \tag{3.119}$$

Namely, if $\psi(x)$ satisfies the Dirac equation $(i\partial\!\!\!/ - m)\psi(x) = 0$, then $\psi'(x')$ defined above satisfies the Dirac equation $(i\partial\!\!\!/' - m)\psi'(x') = 0$.

Five types of bilinear covariants

Now we will write down five types of bilinear covariants and check the transformation properties under proper and orthochronous transformations and under the parity transformation S_P. In this section, we will denote a proper and orthochronous transformation by S for simplicity.

All we use in the following is

$$\begin{gathered} \bar{S}\gamma^\mu S = \Lambda^\mu{}_\nu \gamma^\nu \,, \qquad S_P = \gamma^0 \,, \\ \bar{S}S = 1 \,, \qquad \bar{S}_P S_P = 1 \,. \end{gathered} \tag{3.120}$$

Transformed quantities, either by a proper and orthochronous transformation S or by the parity transformation S_P, are denoted with a prime.

1. **Scalar**

$$f = \bar{a}b \,: \quad \text{transforms as} \quad f' = f\,(S)\,, \quad f' = f\,(S_P)\,. \tag{3.121}$$

We have already seen that $f' = f$ under a proper and orthochronous transformation S (3.104). Under the parity transformation, we have

$$f' = \overline{a'}b' = \overline{S_P a}\, S_P b = \bar{a}\, \underbrace{\bar{S}_P S_P}_{1}\, b = \bar{a}b = f \tag{3.122}$$

2. **Vector**

$$f^\mu = \bar{a}\gamma^\mu b \,: \quad \text{transforms as} \quad f'^\mu = \Lambda^\mu{}_\nu f^\nu\,(S)\,, \quad f'^\mu = f_\mu\,(S_P) \tag{3.123}$$

Again, we have already seen in (3.108) that the set of four quantities, $\bar{a}\gamma^\mu b$ ($\mu = 0, 1, 2, 3$), as a group transforms as a 4-vector (thereby the name 'bilinear *covariants*'). Under S_P, it transforms as

$$f'^\mu = \overline{a'}\gamma^\mu b' = \overline{S_P a}\gamma^\mu S_P b = \bar{a}\, \underbrace{\gamma^0\gamma^\mu\gamma^0}_{\gamma_\mu}\, b = f_\mu \,. \tag{3.124}$$

Note that the superscript μ changed to a subscript; namely, the sign of the time component is unchanged while that of space components flipped, which is the same as the transformation of the familiar energy-momentum 4-vector $P^\mu = (E, \vec{P})$ under space inversion. This use of the Lorentz index is a bit sloppy, since it is mixing up the metric $g_{\mu\nu}$ and the Lorentz transformation P which happen to have the same matrix value. The correct expression would be $f'^\mu = P^\mu{}_\nu f^\nu$. Still, it makes expressions compact that we will use it with this point in mind.

3. **Tensor**

$$f^{\mu\nu} = \bar{a}\sigma^{\mu\nu}b \ : \ \text{transforms as } f'^{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta f^{\alpha\beta} \ (S) \, , \ f'^{\mu\nu} = f_{\mu\nu} \ (S_P) \quad (3.125)$$

where

$$\boxed{\sigma^{\mu\nu} \stackrel{\text{def}}{\equiv} \frac{i}{2}[\gamma^\mu, \gamma^\nu] = \begin{cases} i\,\gamma^\mu\gamma^\nu & (\mu \neq \nu) \\ 0 & (\mu = \nu) \end{cases}} \quad (3.126)$$

Even though $\Gamma = \gamma^\mu\gamma^\nu$ would do just fine, defining $\sigma^{\mu\nu}$ this way makes it explicitly antisymmetric with respect to the μ and ν indexes, and also the addition of i makes it hermitian for $\mu, \nu = 1, 2, 3$. Note that we have $\sigma^{\mu\nu} = 2iB^{\mu\nu}$. Since $\sigma^{\mu\nu} = -\sigma^{\nu\mu}$, $f^{\mu\nu}$ is an antisymmetric tensor, and there are six independent quantities that transform as a set.

The transformation properties can be verified straightforwardly: using the linearity of the commutator and

$$\bar{S}\,[\gamma^\mu, \gamma^\nu]S = \bar{S}(\overbrace{\gamma^\mu\gamma^\nu}^{S\bar{S}} - \overbrace{\gamma^\nu\gamma^\mu}^{S\bar{S}})S = [\bar{S}\gamma^\mu S, \bar{S}\gamma^\nu S] \, , \quad (3.127)$$

the transformation under S is

$$\begin{aligned} f'^{\mu\nu} = \bar{a}'\sigma^{\mu\nu}b' \ &= \ \bar{a}\bar{S}\,\frac{i}{2}\,[\gamma^\mu, \gamma^\nu]Sb \\ &= \ \bar{a}\,\frac{i}{2}\,[\bar{S}\gamma^\mu S, \bar{S}\gamma^\nu S]b \\ &= \ \bar{a}\,\frac{i}{2}\,[\Lambda^\mu{}_\alpha\gamma^\alpha, \Lambda^\nu{}_\beta\gamma^\beta]b \\ &= \ \Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta\,\bar{a}\,\frac{i}{2}\,[\gamma^\alpha, \gamma^\beta]b = \Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta f^{\alpha\beta} \end{aligned} \quad (3.128)$$

For $S_P (= \bar{S}_P = \gamma^0)$, it suffices to check only the non-zero components (namely, for $\mu \neq \nu$):

$$f'^{\mu\nu} = \bar{a}'\sigma^{\mu\nu}b' \ = \ \bar{a}\bar{S}_P\,\sigma^{\mu\nu}S_P\,b$$

$$= \bar{a}\gamma^0(i\underbrace{\gamma^\mu\gamma^\nu)\gamma^0}_{\gamma^0\gamma_\mu\gamma_\nu} b$$
$$= \bar{a}\,(i\gamma_\mu\gamma_\nu)b = f_{\mu\nu}\,, \tag{3.129}$$

where the indexes μ and ν became subscripts when γ^0 was moved over $\gamma^\mu\gamma^\nu$ because γ^0 commutes with γ^0 but anticommutes with γ^i $(i = 1, 2, 3)$.

4. **Pseudoscalar**

$$f = \bar{a}\gamma_5 b\,: \quad \text{transforms as} \quad f' = f\,(S)\,, \quad f' = -f\,(S_P) \tag{3.130}$$

Namely, a pseudoscalar transforms under proper and orthochronous transformation just like a scalar (i.e. it does not change its value), but changes sign under space inversion. The γ_5 matrix in the expression above is defined as

$$\boxed{\gamma_5 \stackrel{\text{def}}{=} i\,\gamma^0\gamma^1\gamma^2\gamma^3}\,, \tag{3.131}$$

which, in the Dirac representation, is

$$\gamma_5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\,. \tag{3.132}$$

The index '5' is not a Lorentz index and thus there is no distinction between superscript and subscript. It is designed to anticommute with any one of γ^μ $(\mu = 0, 1, 2, 3)$:

$$\gamma^\mu\gamma_5 = \gamma^\mu(i\,\gamma^0\gamma^1\gamma^2\gamma^3) = -(i\,\gamma^0\gamma^1\gamma^2\gamma^3)\gamma^\mu = -\gamma_5\gamma^\mu \tag{3.133}$$

where the minus sign arises because, when γ^μ moves over the four gamma matrices $\gamma^0\gamma^1\gamma^2\gamma^3$, one is γ^μ itself and three others will anticommute with γ^μ. It is also easy to see that γ_5 is hermitian and the square is 1. Namely,

$$\boxed{\begin{array}{c} \{\gamma^\mu, \gamma_5\} = 0 \quad (\mu = 0, 1, 2, 3) \\ \gamma_5^\dagger = \gamma_5\,, \quad {\gamma_5}^2 = 1 \end{array}}\,. \tag{3.134}$$

It follows that γ_5 commutes with $B^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu]$ and thus with any proper and orthochronous transformation S:

$$\gamma_5 B^{\mu\nu} = \gamma_5\frac{1}{4}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) = \frac{1}{4}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\gamma_5 = B^{\mu\nu}\gamma_5 \tag{3.135}$$

$$\gamma_5 S = \gamma_5\, e^{\frac{1}{2}a_{\mu\nu}B^{\mu\nu}} = \gamma_5 \sum_k \frac{\left(\frac{1}{2}a_{\mu\nu}B^{\mu\nu}\right)^k}{k\,!} = S\gamma_5\,. \tag{3.136}$$

Namely,

$$\boxed{[S, \gamma_5] = 0}\,. \tag{3.137}$$

The transformation properties can now be verified easily. Under a proper and orthochronous transformation, we have

$$f' = (\bar{a}\bar{S})\underbrace{\gamma_5(S\,b)}_{S\gamma_5} = \bar{a}\,\underbrace{\bar{S}S}_{1}\,\gamma_5 b = f\,, \tag{3.138}$$

and under S_P, it transforms as

$$f' = (\bar{a}\bar{S}_P)\gamma_5(S_P b) = (\bar{a}\gamma^0)\underbrace{\gamma_5(\gamma^0 b)}_{-\gamma^0\gamma_5} = -\bar{a}\gamma_5 b = -f\,. \tag{3.139}$$

5. **Pseudovector (or 'axial vector')**

$$f^\mu = \bar{a}\gamma^\mu\gamma_5 b: \quad \text{transforms as} \quad f'^\mu = \Lambda^\mu{}_\nu f^\nu\,(S)\,, \quad f'^\mu = -f_\mu\,(S_P) \tag{3.140}$$

Namely, a pseudovector transforms just like a vector under proper and orthochronous transformations, and under space inversion it transforms with signs opposite to those of a vector; i.e. the time component changes sign and the space components stay the same.

Under a proper and orthochronous transformation S, we have indeed

$$\begin{aligned} f'^\mu &= (\bar{a}\bar{S})\gamma^\mu\underbrace{\gamma_5(S\,b)}_{S\gamma_5} = \bar{a}\,\underbrace{\bar{S}\gamma^\mu S}_{\Lambda^\mu{}_\nu\gamma^\nu}\gamma_5 b \\ &= \Lambda^\mu{}_\nu(\bar{a}\gamma^\nu\gamma_5 b) = \Lambda^\mu{}_\nu f^\nu\,, \end{aligned} \tag{3.141}$$

as stated above, and under the space inversion S_P,

$$f' = (\bar{a}\bar{S}_P)\gamma^\mu\gamma_5(S_P b) = (\bar{a}\gamma^0)\underbrace{\gamma^\mu\gamma_5(\gamma^0\,b)}_{-\gamma^0\gamma_\mu\gamma_5} = -\bar{a}\gamma_\mu\gamma_5 b = -f_\mu\,, \tag{3.142}$$

where the index μ has changed from a superscript to a subscript when γ^0 is moved over γ^μ. The key points are that γ_5 commutes with a proper and orthochronous transformation S, thus the transformation under S is the same as a vector, and since γ_5 anticommutes with $S_P = \gamma^0$, the changes of sign under S_P are opposite to those of a vector.

And this is all. The five types of bilinear covariants are summarized in Table 3.2. For a given pair of spinors a and b, here are 16 such quantities formed by 16 matrices

Type	Γ_i	# of Γ_i	under S	under S_P
(S) Scalar	I	1	$f' = f$	$f' = f$
(V) Vector	γ^μ	4	$f'^\mu = \Lambda^\mu{}_\nu f^\nu$	$f'^\mu = f_\mu$
(T) Tensor	$\sigma^{\mu\nu}$	6	$f'^{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta f^{\alpha\beta}$	$f'^{\mu\nu} = f_{\mu\nu}$
(A) Axial vector	$\gamma^\mu\gamma_5$	4	$f'^\mu = \Lambda^\mu{}_\nu f^\nu$	$f'^\mu = -f_\mu$
(P) Pseudoscalar	γ_5	1	$f' = f$	$f' = -f$

Table 3.2: Bilinear covariants $\bar{a}\Gamma_i b$. S is a proper and orthochronous transformation, and S_P is the space inversion.

Γ_i ($i = 1, \ldots, 16$). It is straightforward to show that these matrices are independent and complete; namely, a bilinear quantity formed by an arbitrary complex 4×4 matrix $\bar{a}Mb$ can be written uniquely in terms of these bilinear covariants with a set of complex coefficients c_i:

$$\bar{a}Mb = \sum_{i=1}^{16} c_i \, \bar{a}\Gamma_i b \,, \tag{3.143}$$

and thus we know exactly how it transforms under Lorentz transformations (or more precisely, under proper and orthochronous transformations and space inversion).

You may be wondering what happened to the *pseudotensor* which would be $f^{\mu\nu} = \bar{a}\sigma^{\mu\nu}\gamma_5 b$. It actually has the 'desired' transformation properties: $f'^{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta f^{\alpha\beta}$ and $f'^{\mu\nu} = -f_{\mu\nu}$ as can be readily verified, but we have already exhausted all 16 independent $\Gamma's$, and $\sigma^{\mu\nu}\gamma_5$ is already covered by the ones we have listed: it is easily shown that

$$\sigma^{0i}\gamma_5 = i\,\sigma^{jk}, \quad \sigma^{ij}\gamma_5 = i\,\sigma^{k0}, \quad (i, j, k : \text{cyclic}) \,. \tag{3.144}$$

Namely, a pseudotensor quantity can be constructed out of a pair of spinors, but it is just a rearrangement of the components of the tensor bilinear covariants. In fact, any product of γ matrixes can be uniquely reduced to the following form by moving the same γ matrices next to each other and using $\gamma^{\mu 2} = 1$ or -1:

$$c\gamma^{\mu_1} \cdots \gamma^{\mu_n} \qquad (0 \le n \le 4, \mu_1 < \ldots < \mu_n) \tag{3.145}$$

where c is a constant and all γ^{μ_i} are different. Apart from the constant, $n = 0$ means it is the identity, $n = 1$ means it is one of the 4 vectors γ^μ, $n = 2$ means it is one of the 6 tensors $i\gamma^\mu\gamma^\nu$, $n = 3$ means it is one of the 4 axial vectors $\gamma^\mu\gamma_5$, and $n = 4$ means it is the pseudoscalar γ_5. Thus, any product of γ matrixes is one of the 16 matrices listed in the table up to a constant.

Exercise 3.6 *Linear independence of bilinear covariants.*
Follow the steps below to prove that the 16 4×4 *matrixes* Γ_i *of Table 3.2 are linearly independent. Use only representation-independent relations. All you need should be* $\{\gamma^\mu, \gamma^\nu\} = 2\, g^{\mu\nu}$ *and* $\{\gamma^\mu, \gamma_5\} = 0$.
(*a*) *Show that* $(\Gamma_i)^2 = +1$ *or* -1 *for* $i = 1 \ldots 16$.
(*b*) *Verify that for any* Γ_i $(i \neq 1)$ *there is at least one* Γ_k *that anticommutes with it:*

$$\Gamma_i \Gamma_k = -\Gamma_k \Gamma_i \,. \tag{3.146}$$

Using this, show that for $i \neq 1$, $\mathrm{Tr}\Gamma_i = 0$. (*hint: express* $\mathrm{Tr}(\Gamma_k \Gamma_i \Gamma_k)$ *in two ways, one using* $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$, *and the other using the above anticommutation relation.*)
(*c*) *Show that, for any pair* $i \neq j$, *the product of the two* Γ*'s is just another* Γ_k $(k \neq 1)$ *times a constant:*

$$\Gamma_i \Gamma_j = c\Gamma_k \quad (i \neq j, k \neq 1) \tag{3.147}$$

(*d*) *Suppose a set of constants* a_i $(i = 1 \ldots 16)$ *exists to satisfy*

$$\sum_{i=1}^{16} a_i \Gamma_i = 0 \,. \tag{3.148}$$

Take the trace of this equation to show $a_1 = 0$. *Multiply by* Γ_k $(k \neq 1)$ *and then take the trace to show that* $a_k = 0$. *Thus all a's are zero. This completes the proof of independence.*

What is the importance of the bilinear covariants? We have already encountered the conserved current $j^\mu = \bar{\psi}\gamma^\mu\psi$ which is a bilinear covariant; the usefulness of the bilinear covariants, however, goes far beyond the probability current. In general, when we study an interaction of two spin-1/2 particles 1 and 2 creating another particle 3, we consider the *overlap* of the waves $\psi_1(x)$ and $\psi_2(x)$ of particles 1 and 2 acting as a *source* of the wave of particle 3. The stronger the overlap, the more intense the source is. But what do we mean by the overlap of ψ_1 and ψ_2 which have four components each? The classification of the bilinear covariants tells us that there are only five ways to define the overlap. And the transformation properties of the overlap should be consistent with the transformation properties of the particle created; for example, if a particle is created by a vector bilinear covariant, then the particle has to be represented by a vector field, etc.

Later on when we form Lagrangians of particles and interactions, we will see that spin-1/2 fields always appear in pairs as bilinear covariants. This is because in order to form a Lagrangian, which is a Lorentz scalar, 4-component spinors have to be first combined into bilinear covariants which have definite transformation properties which can then be combined to form a scalar quantity.

3.5 Representations of the γ matrices

We mentioned earlier that the Dirac representation of the γ matrices is not the only explicit expression that satisfies the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\, g^{\mu\nu}$. An important theorem in this regard is the *Pauli's fundamental theorem* which states:

> If two sets of 4×4 matrices γ^μ and γ'^μ satisfy
>
> $$\{\gamma^\mu, \gamma^\nu\} = 2\, g^{\mu\nu}, \quad \{\gamma'^\mu, \gamma'^\nu\} = 2\, g^{\mu\nu}\,, \tag{3.149}$$
>
> then there exists a matrix V such that
>
> $$\gamma'^\mu = V \gamma^\mu V^{-1} \tag{3.150}$$
>
> and V is unique up to a multiplicative constant.

Furthermore, if γ^μ and γ'^μ are to satisfy the property $\overline{\gamma^\mu} = \gamma^\mu$ and $\overline{\gamma'^\mu} = \gamma'^\mu$ in addition to the Clifford algebra, then the matrix V can be taken to be unitary. This is seen as follows. First, according to the above theorem, there exists a matrix V that satisfies $\gamma'^\mu = V\gamma^\mu V^{-1}$. Noting that $\overline{\gamma^\mu} = \gamma^\mu$ means $\gamma^{\mu\dagger} = \gamma_\mu$ (and similarly for γ'^μ), $\gamma'^{\mu\dagger} = \gamma'_\mu$ can be written as

$$\begin{aligned} & V^{-1} \times \qquad (V\gamma^\mu V^{-1})^\dagger \;=\; V\gamma_\mu V^{-1} \qquad \times V \\ \rightarrow \quad & V^{-1}(V^{-1})^\dagger \gamma^{\mu\dagger} V^\dagger V \;=\; \gamma_\mu \\ \rightarrow \quad & (V^\dagger V)^{-1} \gamma_\mu (V^\dagger V) \;=\; \gamma_\mu\,, \end{aligned} \tag{3.151}$$

where we have used $\gamma^{\mu\dagger} = \gamma_\mu$ and $(V^{-1})^\dagger = (V^\dagger)^{-1}$. Since Γ_i $(i = 1, \cdots, 16)$ that we have seen in Table 3.2 are products of γ^μ's, they also satisfy

$$(V^\dagger V)^{-1} \Gamma_i (V^\dagger V) = \Gamma_i \quad \rightarrow \quad [\Gamma_i, V^\dagger V] = 0 \quad (i = 1, \cdots, 16)\,. \tag{3.152}$$

Since Γ_i $(i = 1, \cdots, 16)$ form a complete set, $V^\dagger V$ commutes with any 4×4 matrix, and thus $V^\dagger V$ should be proportional to the identity matrix. By rescaling, V can be taken to be unitary:

$$V^\dagger V = a\ (\text{const}), \quad V' \stackrel{\text{def}}{\equiv} a^{1/2} V \quad \rightarrow \quad V'^\dagger V' = 1\,. \tag{3.153}$$

Such unitary matrix V' is unique up to an overall phase.

Suppose a spinor wave function $\psi(x)$ satisfies the Dirac equation $(i\,\partial\!\!\!/ - m)\psi = 0$, then

$$\begin{aligned} & \overbrace{V} \qquad (i\gamma^\mu \partial_\mu - m) \overbrace{V^{-1} V} \psi \;=\; 0 \end{aligned}$$

$$\begin{aligned} \rightarrow \quad & (i\underbrace{V\gamma^\mu V^{-1}}_{\gamma'^\mu}\partial_\mu - m)V\psi \;=\; 0 \\ \rightarrow \quad & (i\gamma'^\mu\partial_\mu - m)\psi' \;=\; 0 \quad \text{with} \quad \boxed{\psi' = V\psi}\,; \end{aligned} \tag{3.154}$$

namely, the new wave function $V\psi(x)$ satisfies the Dirac equation with the new gamma matrices given by $\gamma'^\mu = V\gamma^\mu V^{-1}$. Note that the two wave functions $\psi(x)$ and $V\psi(x)$ represent exactly the same physical state (same $E, \vec{P}$, spin etc.), but simply given in different representation where the meaning of the spinor indexes are modified (which indexes correspond to spin-up, down etc.). This is in contrast to the Lorentz transformation where the physical parameters of the particle such as E, $\vec{P}$ and spin are transformed to different values.

Other representations often used are the Weyl representation or the 'chiral representation':

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{3.155}$$

which decouples two spin states (left and right handed spins) for massless particles as the top two and the bottom two components of $\psi(x)$, and the Majorana representation, which is entirely imaginary:

$$\gamma^0 = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} i\sigma_1 & 0 \\ 0 & i\sigma_1 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} i\sigma_3 & 0 \\ 0 & i\sigma_3 \end{pmatrix},$$

$$\gamma^5 = \begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix}. \tag{3.156}$$

The matrix γ_5 is always defined as $\gamma_5 \equiv i\,\gamma^0\gamma^1\gamma^2\gamma^3$.

Exercise 3.7 *Verify that γ matrixes in Weyl and Majorana representations indeed satisfy the anticommutation relations (3.37). Check for all combinations of (μ, ν), but do so systematically.*

3.6 Spin of the electron

Review of spin-1/2 formalism

Let's review the spin-1/2 formalism in non-relativistic quantum mechanics. We define $|\uparrow\rangle$ to be the state with spin in the $+z$ direction, and $|\downarrow\rangle$ to be the state with spin in the $-z$ direction. In terms of two-component column vectors, they can be written as

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{3.157}$$

In the space spanned by $|\uparrow\rangle$ and $|\downarrow\rangle$, the angular momentum operator $\vec{J}$ is represented by the Pauli matrices:

$$J_i = \frac{\sigma_i}{2} \quad (i = 1, 2, 3 \text{ or } x, y, z) \tag{3.158}$$

and they satisfy the commutation relations of angular momentum operators $[J_i, J_j] = i\,\epsilon_{ijk} J_k$ as shown earlier in (3.26). In the space spanned by states with a given angular momentum j, the square of the angular momentum operator $\vec{J}^2$ is given by

$$\vec{J}^2 \equiv J_1^2 + J_2^2 + J_3^2 = j(j+1)\,. \tag{3.159}$$

Using $\sigma_i^2 = 1$ (3.21), we have

$$\left(\frac{\vec{\sigma}}{2}\right)^2 = \left(\frac{\sigma_1}{2}\right)^2 + \left(\frac{\sigma_2}{2}\right)^2 + \left(\frac{\sigma_3}{2}\right)^2 = \frac{3}{4} = \frac{1}{2}\left(\frac{1}{2}+1\right) \tag{3.160}$$

which shows that $\vec{\sigma}/2$ is indeed a spin-1/2 representation of angular momentum.

The eigenstates with spin polarized along an arbitrary unit vector $\vec{s}$ can be obtained by applying a rotation to the states $|\uparrow\rangle$ and $|\downarrow\rangle$ of (3.157). In general, the rotation operator $R(\vec{\theta})$ by an angle $\theta \equiv |\vec{\theta}|$ around the axis $\hat{\theta} \equiv \vec{\theta}/\theta$ is generated by the angular momentum operators as

$$R(\vec{\theta}) = e^{-i\,\vec{\theta}\cdot\vec{J}}\,. \tag{3.161}$$

For rotations in the 3-dimensional space, the generators L_i $(i = 1, 2, 3)$ we have obtained in (1.95) can be redefined to be consistent with this form:

$$J_i \stackrel{\text{def}}{\equiv} iL_i \;(i = 1, 2, 3) \quad \rightarrow \quad e^{\theta_i L_i} = e^{-i\theta_i J_i}\,, \tag{3.162}$$

then the commutation relation (1.98) of L_i's indeed leads to that of angular momentum:

$$[L_i\,, L_j\,] = \epsilon_{ijk} L_k \quad \rightarrow \quad [J_i\,, J_j\,] = i\,\epsilon_{ijk} J_k\,. \tag{3.163}$$

In the spin-1/2 space, the corresponding rotation $u(\vec{\theta})$ is a 2×2 matrix obtained by the replacement $\vec{J} \rightarrow \vec{\sigma}/2$ in (3.161):

$$u(\vec{\theta}) = e^{-i\,\frac{\vec{\theta}}{2}\cdot\vec{\sigma}} = \cos\frac{\theta}{2} - i\,(\hat{\theta}\cdot\vec{\sigma})\sin\frac{\theta}{2} \tag{3.164}$$

where we have used the identity

$$e^{i\,\vec{a}\cdot\vec{\sigma}} = \cos a + i\,(\hat{a}\cdot\vec{\sigma})\sin a \quad \left(a \equiv |\vec{a}|,\ \hat{a} \equiv \frac{\vec{a}}{a}\right)\,. \tag{3.165}$$

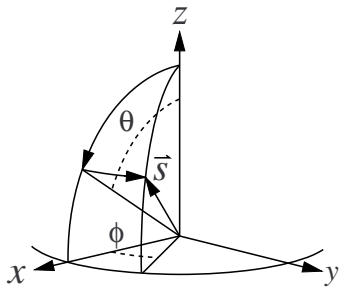


Figure 3.1: The rotation to take the z-axis to a general direction $\vec{s}$.

The rotation matrix (3.164) tells us that a rotation around any axis by 2π changes sign of the state ($u = -1$), and it takes two complete rotations to recover the original state, which is a general feature of half-integer spin states.

The z-axis can be rotated to the direction of $\vec{s}$ by first rotating around the y-axis by θ and then around the original z-axis (not the rotated one) by ϕ, where (θ, ϕ) are the polar angles of the direction $\vec{s}$ (Figure 3.1):

$$\vec{s} = (s_x, s_y, s_z) = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)\,. \tag{3.166}$$

The rotation matrix is then

$$R(\theta,\phi) = e^{-i\phi J_z} e^{-i\theta J_y} \quad \rightarrow \quad u(\theta,\phi) = e^{-i\frac{\phi}{2}\sigma_z} e^{-i\frac{\theta}{2}\sigma_y}\,, \tag{3.167}$$

and the resulting eigenvectors are (up to a common overall phase)

$$\begin{aligned} \chi_+ &= u(\theta,\phi)\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2(1+s_z)}}\begin{pmatrix}1+s_z\\ s_+\end{pmatrix} \\ \chi_- &= u(\theta,\phi)\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{\sqrt{2(1-s_z)}}\begin{pmatrix}s_z-1\\ s_+\end{pmatrix}\,, \end{aligned} \tag{3.168}$$

where

$$s_\pm \stackrel{\text{def}}{\equiv} s_x \pm i s_y \tag{3.169}$$

and the orthonormality is given by

$$\chi_+^\dagger\chi_+ = \chi_-^\dagger\chi_- = 1\,, \quad \chi_+^\dagger\chi_- = \chi_-^\dagger\chi_+ = 0\,. \tag{3.170}$$

On the other hand, the spin component in an arbitrary direction $\vec{s}$ is represented by the operator (in the unit of the absolute value of the spin which is 1/2)

$$\vec{s}\cdot\vec{\sigma} = s_x\begin{pmatrix}0&1\\1&0\end{pmatrix} + s_y\begin{pmatrix}0&-i\\ i&0\end{pmatrix} + s_z\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

$$= \begin{pmatrix} s_z & s_x - is_y \\ s_x + is_y & -s_z \end{pmatrix}$$
$$= \begin{pmatrix} s_z & s_- \\ s_+ & -s_z \end{pmatrix}. \tag{3.171}$$

Using the formula

$$(\vec{a}\cdot\vec{\sigma})(\vec{b}\cdot\vec{\sigma}) = \vec{a}\cdot\vec{b} + i\,\vec{\sigma}\cdot(\vec{a}\times\vec{b}) \tag{3.172}$$

we have

$$(\vec{s}\cdot\vec{\sigma})^2 = \vec{s}^2 + i\,\vec{\sigma}\cdot(\underbrace{\vec{s}\times\vec{s}}_{0}) = 1\,, \tag{3.173}$$

which means that the eigenvalues of $\vec{s}\cdot\vec{\sigma}$ is ± 1. In fact, it is easily verified that the states obtained in (3.168) are indeed eigenvectors of $\vec{s}\cdot\vec{\sigma}$:

$$(\vec{s}\cdot\vec{\sigma})\chi_\pm = \pm\chi_\pm\,. \tag{3.174}$$

Using the fact that $\vec{s}\cdot\vec{\sigma}$ has the eigenvalues ± 1, we can construct projection operators which project out $\chi_\pm$ from any vector. Define the operators $P_\pm$ by

$$P_\pm \stackrel{\text{def}}{\equiv} \frac{1 \pm \vec{s}\cdot\vec{\sigma}}{2}\,. \tag{3.175}$$

Then using $(\vec{s}\cdot\vec{\sigma})\chi_\pm = \pm\chi_\pm$

$$\begin{aligned} P_\pm\chi_\pm &= \frac{1 \pm \vec{s}\cdot\vec{\sigma}}{2}\chi_\pm = \frac{1 \pm (\pm 1)}{2}\chi_\pm = \chi_\pm \\ P_\pm\chi_\mp &= \frac{1 \pm \vec{s}\cdot\vec{\sigma}}{2}\chi_\mp = \frac{1 \pm (\mp 1)}{2}\chi_\mp = 0\,. \end{aligned} \tag{3.176}$$

Writing any vector v as a linear combination of $\chi_\pm$

$$v = c_+\chi_+ + c_-\chi_- \qquad (c_\pm : \text{complex constants}) \tag{3.177}$$

we see that $P_\pm$ indeed projects out $\chi_\pm$ out of v:

$$P_+ v = c_+\chi_+, \quad P_- v = c_-\chi_-\,. \tag{3.178}$$

One can also easily verify that the operators $P_\pm$ satisfy the properties of projection operators:

$$P_\pm^2 = P_\pm, \quad P_+P_- = P_-P_+ = 0\,. \tag{3.179}$$

Exercise 3.8 *Non-relativistic spin-1/2 states.*
(a) Show that $u(\theta,\phi)$ can be written as

$$u(\theta,\phi) = e^{-i\frac{\phi}{2}} \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i\phi} & \cos\frac{\theta}{2}e^{i\phi} \end{pmatrix}, \tag{3.180}$$

and verify (3.168). (hint: Use $e^{i\,\vec{a}\cdot\vec{\sigma}} = \cos a + i\,(\hat{a}\cdot\vec{\sigma})\sin a$.*)*
(b) For any 2-component vector v*,* $P_\pm(\vec{s})v$ *are eigenvectors of* $\vec{s}\cdot\vec{\sigma}$ *with eigenvalues* ± 1 *assuming that they are not zero. One can use this feature to construct the eigenvectors of* $\vec{s}\cdot\vec{\sigma}$*. Take any state, say the state with spin in* $+z$ *direction, as the vector* v *to obtain the same result for* $\chi_\pm$ *as above.*

Electron spin

From the spinor-space representation of the proper and orthochronous Lorentz group (3.97), a pure rotation by $\vec{\theta}$, $U(\vec{\theta})$, is obtained by extracting the terms that correspond to the generators of rotation $M^{ij} = L_k$ (i, j, k: cyclic):

$$U(\vec{\theta}) = \exp\left(\frac{1}{2}\sum_{i,j} a_{ij}B^{ij}\right) = \exp\left(\sum_{i<j} a_{ij}B^{ij}\right) = e^{-i\theta_i\frac{\Sigma_i}{2}} \tag{3.181}$$

with

$$\Sigma_i \stackrel{\text{def}}{\equiv} 2iB^{jk}\,, \quad \theta_i = a_{jk} \quad (i,j,k : \text{cyclic})\,, \tag{3.182}$$

or using the definition of $\sigma^{\mu\nu}$ (3.126),

$$\boxed{\Sigma_i \stackrel{\text{def}}{\equiv} \sigma^{jk} = i\gamma^j\gamma^k \quad (i,j,k : \text{cyclic})}\,. \tag{3.183}$$

Comparing (3.181) with the general form of a rotation (3.161), we identify a set of operators that is acting as angular momentum operators:

$$J_i = \frac{\Sigma_i}{2}\,. \tag{3.184}$$

Since B^{jk} and M^{jk} satisfy the same commutation relations, $iL_i = iM^{jk}$ and $\Sigma_i/2 = iB^{jk}$ should satisfy the same commutation relations. On the other hand, we have seen in (3.163) that iL_i ($i = 1, 2, 3$) satisfy the commutation relations of angular momentum, and thus so do $\frac{\Sigma_i}{2}$ ($i = 1, 2, 3$). In fact, we will show below that $\vec{\Sigma}_i$ satisfies exactly the same commutation and anticommutation relations as the Pauli matrices $\vec{\sigma}$. First, the square of Σ_i is unity:

$$\begin{aligned}(\Sigma_i)^2 &= (i\gamma^j\gamma^k)^2 = -\gamma^j\underbrace{\gamma^k\gamma^j}_{-\gamma^j\gamma^k \text{ since } j\neq k}\gamma^k \\ &= (\gamma^j)^2(\gamma^k)^2 = 1\end{aligned} \tag{3.185}$$

where i, j, k are cyclic and no summation over repeated indexes is implied. For cyclic i, j, k, we have $\Sigma_i = i\gamma^j\gamma^k$ and $\Sigma_j = i\gamma^k\gamma^i$. Then,

$$\Sigma_i\Sigma_j = (i\gamma^j\underbrace{\gamma^k)(i\gamma^k}_{-i}\gamma^i) = \gamma^j\gamma^i = i(i\gamma^i\gamma^j)\,; \tag{3.186}$$

thus,

$$\Sigma_i \Sigma_j = i\Sigma_k \quad (i,j,k : \text{cyclic})\,. \tag{3.187}$$

Now, we note that Σ_i is hermitian:

$$\Sigma_i^\dagger = (i\gamma^j\gamma^k)^\dagger = -i\underbrace{\gamma^{k\dagger}}_{-\gamma^k}\underbrace{\gamma^{j\dagger}}_{-\gamma^j} = i\gamma^j\gamma^k = \Sigma_i\,.$$

Then, taking the hermitian conjugate of (3.187), we obtain

$$\Sigma_j^\dagger \Sigma_i^\dagger = -i\Sigma_k^\dagger \quad \to \quad \Sigma_j\Sigma_i = -i\Sigma_k\,.$$

Comparing this with (3.187), we see that Σ_i and Σ_j anticommute. These relations are summarized as

$$\boxed{\{\Sigma_i\,,\Sigma_j\} = 2\,\delta_{ij}\,, \quad [\Sigma_i\,,\Sigma_j] = 2i\epsilon_{ijk}\Sigma_k}\,, \tag{3.188}$$

leading, in particular, to

$$\left[\frac{\Sigma_i}{2}\,,\frac{\Sigma_j}{2}\right] = i\epsilon_{ijk}\frac{\Sigma_k}{2}\,, \quad \left(\frac{\vec{\Sigma}}{2}\right)^2 = \frac{1}{2}\left(\frac{1}{2}+1\right) \tag{3.189}$$

which shows that $\vec{\Sigma}/2$ acts as an angular momentum operator with $j = 1/2$.

In the Dirac representation Σ_i is explicitly written as

$$\begin{aligned}\Sigma_i &= i\,\gamma^j\gamma^k = i\begin{pmatrix}0 & \sigma_j\\ -\sigma_j & 0\end{pmatrix}\begin{pmatrix}0 & \sigma_k\\ -\sigma_k & 0\end{pmatrix} = i\begin{pmatrix}-\sigma_j\sigma_k & 0\\ 0 & -\sigma_j\sigma_k\end{pmatrix}\\ &= i\begin{pmatrix}-i\sigma_i & 0\\ 0 & -i\sigma_i\end{pmatrix} = \begin{pmatrix}\sigma_i & 0\\ 0 & \sigma_i\end{pmatrix}\,,\end{aligned} \tag{3.190}$$

which explicitly shows that Σ_i's satisfy the same relations among themselves as do σ_i's. This is, however, independent of representation as we have seen.

To see the spin structure more clearly, apply the rotation operator U (3.181) to a plane wave

$$\psi(x) = \psi_0 e^{-i\,p\cdot x} \tag{3.191}$$

where ψ_0 is a constant spinor and $p^\mu = (E, \vec{P})$ represents the energy and momentum of the state. We carefully follow $\psi'(x') = U\psi(x)$ and make sure that we write the result in terms of x'. Using $x = \Lambda^{-1}x'$, or

$$x_\mu = (\Lambda^{-1})_\mu{}^\nu x'_\nu = \Lambda^\nu{}_\mu x'_\nu\,, \tag{3.192}$$

where Λ is the 4×4 Lorentz transformation matrix corresponding to the rotation, the transformed wave function $\psi'(x')$ can be written as

$$\begin{aligned} \psi'(x') &= U\psi_0 \exp(-i\, p^\mu x_\mu) = (U\psi_0) \exp(-i\, \overbrace{p^\mu \Lambda^\nu{}_\mu}^{p'^\nu} x'_\nu) \\ &= (U\psi_0) e^{-ip'\cdot x'}\,, \end{aligned} \tag{3.193}$$

where

$$p'^\nu = \Lambda^\nu{}_\mu p^\mu \tag{3.194}$$

is nothing but the rotated energy-momentum 4-vector. If it were a scalar field, then this rotation of the energy-momentum 4-vector (actually, the energy stays the same) would have been the only change. For a spinor field, however, we have additional 'rotation' ($U\psi_0$) associated with some internal structure of the particle, and we know that this internal rotation is exactly like that of a spin-1/2 particle because of the equivalence of Σ_i and σ_i. Thus, we suspect that a particle represented by a spinor field carries an intrinsic spin 1/2.

To obtain a more physical understanding, however, we will now show that the angular momentum is conserved only when the spin is added to the orbital angular momentum. We start from the Dirac equation written as (3.29):

$$i\frac{\partial}{\partial t}\psi = H\psi\,, \quad H = \vec{\alpha}\cdot\vec{p} + \beta m \tag{3.195}$$

with

$$\beta = \gamma^0, \quad \alpha_i = \gamma^0\gamma^i \ (\leftarrow\ \gamma^i \equiv \beta\alpha_i), \quad \vec{p} \equiv -i\vec{\nabla}\,. \tag{3.196}$$

Recall that if an operator O commutes with the Hamiltonian H, then it is a constant of motion. Does it apply to the operator H we have here? Actually it does: suppose states $|a\rangle$ and $|b\rangle$ are solutions of a Schrödinger-form equation $i\frac{\partial}{\partial t}|\ \rangle = H|\ \rangle$. Then we have

$$i|\dot{a}\rangle = H|a\rangle \quad \rightarrow \quad -i\langle\dot{a}| = \langle a|H^\dagger = \langle a|H, \tag{3.197}$$

The time derivative of the matrix element $\langle a|O|b\rangle$ is then

$$\begin{aligned} \frac{\partial}{\partial t}\langle a|O|b\rangle &= \underbrace{\langle\dot{a}|}_{i\langle a|H} O|b\rangle + \langle a|O\underbrace{|\dot{b}\rangle}_{-iH|b\rangle} \\ &= i\langle a|HO|b\rangle - i\langle a|OH|b\rangle \\ &= i\langle a|[H,O]|b\rangle\,; \end{aligned} \tag{3.198}$$

namely, the matrix element $\langle a|O|b\rangle$ is a constant of motion if $[H,O]=0$. This holds as long as the equation is in the Schrödinger form and the 'Hamiltonian' is hermitian.

We first evaluate the commutator of the orbital angular momentum and the Hamiltonian, $[L_i, H]$, with

$$L_i = (\vec{x} \times \vec{p})_i = \epsilon_{ijk}\, x^j p^k \tag{3.199}$$

$$H = \underbrace{\alpha_i}_{\gamma^0\gamma^i}\, p^i + \underbrace{\beta}_{\gamma^0}\, m = \gamma^0(\gamma^i p^i + m)\,, \tag{3.200}$$

where sum over repeated space indexes are implicit regardless of superscript or subscript, and we are consistently using the definition $x^\mu = (x^0, \vec{x})$ and $p^\mu = (p^0, \vec{p})$. Useful relations are

$$[AB, C] = A[B, C] + [A, C]B\,, \qquad [C, AB] = A[C, B] + [C, A]B\,. \tag{3.201}$$

It follows that, in either of the formulas, if A commutes with C, A can simply come out of the commutator to the left, and if B commutes with C, B can simply come out to the right.

Noting that γ^μ commutes with x^i and p^i, and that $[x^i\,, p^j] = i\delta_{ij}$, we have

$$\begin{aligned}
[L_i, H] &= [\epsilon_{ijk}\, x^j p^k\,, \gamma^0(\gamma^l p^l + m)] \\
&= \epsilon_{ijk}\gamma^0 [x^j p^k\,, \gamma^l p^l + \cancel{m}\,] \\
&= \epsilon_{ijk}\gamma^0\gamma^l \underbrace{[x^j p^k\,, p^l]}_{\underbrace{[x^j, p^l]}_{i\delta_{jl}}\, p^k} = i\,\epsilon_{ijk}\underbrace{\gamma^0\gamma^j}_{\alpha_j} p^k \\
&= i(\vec{\alpha} \times \vec{p})_i
\end{aligned} \tag{3.202}$$

This looks non-zero and cannot be simplified further, indicating that the orbital angular momentum is not a constant of motion by itself.

We now evaluate $[\Sigma_i\,, H]$ using $\Sigma_i = i\gamma^j\gamma^k$, $(i, j, k$: cyclic$)$:

$$\begin{aligned}
[\Sigma_i\,, H] &= [i\gamma^j\gamma^k, \gamma^0(\gamma^l p^l + m)] \\
&= i\gamma^0 \underbrace{[\gamma^j\gamma^k, \gamma^l p^l + \cancel{m}\,]}_{\underbrace{[\gamma^j\gamma^k, \gamma^l]}\, p^l} \\
(3.90) \rightarrow\ & -2(g^{jl}g^k{}_\nu - g^{kl}g^j{}_\nu)\gamma^\nu = 2(\delta_{jl}\delta_{k\nu} - \delta_{kl}\delta_{j\nu})\gamma^\nu \\
&= 2i\gamma^0(\gamma^k p^j - \gamma^j p^k) = -2i(\alpha_j p^k - \alpha_k p^j) \\
&= -2i\,(\vec{\alpha} \times \vec{p})_i
\end{aligned} \tag{3.203}$$

Combining (3.202) and (3.203), we have a conserved quantity:

$$[\vec{J}, H] = 0\,, \quad \text{with} \quad \vec{J} = \vec{L} + \frac{\vec{\Sigma}}{2}\,, \tag{3.204}$$

which shows that when the spin operator $\vec{\Sigma}/2$ is added to the orbital angular momentum, the total angular momentum is conserved, indicating that electron does carry an intrinsic spin angular momentum whose absolute value is 1/2. Note also that the above equation clearly identifies Σ_i to be the i-th component of the spin which has to be added to the i-th component of $\vec{L}$; for example, $\Sigma_3 = i\gamma^1\gamma^2$ always represents the z-component of the spin regardless of the representation of the γ matrices.

Let's reflect upon how this descritpion of electron spin has come about. We set out to solve the problems of the Klein-Gordon equation, in particular its negative probability, and in making the equation linear in time and space derivatives in hope of solving it, the equation became 4-component, and it introduced the possibility that the particle has some internal structure. The spin of the electron, or more correctly the magnetic moment of electron, had been 'discovered' by G. Uhlenbeck and S. Goudsmit three years earlier in 1925 in order to explain the anomalous Zeeman effect (splitting of energy levels in alkali atoms in presence of magnetic fields). The electron spin was now beautifully explained by a theory which was consistent with the special relativity and the quantum mechanics. Actually, we have not shown that electron has a magnetic moment yet, which involves interactions with the electromagnetic field $A^\mu(x)$. This will be done later, and it will be shown that the value of the electron magnetic moment is twice as large as the natural value classically expected from its charge and spin. Now, let's move on to finding explicit plane-wave solutions of the Dirac equation.

3.7 Plane-wave solutions of the Dirac equation

In this section, we will construct plane-wave solutions of the Dirac equation. Even though we will use the Dirac representation for explicit expressions, many of the essential relations are independent of the representation as will be shown. Let's start from the solutions for a particle at rest.

Electron at rest ($\vec{p} = 0$)
From the correspondence $\vec{p} \leftrightarrow -i\vec{\nabla}$, the solution for $\vec{p} = 0$ should have no space dependence, or $\partial_i\psi = 0$, $(i = 1, 2, 3)$. Then the Dirac equation $i\gamma^\mu\partial_\mu\psi = m\psi$ becomes

$$i\gamma^0\partial_0\psi = m\psi\,, \quad \text{or} \quad i\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}\begin{pmatrix} \dot{\psi}_1 \\ \dot{\psi}_2 \\ \dot{\psi}_3 \\ \dot{\psi}_4 \end{pmatrix} = m\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}. \tag{3.205}$$

The four equations are

$$\begin{cases} i\dot{\psi}_1 &= m\psi_1 \\ i\dot{\psi}_2 &= m\psi_2 \\ i\dot{\psi}_3 &= -m\psi_3 \\ i\dot{\psi}_4 &= -m\psi_4 \end{cases} \tag{3.206}$$

There are thus four independent solutions:

$$\psi^{(1)} = \omega^{(1)} e^{-imt}, \;\; \psi^{(2)} = \omega^{(2)} e^{-imt}, \;\; \psi^{(3)} = \omega^{(3)} e^{+imt}, \;\; \psi^{(4)} = \omega^{(4)} e^{+imt},$$
$$\omega^{(1)} \equiv \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \omega^{(2)} \equiv \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \omega^{(3)} \equiv \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \omega^{(4)} \equiv \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{3.207}$$

Since we define the eigenvalue of the operator $i\partial/\partial t$ to be the energy, $\psi^{(1)}$ and $\psi^{(2)}$ are positive energy solutions and $\psi^{(3)}$ and $\psi^{(4)}$ are negative energy solutions.

That this is a complete set of solutions for $\vec{p} = 0$ can be seen as follows. At a given time, say $t = 0$, the most general form of ψ is $\psi_i(x)|_{t=0} = a_i$ where a_i $(i = 1, 2, 3, 4)$ are some complex constants (there is no space dependence since $\vec{p} = 0$). Then, in order for this wave function to be a solution of the Dirac equation, the time dependence of each component in the past and future is uniquely determined by (3.206); namely, the first and second components has to vary as e^{-imt} and the third and fourth as e^{+imt}:

$$\psi(x)|_{t=0} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} \quad \rightarrow \quad \psi(x) = \begin{pmatrix} a_1 e^{-imt} \\ a_2 e^{-imt} \\ a_3 e^{+imt} \\ a_4 e^{+imt} \end{pmatrix}, \tag{3.208}$$

which is thus a completely general $\vec{p} = 0$ solution of the Dirac equation. This can of course be written as a linear combination of the four solutions (3.207) which are thus complete:

$$\psi(x) = a_i \psi^{(i)}(x)\,, \tag{3.209}$$

where the spinor index i is summed over 1 through 4.

Leaving the discussion of the negative energy states to a later time when we describe the hole theory, let's look at how $\psi^{(i)}$'s respond to the spin z-component operator Σ_3, which, in the Dirac representation, is written as [see (3.190)]

$$\Sigma_3 = \begin{pmatrix} \sigma_3 & \\ & \sigma_3 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}. \tag{3.210}$$

Clearly, $\psi^{(i)}$'s are eigenvectors of the spin z-component with eigenvalues given by

$$\psi^{(1)} : \Sigma_3 = +1, \quad \psi^{(2)} : \Sigma_3 = -1, \quad \psi^{(3)} : \Sigma_3 = +1, \quad \psi^{(4)} : \Sigma_3 = -1\,. \tag{3.211}$$

Eigenstates of spin component in any arbitrary direction $\vec{s}$ can be obtained by rotating these solutions, and we know exactly how to rotate spinors. The operator to rotate the z-axis to an arbitrary direction $\vec{s}$ can be taken as in (3.167): $R(\theta,\phi) = e^{-i\phi J_z} e^{-i\theta J_y}$ where (θ,ϕ) are the polar angles of the direction $\vec{s}$. The corresponding rotation in the spinor space is given by the replacement $\vec{J} \to \vec{\Sigma}/2$ (3.184):

$$U(\theta,\phi) = e^{-i\frac{\phi}{2}\Sigma_z} e^{-i\frac{\theta}{2}\Sigma_y}\,, \tag{3.212}$$

which can be rewritten as follows: First, due to the block-diagonal form of (3.190), we have

$$\Sigma_i = \begin{pmatrix} \sigma_i & \\ & \sigma_i \end{pmatrix} \quad \to \quad (c\Sigma_i)^k = \begin{pmatrix} (c\,\sigma_i)^k & \\ & (c\,\sigma_i)^k \end{pmatrix}, \tag{3.213}$$

where c is any constant number. Then, the exponential can go inside the block-diagonal subsections:

$$\begin{aligned} e^{c\Sigma_i} &= \sum_k \frac{1}{k!}\,(c\Sigma_i)^k = \sum_k \frac{1}{k!} \begin{pmatrix} (c\,\sigma_i)^k & \\ & (c\,\sigma_i)^k \end{pmatrix} \\ &= \begin{pmatrix} e^{c\,\sigma_i} & \\ & e^{c\,\sigma_i} \end{pmatrix} \end{aligned} \tag{3.214}$$

Thus, the spinor operator $U(\theta,\phi)$ can be written as

$$\begin{aligned} U(\theta,\phi) &= e^{-i\frac{\phi}{2}\Sigma_z} e^{-i\frac{\theta}{2}\Sigma_y} = \begin{pmatrix} e^{-i\frac{\phi}{2}\sigma_z} & \\ & e^{-i\frac{\phi}{2}\sigma_z} \end{pmatrix} \begin{pmatrix} e^{-i\frac{\theta}{2}\sigma_y} & \\ & e^{-i\frac{\theta}{2}\sigma_y} \end{pmatrix} \\ &= \begin{pmatrix} e^{-i\frac{\phi}{2}\sigma_z} e^{-i\frac{\theta}{2}\sigma_y} & \\ & e^{-i\frac{\phi}{2}\sigma_z} e^{-i\frac{\theta}{2}\sigma_y} \end{pmatrix} \\ &= \begin{pmatrix} u(\theta,\phi) & \\ & u(\theta,\phi) \end{pmatrix} \end{aligned} \tag{3.215}$$

where $u(\theta,\phi)$ is the 2×2 matrix defined in (3.167). Applying this rotation to $\psi^{(1)}$,

$$\psi'^{(1)} = U(\theta,\phi)\psi^{(1)} = \begin{pmatrix} u(\theta,\phi)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{pmatrix} e^{-imt} = \begin{pmatrix} \chi_+ \\ 0 \end{pmatrix} e^{-imt}\,, \tag{3.216}$$

where χ_+ is defined by (3.168). Similarly, applying the rotation $U(\theta,\phi)$ to $\psi^{(i)}$ $(i = 2,3,4)$, we obtain a set of four states polarized along a general direction $\vec{s}$ (and $\vec{p}=0$):

$$\begin{aligned} \psi^{(1)}_{\vec{s}} &= N \begin{pmatrix} \chi_+ \\ 0 \end{pmatrix} e^{-imt}\ (\text{spin } +\vec{s}), \qquad \psi^{(2)}_{\vec{s}} = N \begin{pmatrix} \chi_- \\ 0 \end{pmatrix} e^{-imt}\ (\text{spin } -\vec{s}), \\ \psi^{(3)}_{\vec{s}} &= N \begin{pmatrix} 0 \\ \chi_+ \end{pmatrix} e^{+imt}\ (\text{spin } +\vec{s}), \qquad \psi^{(4)}_{\vec{s}} = N \begin{pmatrix} 0 \\ \chi_- \end{pmatrix} e^{+imt}\ (\text{spin } -\vec{s})\,, \end{aligned} \tag{3.217}$$

where we have omitted the primes on ψ's, and N is a normalization factor to be chosen later. It can be easily checked that these are indeed eigenvectors of the operator $\vec{s}\cdot\vec{\Sigma}$, which should represent the component of spin along the direction $\vec{s}$, and that they have the stated eigenvalues: Using (3.190), we have

$$\vec{s}\cdot\vec{\Sigma} = \begin{pmatrix} \vec{s}\cdot\vec{\sigma} & \\ & \vec{s}\cdot\vec{\sigma} \end{pmatrix} . \tag{3.218}$$

Applying this to $\psi^{(1)}_{\vec{s}}$ of (3.217), for example,

$$(\vec{s}\cdot\vec{\Sigma})\psi^{(1)}_{\vec{s}} = \begin{pmatrix} (\vec{s}\cdot\vec{\sigma})\chi_+ \\ 0 \end{pmatrix} e^{-imt} = \begin{pmatrix} \chi_+ \\ 0 \end{pmatrix} e^{-imt} = +\psi^{(1)}_{\vec{s}} , \tag{3.219}$$

where we have used (3.174). Similarly, other states can be shown to be eigenvectors of $\vec{s}\cdot\vec{\Sigma}$; thus, we have

$$(\vec{s}\cdot\vec{\Sigma})\psi^{(1,3)}_{\vec{s}} = +\psi^{(1,3)}_{\vec{s}} , \quad (\vec{s}\cdot\vec{\Sigma})\psi^{(2,4)}_{\vec{s}} = -\psi^{(2,4)}_{\vec{s}} . \tag{3.220}$$

Electron in motion

Now that we have a set of solutions for an electron at rest with its spin polarized in an arbitrary direction, the next step is to construct solutions for an electron in motion. One could solve the Dirac equation explicitly without dropping the space derivatives, but we are now well equipped to deal with it more systematically: we know solutions at rest and we know exactly how to boost them.

The matrix S in the spinor space corresponding to a Lorentz transformation in space-time Λ which is a boost in a direction $\hat{\xi}$ by velocity $\beta = \tanh|\vec{\xi}|$ is

$$\Lambda = e^{\xi_i K_i} = e^{\xi_i M^{0i}} \quad \to \quad S = e^{\xi_i B^{0i}} , \tag{3.221}$$

with

$$B^{0i} = \frac{1}{2}\gamma^0\gamma^i = \frac{1}{2}\alpha_i . \tag{3.222}$$

Writing $\vec{\xi} = \xi\hat{\xi}$ ($\xi \equiv |\vec{\xi}|$, $\hat{\xi} = \vec{\xi}/\xi$),

$$\begin{aligned} (\hat{\xi}\cdot\vec{\alpha})^2 &= \hat{\xi}_i\alpha_i\hat{\xi}_j\alpha_j = \hat{\xi}_i\hat{\xi}_j\alpha_i\alpha_j \\ &= \frac{1}{2}(\hat{\xi}_i\hat{\xi}_j \overbrace{\alpha_i\alpha_j + \hat{\xi}_j\hat{\xi}_i}^{(i\leftrightarrow j)}\alpha_j\alpha_i) \\ &= \frac{1}{2}\hat{\xi}_i\hat{\xi}_j(\underbrace{\alpha_i\alpha_j + \alpha_j\alpha_i}_{2\delta_{ij}}) = |\hat{\xi}|^2 = 1 . \end{aligned} \tag{3.223}$$

The matrix S can then be written as

$$\begin{aligned} S &= e^{\frac{1}{2}\vec{\xi}\cdot\vec{\alpha}} = e^{\frac{\xi}{2}(\hat{\xi}\cdot\vec{\alpha})} \\ &= 1 + \left(\frac{\xi}{2}\right)(\hat{\xi}\cdot\vec{\alpha}) + \frac{1}{2!}\left(\frac{\xi}{2}\right)^2 \underbrace{(\hat{\xi}\cdot\vec{\alpha})^2}_{1} + \frac{1}{3!}\left(\frac{\xi}{2}\right)^3 \underbrace{(\hat{\xi}\cdot\vec{\alpha})^3}_{(\hat{\xi}\cdot\vec{\alpha})} + \cdots \\ &= \left[1 + \frac{1}{2!}\left(\frac{\xi}{2}\right)^2 + \cdots\right] + \left[\left(\frac{\xi}{2}\right) + \frac{1}{3!}\left(\frac{\xi}{2}\right)^3 + \cdots\right](\hat{\xi}\cdot\vec{\alpha}) \\ &= \cosh\frac{\xi}{2} + (\hat{\xi}\cdot\vec{\alpha})\sinh\frac{\xi}{2}\,. \end{aligned} \tag{3.224}$$

Using

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \quad \rightarrow \quad \hat{\xi}\cdot\vec{\alpha} = \begin{pmatrix} 0 & \hat{\xi}\cdot\vec{\sigma} \\ \hat{\xi}\cdot\vec{\sigma} & 0 \end{pmatrix} \tag{3.225}$$

we have

$$\begin{aligned} S = e^{\frac{1}{2}\vec{\xi}\cdot\vec{\alpha}} &= \begin{pmatrix} \cosh\frac{\xi}{2} & (\hat{\xi}\cdot\vec{\sigma})\sinh\frac{\xi}{2} \\ (\hat{\xi}\cdot\vec{\sigma})\sinh\frac{\xi}{2} & \cosh\frac{\xi}{2} \end{pmatrix} \\ &= \cosh\frac{\xi}{2}\begin{pmatrix} 1 & (\hat{\xi}\cdot\vec{\sigma})\tanh\frac{\xi}{2} \\ (\hat{\xi}\cdot\vec{\sigma})\tanh\frac{\xi}{2} & 1 \end{pmatrix}. \end{aligned} \tag{3.226}$$

It is more convenient to express it in terms of the energy and momentum of the particle. Recall that the boost parameter ξ is related to γ and η of the boost by

$$\gamma = \cosh\xi\,, \quad \eta = \sinh\xi\,, \tag{3.227}$$

then using the hyperbolic half-angle formulas,

$$\cosh\frac{\xi}{2} = \sqrt{\frac{\cosh\xi + 1}{2}} = \sqrt{\frac{\gamma+1}{2}} \tag{3.228}$$

$$\tanh\frac{\xi}{2} = \frac{\sinh\xi}{\cosh\xi + 1} = \frac{\eta}{\gamma+1}\,. \tag{3.229}$$

The energy-momentum 4-vector $p^\mu \equiv (E, \vec{p})$ acquired by a rest mass m after the boost is

$$\boxed{p^\mu = m\eta^\mu \quad \text{with} \quad \eta^\mu \stackrel{\text{def}}{\equiv} (\gamma, \vec{\eta})}\,. \tag{3.230}$$

where $\vec{\eta}$ is the vector with length η pointing to the direction of the boost. We then have $E = m\gamma$ and $p \equiv |\vec{p}| = m\eta$, and thus we can write

$$\cosh\frac{\xi}{2} = \sqrt{\frac{\gamma+1}{2}} = \sqrt{\frac{E+m}{2m}}\,, \quad \tanh\frac{\xi}{2} = \frac{\eta}{\gamma+1} = \frac{p}{E+m}\,. \tag{3.231}$$

Since $\vec{\xi}$ is in the direction of the boost as discussed just below (1.139), we have $\vec{p} = \hat{\xi} p$, and thus

$$(\hat{\xi} \cdot \vec{\sigma}) \tanh \frac{\xi}{2} = (\hat{\xi} \cdot \vec{\sigma}) \frac{p}{E+m} = \frac{\vec{p} \cdot \vec{\sigma}}{E+m}\,, \tag{3.232}$$

The boost matrix in the spinor space (3.226) can now be written as

$$\boxed{S = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} 1 & \dfrac{\vec{p} \cdot \vec{\sigma}}{E+m} \\ \dfrac{\vec{p} \cdot \vec{\sigma}}{E+m} & 1 \end{pmatrix}}, \tag{3.233}$$

which is a boost under which a rest mass m (> 0) acquires an energy-momentum $p^\mu = (E, \vec{p})$. Note that $p^0 \equiv E$ as defined is *always positive*. Using [see (3.171)]

$$\vec{p} \cdot \vec{\sigma} = \begin{pmatrix} p_z & p_- \\ p_+ & -p_z \end{pmatrix} \qquad (p_\pm = p_x \pm i p_y)\,, \tag{3.234}$$

it could be written in a fully 4×4 form:

$$S = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} 1 & 0 & \frac{p_z}{E+m} & \frac{p_-}{E+m} \\ 0 & 1 & \frac{p_+}{E+m} & \frac{-p_z}{E+m} \\ \frac{p_z}{E+m} & \frac{p_-}{E+m} & 1 & 0 \\ \frac{p_+}{E+m} & \frac{-p_z}{E+m} & 0 & 1 \end{pmatrix}. \tag{3.235}$$

All we need now is to apply S to the solutions at rest $\psi^{(i)}_{\vec{s}}$, and strictly follow the definition

$$\psi'(x') = S\psi(x), \quad x' = \Lambda x\,, \tag{3.236}$$

to write it as a function of x'. In doing so, we need to write the exponent imt in terms of x'. This can easily done by noting that

$$mt = p \cdot x, \qquad p = (m, \vec{0}). \tag{3.237}$$

Then we have the Lorentz invariance relation

$$p \cdot x = p' \cdot x'\,, \qquad (x' = \Lambda x\,, \quad p' = \Lambda x')\,. \tag{3.238}$$

Applying S to the solution $\psi(x) = \psi^{(1)}_{\vec{s}}(x)$ with $\vec{p} = 0$ and spin $+\vec{s}$ (3.217), the boosted wave function, which we will denote as $\psi^{(1)}_{\vec{s},\vec{p}}(x')$, is then

$$\begin{aligned} \psi'(x') = S\psi^{(1)}_{\vec{s}}(x) &= N\sqrt{\frac{E+m}{2m}} \begin{pmatrix} 1 & \dfrac{\vec{p} \cdot \vec{\sigma}}{E+m} \\ \dfrac{\vec{p} \cdot \vec{\sigma}}{E+m} & 1 \end{pmatrix} \begin{pmatrix} \chi_+ \\ \\ 0 \end{pmatrix} e^{-imt} \\ &= N\sqrt{\frac{E+m}{2m}} \begin{pmatrix} \chi_+ \\ \dfrac{\vec{p} \cdot \vec{\sigma}}{E+m}\chi_+ \end{pmatrix} e^{-ip'\cdot x'} \overset{\text{def}}{\equiv} \psi^{(1)}_{\vec{s},\vec{p}}(x')\,. \end{aligned} \tag{3.239}$$

At this point, we choose the normalization by $N = \sqrt{2m}$, which will avoid the divergence in the limit $m \to 0$. Similarly applying the same boost S to the rest of the states, we obtain (omitting the primes)

$$\begin{aligned}
&\psi^{(1)}_{\vec{s},\vec{p}}(x) = \omega_{p,\vec{s}}\, e^{-ip\cdot x}\,, && \psi^{(2)}_{\vec{s},\vec{p}}(x) = \omega_{p,-\vec{s}}\, e^{-ip\cdot x}\,,\\
&\omega_{p,\vec{s}} = \sqrt{E+m}\begin{pmatrix} \chi_+ \\ \dfrac{\vec{p}\cdot\vec{\sigma}}{E+m}\chi_+ \end{pmatrix}, && \omega_{p,-\vec{s}} = \sqrt{E+m}\begin{pmatrix} \chi_- \\ \dfrac{\vec{p}\cdot\vec{\sigma}}{E+m}\chi_- \end{pmatrix},\\
&\psi^{(3)}_{\vec{s},\vec{p}}(x) = \omega_{-p,\vec{s}}\, e^{ip\cdot x}\,, && \psi^{(4)}_{\vec{s},\vec{p}}(x) = \omega_{-p,-\vec{s}}\, e^{ip\cdot x}\,,\\
&\omega_{-p,\vec{s}} = \sqrt{E+m}\begin{pmatrix} \dfrac{\vec{p}\cdot\vec{\sigma}}{E+m}\chi_+ \\ \chi_+ \end{pmatrix}, && \omega_{-p,-\vec{s}} = \sqrt{E+m}\begin{pmatrix} \dfrac{\vec{p}\cdot\vec{\sigma}}{E+m}\chi_- \\ \chi_- \end{pmatrix},
\end{aligned} \tag{3.240}$$

with

$$p^\mu \equiv (E,\vec{p}) \equiv m\eta^\mu\,, \qquad \text{in particular,} \quad p^0 \equiv E \geq 0\,, \tag{3.241}$$

where we have labeled the constant spinors ω by the spin direction and the eigenvalue of the operator $i\partial^\mu$ ($p \equiv p^\mu$). Thus, we have here a set of four solutions corresponding to a moving electron. The rule is that the constant spinor $\omega_{p,\pm\vec{s}}$ has to be attached to the space-time dependence $e^{-ip\cdot x}$ in order for the wave function to become a solution of the Dirac equation, where the parameters p^μ appearing in ω and the exponent have to be the same, and $\omega_{-p,\pm\vec{s}}$ has to be attached to $e^{ip\cdot x}$ to be a solution of the Dirac equation. Again, we emphasize that the 4-vector p^μ above is simply m times the boost 4-vector η^μ, and $p^0 \equiv E$ is always positive by definition. Thus, as long as the energy is defined by $i\partial_0$, the solutions $\psi^{(1,2)}_{\vec{s},\vec{p}}$ have a positive energy and $\psi^{(3,4)}_{\vec{s},\vec{p}}$ have a negative energy.

Conserved current of the plane wave solutions

Let's evaluate the conserved current $j^\mu = \bar{\psi}\gamma^\mu\psi$ for the rest frame solutions (3.217) with $N = \sqrt{2m}$. The time component, which should be the probability density, for the solution $\psi^{(1)}_{\vec{s}}$ is

$$\begin{aligned}
j^0 &= \bar{\psi}^{(1)}_{\vec{s}}\gamma^0\psi^{(1)}_{\vec{s}} = \psi^{(1)\dagger}_{\vec{s}}\psi^{(1)}_{\vec{s}}\\
&= 2m\,\chi_+^\dagger\chi_+ = 2m\,,
\end{aligned} \tag{3.242}$$

and the result is the same for all other solutions. The space component vanishes as expected for a particle at rest:

$$\begin{aligned}
j^k &= \overline{\psi^{(1)}_{\vec{s}}}\gamma^k\psi^{(1)}_{\vec{s}} = \psi^{(1)\dagger}_{\vec{s}}\underbrace{\gamma^0\gamma^k}_{\alpha_k}\psi^{(1)}_{\vec{s}}\\
&= 2m\begin{pmatrix}\chi_+^\dagger & 0\end{pmatrix}\begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0\end{pmatrix}\begin{pmatrix}\chi_+ \\ 0\end{pmatrix} = 0\,,
\end{aligned} \tag{3.243}$$

which also vanishes for other solutions. Thus, we have for all the four solutions with $\vec{p} = 0$,

$$j^\mu = (2m, \vec{0}) \quad \text{for} \quad \psi_{\vec{s}}^{(i)} \quad (i = 1, 2, 3, 4)\,. \tag{3.244}$$

Since we know that j^μ transforms as a vector, we should have for the boosted solutions,

$$j^\mu = 2m\eta^\mu = 2p^\mu \tag{3.245}$$

where η^μ is the boost 4-velocity as before, and thus p^0 is positive by definition. This can be explicitly verified from $j^\mu = \bar{\psi}\gamma^\mu\psi$ and (3.240).

Our normalization of the wave function is such that for the solutions at rest, there are $2m$ particles per unit volume. When the states are boosted, we see that the probability density becomes denser by the contraction factor γ to become $2m\gamma = 2E$. These are all things we already know from general analysis. Here, however, we have verified them using explicit plane-wave solutions.

Negative energy solutions - the hole theory

We have defined the energy as the eigenvalue of the time derivative operator $i\partial^0$, and according to this definition, two of our plane-wave solutions (3.240) have negative energy. One cannot simply ignore the negative energy solutions, since when interactions are included in the theory, it becomes unavoidable to have transitions to negative energy states. As it turns out, such transitions cannot be excluded without violating the conservation of probability, and the true solution lies in the Quantum Field Theory. For now we will follow Dirac's argument to wiggle out of the problem. The main aim of this exercise is to assign proper quantum numbers to the negative energy solutions which happen to represent the antiparticle of the electron - the positron.

Consider an atom in its ground state with many electrons. Due to the Pauli's exclusion principle, another electron cannot fill the low-energy states which are already occupied. If one of the electrons is kicked out of the atom ('ionization'), it will leave a positively charged atom. Now imagine the vacuum as something like a gigantic atom where all the negative energy states are filled up. Then a positive energy electron cannot drop into one of the negative energy states since it is already filled up.

If a negative energy electron is 'excited' to a positive energy state, it will leave a 'hole'. If the original negative energy electron had 4-momentum $-p^\mu$ defined as the eigenvalues of $i\partial^\mu$ (with $p^0 \geq 0$) and spin $-\vec{s}$, then the hole would look like it has a 4-momentum p^μ and spin $\vec{s}$ relative to the vacuum. Also, the charge of the hole would look like the opposite of that of the electron. The mass of the hole then should be the same as that of the electron since $p^2 = (-p)^2 = m^2$. Thus, the hole has the same mass, same absolute spin, and opposite charge to those of the electron. Such particle is called the *antiparticle* of the electron, or 'positron' denoted as e^+, while electron is denoted as e^-. The correspondence between the missing negative energy

electron and the resulting positron is summarized as

$$\text{When missing } e^- \text{ of } \begin{cases} \text{energy} & -E \\ \text{momentum} & -\vec{p} \\ \text{spin} & -\vec{s}, \end{cases} \quad \Longleftrightarrow \text{ all signs flipped} \quad \text{it is equivalent to } e^+ \text{ of } \begin{cases} \text{energy} & E \\ \text{momentum} & \vec{p} \\ \text{spin} & \vec{s}. \end{cases} \tag{3.246}$$

Since in reality we deal with positrons with positive energies rather than electrons with negative energy, it is convenient to label the spinors in (3.240) by physical quantities. Using u for electron spinors and v for positron spinors, we define

$$\begin{aligned} u_{\vec{p},\vec{s}} &\equiv \omega_{p,\vec{s}}\,, \qquad u_{\vec{p},-\vec{s}} \equiv \omega_{p,-\vec{s}}\,, \\ v_{\vec{p},-\vec{s}} &\equiv \omega_{-p,\vec{s}}\,, \qquad v_{\vec{p},\vec{s}} \equiv \omega_{-p,-\vec{s}}\,. \end{aligned} \tag{3.247}$$

Now for both u and v spinors, the labeling corresponds to physical quantities of electron and positron, respectively. Of course, in order to be a solution of the Dirac equation, the $u_{\vec{p},\pm\vec{s}}$ spinor has to be attached to $e^{-ip\cdot x}$ and the $v_{\vec{p},\pm\vec{s}}$ spinor has to be attached to $e^{ip\cdot x}$, where p^0 is given by $p^0 = \sqrt{\vec{p}^2 + m^2}$ which is always positive. Since p^0 is uniquely defined for a given $\vec{p}$, we have chosen to label the u, v spinors by $\vec{p}$ rather than by p^μ. The four plane-wave solutions are now

$$\boxed{\begin{array}{cc} u_{\vec{p},\pm\vec{s}}\, e^{-ip\cdot x}\,, & v_{\vec{p},\pm\vec{s}}\, e^{ip\cdot x}\,, \\ \begin{pmatrix} e^- \text{ with spin } \pm\vec{s} \\ \text{4-momentum } p^\mu \end{pmatrix} & \begin{pmatrix} e^+ \text{ with spin } \pm\vec{s} \\ \text{4-momentum } p^\mu \end{pmatrix} \end{array}} \tag{3.248}$$

where the u, v spinors are defined by (3.247) and (3.240); namely,

$$u_{\vec{p},\pm\vec{s}} = \sqrt{E+m}\begin{pmatrix} \chi_\pm \\ \dfrac{\vec{p}\cdot\vec{\sigma}}{E+m}\chi_\pm \end{pmatrix}, \quad v_{\vec{p},\pm\vec{s}} = \sqrt{E+m}\begin{pmatrix} \dfrac{\vec{p}\cdot\vec{\sigma}}{E+m}\chi_\mp \\ \chi_\mp \end{pmatrix}. \tag{3.249}$$

3.8 Energy and spin projection operators

In the previous section we have used the Dirac representation. Almost all discussions in this section and the next, at least all the boxed formulas, are independent of representation.

Dirac equation in momentum space

Since $u_{\vec{p},\pm\vec{s}}\, e^{-ip\cdot x}$ and $v_{\vec{p},\pm\vec{s}}\, e^{ip\cdot x}$ satisfy the Dirac equation, we can extract the matrix equations for the u and v spinors:

$$0 \;=\; (i\gamma^\mu\partial_\mu - m)u_{\vec{p},\pm\vec{s}}\, e^{-ip\cdot x} = (\gamma^\mu p_\mu - m)u_{\vec{p},\pm\vec{s}}\, e^{-ip\cdot x}$$

$$0 \;=\; (i\gamma^\mu\partial_\mu - m)v_{\vec{p},\pm\vec{s}}\,e^{ip\cdot x} = (-\gamma^\mu p_\mu - m)v_{\vec{p},\pm\vec{s}}\,e^{ip\cdot x}$$

$$\rightarrow \boxed{(\not{p} - m)u_{\vec{p},\pm\vec{s}} = 0\,, \quad (\not{p} + m)v_{\vec{p},\pm\vec{s}} = 0\,.} \tag{3.250}$$

These equations hold for any $\vec{p}$ and $\vec{s}$ as long as the u, v spinors are constructed as defined by (3.249) and p^μ that appears in $\not{p}$ is given by $p^\mu = (p^0, \vec{p})$ where $\vec{p}$ is the same $\vec{p}$ that appears in (3.249) and $p^0 \stackrel{\text{def}}{\equiv} \sqrt{\vec{p}^2 + m^2} \geq 0$. These equations for the u, v spinors are sometimes referred to as the Dirac equations in momentum space. Corresponding equations for the adjoint spinors $\bar{u}$ and $\bar{v}$ can be obtained by simply taking the spinor adjoint of (3.250). Noting that for any *real* 4-vector a^μ, we have

$$\overline{\not{a}} = \overline{\gamma^\mu a_\mu} = \underbrace{\overline{\gamma^\mu}}_{\gamma^\mu}\, a_\mu = \not{a}\,, \quad (a:\ \text{real}) \tag{3.251}$$

the adjoint of (3.250) is

$$\overline{(\not{p} - m)u_{\vec{p},\pm\vec{s}}} = \overline{u_{\vec{p},\pm\vec{s}}}(\underbrace{\overline{\not{p}}}_{\not{p}} - m) = 0\,, \quad \overline{(\not{p} + m)v_{\vec{p},\pm\vec{s}}} = \overline{v_{\vec{p},\pm\vec{s}}}(\underbrace{\overline{\not{p}}}_{\not{p}} + m) = 0\,,$$

$$\rightarrow \boxed{\overline{u_{\vec{p},\pm\vec{s}}}\,(\not{p} - m) = 0\,, \quad \overline{v_{\vec{p},\pm\vec{s}}}\,(\not{p} + m) = 0\,.} \tag{3.252}$$

Now note that the equations (3.250) can be written as

$$\frac{\not{p}}{m}\,u_{\vec{p},\pm\vec{s}} = u_{\vec{p},\pm\vec{s}}\,, \quad \frac{\not{p}}{m}\,v_{\vec{p},\pm\vec{s}} = -v_{\vec{p},\pm\vec{s}}\,, \tag{3.253}$$

which means that the u and v spinors are eigenvectors of the matrix $\not{p}/m$ with eigenvalues $+1$ and -1, respectively. Namely, the operator $\not{p}/m$ represents the sign of the energy, or equivalently, whether the spinor is an electron solution or a positron solution. Using this, one can construct an *energy projection operator* that projects out a spinor of a given 4-momentum and a given energy sign:

$$\boxed{\Lambda_\pm(p) \stackrel{\text{def}}{\equiv} \frac{1}{2}\left(1 \pm \frac{\not{p}}{m}\right).} \tag{3.254}$$

Then, it is easy to verify that (for a given $\vec{p}$)

$$\begin{aligned} &\Lambda_+(p)\,u_{\vec{p},\pm\vec{s}} = u_{\vec{p},\pm\vec{s}}\,, \qquad &&\Lambda_-(p)\,u_{\vec{p},\pm\vec{s}} = 0\,, \\ &\Lambda_-(p)\,v_{\vec{p},\pm\vec{s}} = v_{\vec{p},\pm\vec{s}}\,, \qquad &&\Lambda_+(p)\,v_{\vec{p},\pm\vec{s}} = 0\,, \end{aligned} \tag{3.255}$$

Thus, applying $\Lambda_+(p)$ to any 4-component quantity a projects out an electron solution with 4-momentum p^μ, and applying $\Lambda_-(p)$ projects out a positron solution with the

same 4-momentum. It is also straightforward to show that they satisfy the property of projection operators:

$$\begin{gathered}\Lambda_\pm^2(p) = \Lambda_\pm(p)\,, \quad \Lambda_+(p) + \Lambda_-(p) = 1\,, \\ \Lambda_+(p)\Lambda_-(p) = \Lambda_-(p)\Lambda_+(p) = 0\,.\end{gathered} \tag{3.256}$$

In proving the above, it is useful to note

$$\not{a}\not{b} + \not{b}\not{a} = a_\mu\gamma^\mu b_\nu\gamma^\nu + b_\nu\gamma^\nu a_\mu\gamma^\mu = a_\mu b_\nu \underbrace{(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu)}_{2\,g^{\mu\nu}} = 2a\cdot b\,. \tag{3.257}$$

where a and b are c-number 4-vectors. Applying this to $a = b = p$, we get

$$\not{p}^2 = p^2 = m^2\,. \tag{3.258}$$

Then, we have for example,

$$\Lambda_+(p)^2 = \frac{1}{4}\left(1 + \frac{\not{p}}{m}\right)^2 = \frac{1}{4}\left(1 + 2\frac{\not{p}}{m} + \underbrace{\frac{\not{p}^2}{m^2}}_{1}\right) = \frac{1}{2}\left(1 + \frac{\not{p}}{m}\right) = \Lambda_+(p)\,, \tag{3.259}$$

and other relations of (3.256) can be shown similarly.

Spin projection operators

We have just seen that the matrix $\not{p}/m$ acts as an energy sign operator for a given 4-momentum p^μ. We now look for an operator that represents the *physical spin* in a given direction $\vec{s}$, where by 'physical spin' we mean the spin we measure experimentally, in particular, for a positron we want a projection operator that projects out a given spin of the positron and not that of the missing electron in the sea of negative energy. In doing so, there are two problems we have to deal with: one is that for a positron, we want to make sure that it is the physical spin that is measured and not the eigenvalue of $\vec{s}\cdot\vec{\Sigma}$, and the other is that we want the spin measured in the rest frame of the particle even for a solution with a finite momentum.

For the first problem all we have to do is to somehow flip the sign of the spin for a positron. The second problem needs some care, since when an eigenspinor of $\vec{s}\cdot\vec{\Sigma}$ is boosted, it is in general no longer an eigenspinor of $\vec{s}\cdot\vec{\Sigma}$. Take for example the $\vec{p} = 0$ solution $u_{\vec{0},+\vec{s}}$ which is an eigenspinor of $\vec{s}\cdot\vec{\Sigma}$ with eigenvalue $+1$. If we apply the operator $\vec{s}\cdot\vec{\Sigma}$ to the boosted spinor $u_{\vec{p},+\vec{s}}$, we get (dropping the normalization factor $\sqrt{E+m}$ for simplicity)

$$(\vec{s}\cdot\vec{\Sigma})\,u_{\vec{p},+\vec{s}} = \begin{pmatrix} \vec{s}\cdot\vec{\sigma} & 0 \\ 0 & \vec{s}\cdot\vec{\sigma} \end{pmatrix} \begin{pmatrix} \chi_+ \\ \dfrac{\vec{p}\cdot\vec{\sigma}}{E+m}\chi_+ \end{pmatrix} = \begin{pmatrix} (\vec{s}\cdot\vec{\sigma})\chi_+ \\ \dfrac{(\vec{s}\cdot\vec{\sigma})(\vec{p}\cdot\vec{\sigma})}{E+m}\chi_+ \end{pmatrix}. \tag{3.260}$$

Since $(\vec{s}\cdot\vec{\sigma})\chi_+ = \chi_+$, if $\vec{s}\cdot\vec{\sigma}$ and $\vec{p}\cdot\vec{\sigma}$ commute, then $u_{\vec{p},+\vec{s}}$ becomes an eigenvector of $\vec{s}\cdot\vec{\Sigma}$. However, we have

$$[\vec{s}\cdot\vec{\sigma},\vec{p}\cdot\vec{\sigma}] = [s^i\sigma_i\,, p^j\sigma_j] = s^i p^j \underbrace{[\sigma_i\,,\sigma_j]}_{2i\epsilon_{ijk}\sigma_k} = 2i(\vec{s}\times\vec{p})\cdot\vec{\sigma} \tag{3.261}$$

which is in general not zero; thus, $u_{\vec{p},+\vec{s}}$ is in general *not* an eigenspinor of $\vec{s}\cdot\vec{\Sigma}$ (unless $\vec{s}\times\vec{p}=0$). Our strategy then is to go to the rest frame of the particle, construct an operator that reflects the physical spin there, express it in a Lorentz-invariant form, and then carefully 'boost' it.

Let's start with constructing the eigenstates of $\vec{s}\cdot\vec{\Sigma}$ in the rest frame in a way that is independent of representation. In the Dirac representation, the $\vec{p}=0$ solutions $u_{\vec{0},\pm\vec{s}}$ and $v_{\vec{0},\pm\vec{s}}$ look like

$$u_{\vec{0},\pm\vec{s}} \propto \begin{pmatrix}\chi_\pm\\ 0\end{pmatrix}\,, \quad v_{\vec{0},\pm\vec{s}} \propto \begin{pmatrix}0\\ \chi_\mp\end{pmatrix}\,, \tag{3.262}$$

and they are eigenstates of $\vec{s}\cdot\vec{\Sigma}$:

$$(\vec{s}\cdot\vec{\Sigma})\,u_{\vec{0},\pm\vec{s}} = \pm u_{\vec{0},\pm\vec{s}}\,, \quad (\vec{s}\cdot\vec{\Sigma})\,v_{\vec{0},\pm\vec{s}} = \mp v_{\vec{0},\pm\vec{s}}\,. \tag{3.263}$$

These relations are independent of representation. In fact, if we change the representation by a unitary matrix V, then by (3.150) and (3.154), the spinor and Σ_i in the new representation are given by

$$\begin{aligned} u'_{\vec{0},\pm\vec{s}} &= V u_{\vec{0},\pm\vec{s}}\,,\\ \Sigma'_i = i\gamma'^j\gamma'^k &= i(V\gamma^j \underbrace{V^\dagger)(V}_{1}\gamma^k V^\dagger) = V i\,\gamma^j\gamma^k V^\dagger = V\Sigma_i V^\dagger\,. \end{aligned} \tag{3.264}$$

If $(\vec{s}\cdot\vec{\Sigma})\,u_{\vec{0},\pm\vec{s}} = \pm u_{\vec{0},\pm\vec{s}}$ holds in one representation, then multiplying V from the left,

$$\begin{aligned} \overbrace{V}\ \ \overbrace{V^\dagger V}\ (\vec{s}\cdot\vec{\Sigma})\,u_{\vec{0},\pm\vec{s}} &= \overbrace{\pm u_{\vec{0},\pm\vec{s}}}^{V}\\ \to (\vec{s}\cdot\vec{\Sigma}')\,u'_{\vec{0},\pm\vec{s}} &= \pm u'_{\vec{0},\pm\vec{s}}\,, \end{aligned} \tag{3.265}$$

demonstrating that the same form holds in the new representation.

Note that for the v spinors in (3.263), the sign on the subscript $\vec{s}$ is opposite to the sign of the eigenvalue. We thus want an operator whose eigenvalues for v spinors are sign-flipped with respect to $\vec{s}\cdot\vec{\Sigma}$. This can be accomplished by (3.253) which can be written in the rest frame as

$$\gamma^0 u_{\vec{0},\pm\vec{s}} = u_{\vec{0},\pm\vec{s}}\,, \quad \gamma^0 v_{\vec{0},\pm\vec{s}} = -v_{\vec{0},\pm\vec{s}}\,, \tag{3.266}$$

where we have used $\not{p} = m\gamma^0$ [since $p^\mu = (m, \vec{0})$ in the rest frame]. It will thus flip the sign for the v spinors only. Together with (3.263), we have

$$(\vec{s}\cdot\vec{\Sigma}\underbrace{\gamma^0)\,u_{\vec{0},\pm\vec{s}}}_{u_{\vec{0},\pm\vec{s}}} = \pm u_{\vec{0},\pm\vec{s}}\,, \quad (\vec{s}\cdot\vec{\Sigma}\underbrace{\gamma^0)\,v_{\vec{0},\pm\vec{s}}}_{-v_{\vec{0},\pm\vec{s}}} = \pm v_{\vec{0},\pm\vec{s}}\,, \tag{3.267}$$

which is exactly what we wanted. This, however, works only for the rest frame solutions. In order to extend it to the boosted states, we first show that the operator $\vec{s}\cdot\vec{\Sigma}\gamma^0$ can be written in the rest frame as

$$\vec{s}\cdot\vec{\Sigma}\gamma^0 = \gamma_5\not{p}\not{s} \quad \text{(rest frame)}, \tag{3.268}$$

where the *real* quantity s^μ is defined in the rest frame by

$$s^\mu \stackrel{\text{def}}{\equiv} (0, \vec{s}) \quad \text{(rest frame)}, \tag{3.269}$$

and assumed to transform as a Lorentz 4-vector. Since $\not{s} = s^i\gamma_i$ in the rest frame, and

$$\Sigma_i \equiv i\gamma^j\gamma^k = -i\gamma^1\gamma^2\gamma^3\gamma^i \quad (i,j,k : \text{cyclic}) \tag{3.270}$$

which can be verified for $k = 1, 2, 3$ explicitly, we have

$$\begin{aligned}\gamma_5\not{p}\not{s} &= (i\boxed{\gamma^0}\overrightarrow{\gamma^1\gamma^2\gamma^3)(s^i\gamma_i)}) = s^i\ \underbrace{i\gamma^1\gamma^2\gamma^3\gamma_i}_{-i\gamma^1\gamma^2\gamma^3\gamma^i = \Sigma_i}\gamma^0 \\ &= s^i\Sigma_i\gamma^0 = \vec{s}\cdot\vec{\Sigma}\gamma^0\,.\end{aligned} \tag{3.271}$$

Thus, (3.267) can be written as

$$\gamma_5\not{p}\not{s}\, w_{\vec{0},\pm\vec{s}} = \pm w_{\vec{0},\pm\vec{s}} \quad (w : u \text{ or } v)\,. \tag{3.272}$$

The above equation is for spinors representing a particle at rest. A spinor with momentum $\vec{p}$ was defined by

$$w_{\vec{p},\pm\vec{s}} \stackrel{\text{def}}{\equiv} S(\Lambda)w_{\vec{0},\pm\vec{s}}\,, \tag{3.273}$$

where as before w represents u or v, and Λ is the boost that makes the rest mass m acquire a momentum $\vec{p}$. First, we set $\Lambda \to \Lambda^{-1}$ in $S^{-1}(\Lambda)\gamma^\mu S(\Lambda) = \Lambda^\mu{}_\alpha\gamma^\alpha$ (3.80) and use $S(\Lambda^{-1}) = S^{-1}(\Lambda)$, and thus $S^{-1}(\Lambda^{-1}) = S(\Lambda)$, to get

$$\underbrace{S^{-1}(\Lambda^{-1})\gamma^\mu S(\Lambda^{-1})}_{S(\Lambda)\gamma^\mu S^{-1}(\Lambda)} = \underbrace{(\Lambda^{-1})^\mu{}_\alpha}_{\Lambda_\alpha{}^\mu}\gamma^\alpha\,. \tag{3.274}$$

Multiplying (3.272) by $S(\Lambda)$ on the left, we obtain

$$S(\Lambda) \times \underbrace{\gamma_5 \not{s}\; w_{\vec{0},\pm\vec{s}}}_{S^{-1}(\Lambda)S(\Lambda)} = S(\Lambda) \times \pm w_{\vec{0},\pm\vec{s}} \tag{3.275}$$

Together with $[\gamma_5, S] = 0$ [see (3.137)], this becomes

$$\gamma_5 \;\; s_\mu \underbrace{S(\Lambda)\gamma^\mu S^{-1}(\Lambda)}_{\underbrace{\Lambda_\alpha{}^\mu \gamma^\alpha}_{s'_\alpha \gamma^\alpha = \not{s}'}} \underbrace{S(\Lambda) w_{\vec{0},\pm\vec{s}}}_{w_{\vec{p},\pm\vec{s}}} = \pm \underbrace{S(\Lambda) w_{\vec{0},\pm\vec{s}}}_{w_{\vec{p},\pm\vec{s}}} \,. \tag{3.276}$$

$$(3.274) \;\rightarrow$$

where we have defined the boosted spin 4-vector

$$s'^\mu \stackrel{\text{def}}{\equiv} \Lambda^\mu{}_\nu s^\nu \,. \tag{3.277}$$

Thus, we have

$$\boxed{\gamma_5 \not{s}\; w_{\vec{p},\pm\vec{s}} = \pm w_{\vec{p},\pm\vec{s}} \quad (w : u \text{ or } v)}, \tag{3.278}$$

where we have dropped the prime on s with the understanding that s^μ is always the transformed 4-vector of $(0, \vec{s})$ by the same boost under which the rest mass m acquires the momentum $\vec{p}$. Thus, we see that the operator $\gamma_5 \not{s}$ has eigenvalues ± 1, and properly represents the component of the *physical spin* along the direction $\vec{s}$ in the rest frame (times two, to be precise), where it is understood that the rest frame is reached by a pure boost. And it works for a positron as well as for an electron. Also, note that our derivation was independent of the representation of the γ matrices.

Just like we constructed the energy projection operator from the energy-sign operator $\not{p}/m$, we can construct spin projection operators as

$$\boxed{\Sigma_\pm(s) \stackrel{\text{def}}{\equiv} \frac{1 \pm \gamma_5 \not{s}}{2}}, \tag{3.279}$$

which project out the eigenstates of the spin operator $\gamma_5 \not{s}$ for the given direction $\vec{s}$ and the given boost represented by $\vec{p}$. Again, it can be readily verified that the operators $\Sigma_\pm(s)$ satisfy the properties of projection operators:

$$\begin{gathered} \Sigma_\pm^2(s) = \Sigma_\pm(s) \,, \quad \Sigma_+(s) + \Sigma_-(s) = 1 \,, \\ \Sigma_+(s)\Sigma_-(s) = \Sigma_-(s)\Sigma_+(s) = 0 \,. \end{gathered} \tag{3.280}$$

Orthonormalities of the u, v spinors

In the rest frame, we have $s^\mu = (0, \vec{s})$ and $p^\mu = (m, \vec{0})$. Thus,

$$\boxed{s^2 = -1 \,, \quad s \cdot p = 0}, \tag{3.281}$$

		$\not{p}/m$	
		$+1$	-1
$\gamma_5\not{s}$	$+1$	$u_{\vec{p},\vec{s}}$	$v_{\vec{p},\vec{s}}$
	-1	$u_{\vec{p},-\vec{s}}$	$v_{\vec{p},-\vec{s}}$

Table 3.3: The u and v spinors for a given momentum $\vec{p}$ and a spin quantization axis $\vec{s}$ shown with the eigenvalues of the operators $\not{p}/m$ and $\gamma_5\not{s}$.

which are in Lorentz-invariant form and thus true in any frame. Then from (3.257), we see that $\not{s}$ and $\not{p}$ anticommute:

$$\not{s}\not{p} + \not{p}\not{s} = 2s \cdot p = 0\,. \tag{3.282}$$

For any given set of $\vec{s}$ and $\vec{p}$, the energy-sign operator $\not{p}/m$ and the spin operator $\gamma_5\not{s}$ then commute:

$$\begin{aligned} \underbrace{\not{p}}_{-\gamma_5\not{p}}(\gamma_5\not{s}) = -\gamma_5 \underbrace{\not{p}\not{s}}_{-\not{s}\not{p}} &= (\gamma_5\not{s})\not{p} \\ \rightarrow \quad \left[\frac{\not{p}}{m}, \gamma_5\not{s}\right] &= 0\,. \end{aligned} \tag{3.283}$$

We have seen that the two commuting operators $\gamma_5\not{s}$ and $\not{p}/m$ both have eigenvalues ± 1. Thus, there should be a set of four spinors that are simultaneous eigenspinors of the two operators. In fact, they are nothing but $u_{\vec{p},\pm\vec{s}}$ and $v_{\vec{p},\pm\vec{s}}$. Table 3.3 summarizes their eigenvalues. Now we will show that they are also orthonormal.

First, the operators $\not{p}/m$ and $\gamma_5\not{s}$ are self-adjoint: using $\overline{\not{a}} = \not{a}$ (3.251),

$$\overline{\frac{\not{p}}{m}} = \frac{\not{p}}{m}\,, \qquad \overline{\gamma_5\not{s}} = \overline{\not{s}}\,\underbrace{\overline{\gamma_5}}_{-\gamma_5} = -\not{s}\gamma_5 = \gamma_5\not{s}\,, \tag{3.284}$$

where we have used

$$\overline{\gamma_5} \equiv \gamma^0 \underbrace{\gamma_5^\dagger}_{\gamma_5 \text{ by (3.134)}} \gamma^0 = \gamma^0\gamma_5\gamma^0 = -\gamma_5\,. \tag{3.285}$$

This indicates that the eigenspinors are orthogonal in terms of the inner product defined by the spinor adjoint. In fact, using (3.253),

$$\frac{\not{p}}{m}\,u_{\vec{p},\pm\vec{s}} = u_{\vec{p},\pm\vec{s}} \quad \overset{\overline{v_{\vec{p},\pm\vec{s}}}\,\times}{\rightarrow} \quad \overline{v_{\vec{p},\pm\vec{s}}}\,\frac{\not{p}}{m}\,u_{\vec{p},\pm\vec{s}} = \overline{v_{\vec{p},\pm\vec{s}}}\,u_{\vec{p},\pm\vec{s}}\,. \tag{3.286}$$

On the other hand, taking adjoint of $(\not{p}/m)v_{\vec{p},\pm\vec{s}} = -v_{\vec{p},\pm\vec{s}}$, and using $\overline{\not{p}/m} = \not{p}/m$,

$$\overline{v_{\vec{p},\pm\vec{s}}}\,\frac{\not{p}}{m} = -\overline{v_{\vec{p},\pm\vec{s}}} \quad \overset{\times\, u_{\vec{p},\pm\vec{s}}}{\rightarrow} \quad \overline{v_{\vec{p},\pm\vec{s}}}\,\frac{\not{p}}{m}\,u_{\vec{p},\pm\vec{s}} = -\overline{v_{\vec{p},\pm\vec{s}}}\,u_{\vec{p},\pm\vec{s}}\,. \tag{3.287}$$

Subtracting (3.287) from (3.286), we obtain

$$\overline{v_{\vec{p},\pm\vec{s}}}\,u_{\vec{p},\pm\vec{s}} = 0\,, \tag{3.288}$$

which holds for all spin combinations. Note that it was critical that $\not{p}/m$ was self-adjoint in order to make the left-hand sides of (3.287) and (3.286) identical.

Similarly, two spinors are orthogonal if they have different eigenvalues of $\gamma_5\not{s}$ as can be shown by simply replacing $\not{p}/m$ by $\gamma_5\not{s}$ in the above derivation (3.286) through (3.288). For example, we have

$$\gamma_5\not{s}\;u_{\vec{p},\vec{s}} = u_{\vec{p},\vec{s}}\,, \quad \gamma_5\not{s}\;u_{\vec{p},-\vec{s}} = -u_{\vec{p},-\vec{s}}\,. \tag{3.289}$$

Multiplying the first with $\overline{u_{\vec{p},-\vec{s}}}$ from the left, and multiplying the spinor adjoint of the second with $u_{\vec{p},\vec{s}}$ from the right, we obtain

$$\overline{u_{\vec{p},-\vec{s}}}\,\gamma_5\not{s}\;u_{\vec{p},\vec{s}} = \overline{u_{\vec{p},-\vec{s}}}\,u_{\vec{p},\vec{s}}\,, \quad \overline{u_{\vec{p},-\vec{s}}}\,\gamma_5\not{s}\;u_{\vec{p},\vec{s}} = -\overline{u_{\vec{p},-\vec{s}}}\,u_{\vec{p},\vec{s}}\,, \tag{3.290}$$

where we have used $\overline{\gamma_5\not{s}} = \gamma_5\not{s}$ (3.284). Taking the difference, we get

$$\overline{u_{\vec{p},-\vec{s}}}\,u_{\vec{p},\vec{s}} = 0\,. \tag{3.291}$$

Similarly for v spinors, we obtain

$$\overline{v_{\vec{p},-\vec{s}}}\,v_{\vec{p},\vec{s}} = 0\,. \tag{3.292}$$

Thus, all four spinors in Table 3.3 are orthogonal to each other.

There is another set of four spinors that are orthogonal to each other where the orthogonality is defined by the ordinary inner product $a^\dagger b$. First, replace $\vec{p}$ by $-\vec{p}$ in $(\not{p}/m)v_{\vec{p},\pm\vec{s}} = -v_{\vec{p},\pm\vec{s}}$ to get

$$\frac{\not{p}'}{m}v_{-\vec{p},\pm\vec{s}} = -v_{-\vec{p},\pm\vec{s}} \qquad \text{with} \quad p'^\mu \equiv (p^0, -\vec{p}) = p_\mu\,, \tag{3.293}$$

where the time component of p' did not change since $p'^0 \overset{\text{def}}{\equiv} \sqrt{(-\vec{p})^2 + m^2} = p^0$. We note

$$(\not{p}')^\dagger = (\gamma_\mu p'^\mu)^\dagger = \underbrace{\gamma_\mu^\dagger}_{\gamma^\mu}\underbrace{p'^\mu}_{p_\mu} = \not{p}\,. \tag{3.294}$$

Then, taking the hermitian conjugate of (3.293), we have

$$v^\dagger_{-\vec{p},\pm\vec{s}} \frac{\overbrace{(\not{p}')^\dagger}^{\not{p}}}{m} = -v^\dagger_{-\vec{p},\pm\vec{s}}\,, \tag{3.295}$$

to which we multiply $u_{\vec{p},\pm\vec{s}}$ from the right to get

$$v^\dagger_{-\vec{p},\pm\vec{s}} \frac{\not{p}}{m}\, u_{\vec{p},\pm\vec{s}} = -v^\dagger_{-\vec{p},\pm\vec{s}}\, u_{\vec{p},\pm\vec{s}}\,. \tag{3.296}$$

On the other hand, multiplying $v^\dagger_{-\vec{p},\pm\vec{s}}$ to $(\not{p}/m)u_{\vec{p},\pm\vec{s}} = u_{\vec{p},\pm\vec{s}}$,

$$v^\dagger_{-\vec{p},\pm\vec{s}} \frac{\not{p}}{m}\, u_{\vec{p},\pm\vec{s}} = v^\dagger_{-\vec{p},\pm\vec{s}}\, u_{\vec{p},\pm\vec{s}}\,. \tag{3.297}$$

Taking the difference of (3.296) and (3.297), we see that

$$v^\dagger_{-\vec{p},\pm\vec{s}}\, u_{\vec{p},\pm\vec{s}} = 0\,, \tag{3.298}$$

which holds for any spin combinations.

Now, let's look at the orthogonality of spinors with different spin eigenvalues. First, we can see that $u^\dagger_{\vec{p},+\vec{s}} u_{\vec{p},-\vec{s}} = 0$ explicitly in the Dirac representation as follows. Recalling $u_{\vec{p},\pm\vec{s}} = e^{\vec{\xi}\cdot\vec{\alpha}/2}\, u_{\vec{0},\pm\vec{s}}$ and, since α_i is hermitian,

$$(e^{\frac{1}{2}\vec{\xi}\cdot\vec{\alpha}})^\dagger = e^{\frac{1}{2}\vec{\xi}\cdot\vec{\alpha}^\dagger} = e^{\frac{1}{2}\vec{\xi}\cdot\vec{\alpha}}\,, \tag{3.299}$$

we have

$$\begin{aligned} u^\dagger_{\vec{p},+\vec{s}}\, u_{\vec{p},-\vec{s}} &= u^\dagger_{\vec{0},+\vec{s}} \underbrace{(e^{\frac{1}{2}\vec{\xi}\cdot\vec{\alpha}})^\dagger}_{e^{\frac{1}{2}\vec{\xi}\cdot\vec{\alpha}}} e^{\frac{1}{2}\vec{\xi}\cdot\vec{\alpha}} u_{\vec{0},-\vec{s}} = u^\dagger_{\vec{0},+\vec{s}} e^{\vec{\xi}\cdot\vec{\alpha}}\, u_{\vec{0},-\vec{s}} \\ &= 2m \begin{pmatrix} \chi^\dagger_+ & 0 \end{pmatrix} \begin{pmatrix} \cosh\xi & (\hat{\xi}\cdot\vec{\sigma})\sinh\xi \\ (\hat{\xi}\cdot\vec{\sigma})\sinh\xi & \cosh\xi \end{pmatrix} \begin{pmatrix} \chi_- \\ 0 \end{pmatrix} \\ &= 2m\cosh\xi\ (\chi^\dagger_+\chi_-) = 0\,, \end{aligned} \tag{3.300}$$

where we have used the expression of $e^{\vec{\xi}\cdot\vec{\alpha}/2}$ given in (3.226) with $\vec{\xi}/2$ replaced by $\vec{\xi}$. This is representation-independent, since if we move to a different representation by a unitary matrix V, then the orthogonality remains valid:

$$u'^\dagger_{\vec{p},+\vec{s}}\, u'_{\vec{p},-\vec{s}} = (V u_{\vec{p},+\vec{s}})^\dagger (V u_{\vec{p},-\vec{s}}) = u^\dagger_{\vec{p},+\vec{s}} \underbrace{V^\dagger V}_{1} u_{\vec{p},-\vec{s}} = 0\,. \tag{3.301}$$

Similarly, we can show that $v^\dagger_{\vec{p},+\vec{s}}\, v_{\vec{p},-\vec{s}} = 0$ which is also representation-independent. Thus, for given $\vec{p}$ and $\vec{s}$, we have two sets of orthogonal spinors; one defined by the inner product $\bar{a}b\colon\ (u_{\vec{p},\pm\vec{s}}\,,\ v_{\vec{p},\pm\vec{s}})$, and the other by $a^\dagger b\colon\ (u_{\vec{p},\pm\vec{s}}\,,\ v_{-\vec{p},\pm\vec{s}})$.

How about the normalization? We have seen in (3.245) that the probability density is $j^0 = \psi^\dagger\psi = 2E$ which is always positive. Using the plane wave form (3.248) for ψ, the normalization of u, v spinors are then

$$u^\dagger_{\vec{p},\pm\vec{s}}\, u_{\vec{p},\pm\vec{s}} = v^\dagger_{\vec{p},\pm\vec{s}}\, v_{\vec{p},\pm\vec{s}} = 2E \tag{3.302}$$

Once this is given, the values of $\overline{u_{\vec{p},\pm\vec{s}}}\, u_{\vec{p},\pm\vec{s}}$ and $\overline{v_{\vec{p},\pm\vec{s}}}\, v_{\vec{p},\pm\vec{s}}$ are already fixed as follows: For the solution $\psi = u_{\vec{p}} e^{-ip\cdot x}$, the current is $j^\mu = \bar{u}_{\vec{p}}\gamma^\mu u_{\vec{p}}$ (we have dropped the spin indexes for simplicity). Using $(\not{p}/m)u_{\vec{p}} = u_{\vec{p}}$ (3.253) and its adjoint $\bar{u}_{\vec{p}}(\not{p}/m) = \bar{u}_{\vec{p}}$ this can be written as

$$\begin{aligned}
j^\mu = \bar{u}_{\vec{p}}\gamma^\mu u_{\vec{p}} \ &= \ \bar{u}_{\vec{p}}\gamma^\mu\Big(\frac{\not{p}}{m}u_{\vec{p}}\Big) = \Big(\bar{u}_{\vec{p}}\frac{\not{p}}{m}\Big)\gamma^\mu u_{\vec{p}} \\
&= \ \frac{1}{2}\bar{u}_{\vec{p}}\Big(\gamma^\mu\frac{\not{p}}{m} + \frac{\not{p}}{m}\gamma^\mu\Big)u_{\vec{p}} \\
&= \ \frac{1}{2m}\bar{u}_{\vec{p}}\,[\gamma^\mu(p_\nu\gamma^\nu) + (p_\nu\gamma^\nu)\gamma^\mu]u_{\vec{p}} \\
&= \ \frac{p_\nu}{2m}\bar{u}_{\vec{p}}\,(\underbrace{\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu}_{2g^{\mu\nu}})u_{\vec{p}} \\
&= \ \eta^\mu\bar{u}_{\vec{p}}u_{\vec{p}}\,.
\end{aligned} \tag{3.303}$$

It is instructive to compare this with the current $j^\mu = \rho^0\eta^\mu$ we encountered in (2.11). We see that the scalar quantity $\bar{u}_{\vec{p}}u_{\vec{p}}$ is acting as ρ^0: the density in the rest frame of the flow. For the v spinors, the only difference is the minus sign in $(\not{p}/m)v_{\vec{p}} = -v_{\vec{p}}$ which leads to

$$\bar{v}_{\vec{p}}\gamma^\mu v_{\vec{p}} = -\eta^\mu\bar{v}_{\vec{p}}v_{\vec{p}}\,. \tag{3.304}$$

The time components of (3.303) and (3.304) give

$$u^\dagger_{\vec{p}}u_{\vec{p}} = \gamma\,\bar{u}_{\vec{p}}u_{\vec{p}}\,, \quad v^\dagger_{\vec{p}}v_{\vec{p}} = -\gamma\,\bar{v}_{\vec{p}}v_{\vec{p}}\,, \tag{3.305}$$

where $\gamma \equiv \eta^0 = E/m$. Together with (3.302) this leads to

$$\bar{u}_{\vec{p}}u_{\vec{p}} = 2m\,, \quad \bar{v}_{\vec{p}}v_{\vec{p}} = -2m\,. \tag{3.306}$$

Thus, for given $\vec{p}$ and $\vec{s}$, we have two orthonormality relations for u, v spinors :

$$\boxed{\begin{array}{c} \overline{u_{\vec{p},\vec{s}_1}}\, u_{\vec{p},\vec{s}_2} = 2m\,\delta_{\vec{s}_1,\vec{s}_2}\,, \quad \overline{v_{\vec{p},\vec{s}_1}}\, v_{\vec{p},\vec{s}_2} = -2m\,\delta_{\vec{s}_1,\vec{s}_2}\,, \\ \overline{v_{\vec{p},\vec{s}_1}}\, u_{\vec{p},\vec{s}_2} = 0 \qquad (\vec{s}_1,\vec{s}_2 : \pm\vec{s}) \end{array}} \tag{3.307}$$

as defined by the inner product $\bar{a}b$, and

$$\boxed{\begin{array}{c} u^\dagger_{\vec{p},\vec{s}_1}\, u_{\vec{p},\vec{s}_2} = v^\dagger_{-\vec{p},\vec{s}_1}\, v_{-\vec{p},\vec{s}_2} = 2E\,\delta_{\vec{s}_1,\vec{s}_2}\,, \\ v^\dagger_{-\vec{p},\vec{s}_1}\, u_{\vec{p},\vec{s}_2} = 0 \qquad (\vec{s}_1,\vec{s}_2 : \pm\vec{s}) \end{array}} \tag{3.308}$$

as defined by the inner product $a^\dagger b$. Any spinor, or any set of four complex numbers, can be written uniquely as a linear combination of either of the above orthonormal sets. Note that one can construct such orthonormal sets for any given $\vec{p}$ and $\vec{s}$; thus, there are an infinite number of orthonormal sets since there are infinite ways to take $\vec{p}$ and $\vec{s}$.

Exercise 3.9 *Orthonormality of u, v spinors (by the product rule $a^\dagger b$).*
Use the Dirac representation of the u, v spinors to explicitly verify the orthonormality relations (3.308).

Note that the normalization (3.308) tells us that the norm $\psi^\dagger\psi$ of a Dirac spinor is in general not invariant under a boost since E changes its value. In fact, as we have seen in (3.299), a boost $S = e^{\vec{\xi}\cdot\vec{\alpha}/2}$ is hermitian but *not* unitary. On the other hand, a rotation in the spinor space $U = e^{-i\vec{\theta}\cdot\vec{\Sigma}/2}$ *is* unitary:

$$U^\dagger = \left(e^{-i\frac{\vec{\theta}}{2}\cdot\vec{\Sigma}}\right)^\dagger = e^{i\frac{\vec{\theta}}{2}\cdot\vec{\Sigma}^\dagger} = e^{i\frac{\vec{\theta}}{2}\cdot\vec{\Sigma}} = U^{-1} \tag{3.309}$$

and thus the norm $\psi^\dagger\psi$ is invariant under a rotation:

$$\psi^\dagger\psi \quad \stackrel{U}{\to} \quad (U\psi)^\dagger(U\psi) = \psi^\dagger \underbrace{U^\dagger U}_{1}\psi = \psi^\dagger\psi\,. \tag{3.310}$$

There exist useful relations between the u, v spinors and the energy and spin projection operators that will be used later in actual calculations of transition rates (the proof is left as an exercise):

$$\begin{array}{rcl} u_{\vec{p},\pm\vec{s}}\,\overline{u_{\vec{p},\pm\vec{s}}} &=& 2m\,\Lambda_+(p)\Sigma_\pm(s) \;=\; (\not{p}+m)\dfrac{1\pm\gamma_5\not{s}}{2} \\ v_{\vec{p},\pm\vec{s}}\,\overline{v_{\vec{p},\pm\vec{s}}} &=& -2m\,\Lambda_-(p)\Sigma_\pm(s) \;=\; (\not{p}-m)\dfrac{1\pm\gamma_5\not{s}}{2} \end{array} \tag{3.311}$$

where for any spinors a and b, which are column vectors, a 4×4 matrix ab^T is defined by

$$(ab^T)_{ij} \stackrel{\text{def}}{\equiv} a_i\, b_j\,. \tag{3.312}$$

By this definition, the multiplications of matrices and vectors become 'associative'; for example, if a, b, and c are column vectors, we have

$$[(ab^T)c]_i = (ab^T)_{ij}c_j = a_i b_j c_j = a_i(b^T c) \quad \to \quad (ab^T)c = a(b^T c)\,, \tag{3.313}$$

which has the form

$$\overbrace{\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}}^{(ab^T)} \overbrace{\begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}}^{c} = \overbrace{\begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}}^{a} \overbrace{\begin{pmatrix} \cdot \end{pmatrix}}^{(b^T c)} . \tag{3.314}$$

Exercise 3.10 *u, v spinors and energy-spin projection operators.*
Prove the identities (3.311). (hint: Since, for a given set of $\vec{p}$ and $\vec{s}$, the spinor space is spanned by $u_{\vec{p},\pm\vec{s}}$ and $v_{\vec{p},\pm\vec{s}}$, all that is needed is to show that the LHS and the RHS of the identities behave the same for these 4 spinors.)

Similarly,

$$c^T(ab^T) = (c^T a)b^T . \tag{3.315}$$

In terms of the spinor adjoint, we have

$$(a\bar{b})c = a(\bar{b}c) \qquad \bar{c}(a\bar{b}) = (\bar{c}a)\bar{b} . \tag{3.316}$$

Summing over the sign of spins in (3.311), we obtain

$$\boxed{\begin{array}{rcl} \displaystyle\sum_{\pm\vec{s}} u_{\vec{p},\vec{s}}\,\overline{u_{\vec{p},\vec{s}}} & = & (\not{p} + m) \\ \displaystyle\sum_{\pm\vec{s}} v_{\vec{p},\vec{s}}\,\overline{v_{\vec{p},\vec{s}}} & = & (\not{p} - m) \end{array}} , \tag{3.317}$$

which will be used extensively when we calculate transition rates averaged over spins. Taking the difference of these two equations, we obtain a relation which expresses the completeness of the first orthonormal set:

$$\frac{1}{2m}\sum_{\pm\vec{s}} \left(u_{\vec{p},\vec{s}}\,\overline{u_{\vec{p},\vec{s}}} - v_{\vec{p},\vec{s}}\,\overline{v_{\vec{p},\vec{s}}} \right) = 1 , \tag{3.318}$$

where the right hand side is actually the 4×4 identity matrix. This can be used to write any spinor a as a linear combination of the orthonormal basis $(u_{\vec{p},\pm\vec{s}}\,,\ v_{\vec{p},\pm\vec{s}})$:

$$a = \frac{1}{2m}\sum_{\pm\vec{s}} \left(u_{\vec{p},\vec{s}}\,\overline{u_{\vec{p},\vec{s}}} - v_{\vec{p},\vec{s}}\,\overline{v_{\vec{p},\vec{s}}} \right) a = \frac{1}{2m}\sum_{\pm\vec{s}} \left[u_{\vec{p},\vec{s}}\,(\overline{u_{\vec{p},\vec{s}}}\,a) - v_{\vec{p},\vec{s}}\,(\overline{v_{\vec{p},\vec{s}}}\,a) \right] , \tag{3.319}$$

where we have used the associativity (3.313). Namely, the coefficient of $u_{\vec{p},\vec{s}}$ is $\overline{u_{\vec{p},\vec{s}}}\,a/2m$, and that of $v_{\vec{p},\vec{s}}$ is $-\overline{v_{\vec{p},\vec{s}}}\,a/2m$. This could have been directly obtained

from the orthonormality relations (3.307) as follows: if an arbitrary spinor a is expanded as

$$a = \sum_{\pm\vec{s}} (c_{\pm\vec{s}}\, u_{\vec{p},\vec{s}} + d_{\pm\vec{s}}\, v_{\vec{p},\vec{s}})\,, \tag{3.320}$$

where $c_{\pm\vec{s}}$ and $d_{\pm\vec{s}}$ are complex coefficients, then, multiplying $\overline{u_{\vec{p},\vec{s}}}$ or $\overline{v_{\vec{p},\vec{s}}}$ from the left and using the orthonormality realtions gives

$$c_{\pm\vec{s}} = \frac{\overline{u_{\vec{p},\vec{s}}}\, a}{2m}\,, \quad d_{\pm\vec{s}} = -\frac{\overline{v_{\vec{p},\vec{s}}}\, a}{2m}\,, \tag{3.321}$$

which recovers the expansion (3.319).

Similarly, we can use the second orthonormal basis $(u_{\vec{p},\pm\vec{s}},\ v_{-\vec{p},\pm\vec{s}})$ to expand an arbitrary spinor a as

$$a = \sum_{\pm\vec{s}} (c'_{\pm\vec{s}}\, u_{\vec{p},\vec{s}} + d'_{\pm\vec{s}}\, v_{-\vec{p},\vec{s}})\,. \tag{3.322}$$

Left-multiplying $u^\dagger_{\vec{p},\vec{s}}$ or $v^\dagger_{\vec{p},\vec{s}}$ gives

$$c'_{\pm\vec{s}} = \frac{u^\dagger_{\vec{p},\vec{s}}\, a}{2E}\,, \quad d'_{\pm\vec{s}} = \frac{v^\dagger_{-\vec{p},\vec{s}}\, a}{2E}\,.$$

This suggests the completeness relation

$$\frac{1}{2E} \sum_{\pm\vec{s}} \left(u_{\vec{p},\vec{s}}\, u^\dagger_{\vec{p},\vec{s}} + v_{-\vec{p},\vec{s}}\, v^\dagger_{-\vec{p},\vec{s}} \right) = 1\,. \tag{3.323}$$

In fact, by applying this identity to a, we obtain

$$\frac{1}{2E} \sum_{\pm\vec{s}} \left(u_{\vec{p},\vec{s}}\, (u^\dagger_{\vec{p},\vec{s}} a) + v_{-\vec{p},\vec{s}}\, (v^\dagger_{-\vec{p},\vec{s}} a) \right) = a\,, \tag{3.324}$$

which reproduces the correct coefficients.

3.9 Low energy limit - electron magnetic moment

In order to find out the value of the electron magnetic moment predicted by the Dirac equation, we have to somehow introduce the coupling to the electromagnetic field $A^\mu = (\Phi, \vec{A})$ and then look for a term in the potential which looks like μB where B is the magnetic field and μ is the magnetic moment. For now, we accept that the interaction is introduced by the so-called minimal substitution

$$\partial_\mu \quad \to \quad D_\mu \stackrel{\text{def}}{\equiv} \partial_\mu + ieA_\mu \qquad \text{with} \quad A^\mu(x) = (\Phi(x), \vec{A}(x))\,, \tag{3.325}$$

where e is the charge of electron ($e < 0$). The operator D_μ is sometimes called the covariant derivative. Starting from the Schrödinger form of the Dirac equation (3.29):

$$i\partial_0\psi = (-i\vec{\alpha}\cdot\vec{\nabla} + \beta m)\psi, \tag{3.326}$$

the minimal substitution

$$\partial_\mu = (\partial_0, \vec{\nabla}) \quad \rightarrow \quad D_\mu = (\partial_0 + ie\Phi, \vec{D}) \tag{3.327}$$

with

$$D_i \stackrel{\text{def}}{\equiv} \nabla_i + ie\underbrace{A_i}_{-A^i} = \nabla_i - ieA^i\,, \quad \text{or} \quad \vec{D} \stackrel{\text{def}}{\equiv} \vec{\nabla} - ie\vec{A} \tag{3.328}$$

yields

$$\begin{aligned} (i\partial_0 - e\Phi)\psi &= \Big(-i\vec{\alpha}\cdot\vec{D} + \beta m\Big)\psi \\ \rightarrow \quad i\dot{\psi} &= \Big(-i\vec{\alpha}\cdot\vec{D} + \beta m + e\Phi\Big)\psi\,. \end{aligned} \tag{3.329}$$

We are interested in a low energy electron; namely, $|\vec{p}| \ll m$ and $E \sim m$, or equivalently, $i\partial_i\psi \ll m\psi$ and $i\partial_0\psi \sim m\psi$. Thus, if we write $\psi(x)$ as

$$\psi(x) = \psi_s(x)e^{-imt}\,, \quad \psi_s(x) = \begin{pmatrix}\phi(x)\\ \eta(x)\end{pmatrix} \tag{3.330}$$

where ϕ and η are 2-component functions, then the space-time derivatives of ϕ and η are small: [4]

$$i\partial_\mu\phi \ll m\phi\,, \qquad i\partial_\mu\eta \ll m\eta\,. \tag{3.331}$$

Also, since the potential energies are assumed to be small compared to the rest mass,

$$e\Phi \ll m\,, \qquad eA^i \ll m \quad (i = 1, 2, 3)\,. \tag{3.332}$$

Furthermore, the solution in the Dirac representation for an electron at rest, $u_{\vec{0},\pm\vec{s}}\,e^{-imt}$, has only upper two components being non-zero. Since we are considering small perturbations to this state, we should have

$$\eta \ll \phi\,. \tag{3.333}$$

Substituting (3.330) in (3.329), and using the Dirac representations of $\vec{\alpha}$ and β,

$$\Big(m\psi_s + i\dot{\psi}_s\Big)e^{-imt} = \left[-i\begin{pmatrix} & \vec{\sigma}\cdot\vec{D}\\ \vec{\sigma}\cdot\vec{D} & \end{pmatrix} + m\begin{pmatrix} I & \\ & -I\end{pmatrix} + e\Phi\right]\psi_s e^{-imt} \tag{3.334}$$

[4] In the order-of-magnitude relations shown, absolute values are implicit.

or in terms of ϕ and η,

$$\left\{\begin{array}{rcl} m\phi + i\,\dot{\phi} &=& -i\,\vec{\sigma}\cdot\vec{D}\,\eta + (\quad m + e\Phi)\phi \\ m\eta + i\,\dot{\eta} &=& -i\,\vec{\sigma}\cdot\vec{D}\,\phi + (-m + e\Phi)\eta \end{array}\right. ,$$

$$\rightarrow \quad \left\{\begin{array}{rcl} i\,\dot{\phi} &=& -i\,\vec{\sigma}\cdot\vec{D}\,\eta + e\Phi\phi \\ \underbrace{i\,\dot{\eta}}_{\text{small}^2} &=& -\underbrace{i\,\vec{\sigma}\cdot\vec{D}}_{\text{small}}\phi - 2m\underbrace{\eta}_{\text{small}} + \underbrace{e\Phi\eta}_{\text{small}^2} \end{array}\right. . \tag{3.335}$$

In the second equation, the dominant terms are $i\,\vec{\sigma}\cdot\vec{D}\,\phi$ and $2m\eta$; thus, we have

$$\eta = -\frac{i\,\vec{\sigma}\cdot\vec{D}}{2m}\phi\,. \tag{3.336}$$

Substituting this in the first equation of (3.335),

$$i\,\dot{\phi} = \left[-\frac{(\vec{\sigma}\cdot\vec{D})^2}{2m} + e\Phi\right]\phi\,. \tag{3.337}$$

Now, can we set $(\vec{\sigma}\cdot\vec{D})^2 = \vec{D}^2$? That would be correct if D_i and D_j commute; in this case, however, they do not commute since D_i contains ∂_i and $A^i(x)$ in it. Since D_i is not a matrix, it does commute with σ_i. Noting that

$$\{\sigma_i,\sigma_j\} = 2\,\delta_{ij}\,, \quad [\sigma_i,\sigma_j] = 2\,i\,\epsilon_{ijk}\sigma_k \quad \rightarrow \quad \sigma_i\sigma_j = \delta_{ij} + i\,\epsilon_{ijk}\sigma_k\,, \tag{3.338}$$

we can write $(\vec{\sigma}\cdot\vec{D})^2$ as

$$\begin{array}{rcl} (\vec{\sigma}\cdot\vec{D})^2 &=& (\sigma_iD_i)(\sigma_jD_j) = \sigma_i\sigma_jD_iD_i = (\delta_{ij} + i\,\epsilon_{ijk}\sigma_k)D_iD_j \\ &=& \vec{D}^2 + i\,\sigma_k\,\epsilon_{ijk}D_iD_j\,. \end{array} \tag{3.339}$$

Note that ∂_i applies to everything on its right all the way to ϕ. Using angle brackets as in $\langle\partial_iA^j\rangle$ to indicate the limit of the detivative, we have

$$\partial_iA^j\phi = A^j\partial_i\phi + \langle\partial_iA^j\rangle\phi \quad \rightarrow \quad \partial_iA^j = A^j\partial_i + \langle\partial_iA^j\rangle\,, \tag{3.340}$$

then,

$$\begin{array}{rcl} \epsilon_{ijk}D_iD_j &=& \epsilon_{ijk}\quad \underbrace{(\partial_i - ieA^i)(\partial_j - ieA^j)}_{\underbrace{\partial_i\partial_j - e^2A^iA^j}_{0\ (i\leftrightarrow j\ \text{symmetric})} \underbrace{-ie(A^i\partial_j + \underbrace{\partial_iA^j}_{\langle\partial_iA^j\rangle})}_{A^j\partial_i}} \\ &=& -ie\,\epsilon_{ijk}[\underbrace{A^i\partial_j + A^j\partial_i}_{0\ (i\leftrightarrow j\ \text{symmetric})} + \langle\partial_iA^j\rangle] \\ &=& -ie\,\epsilon_{ijk}\langle\partial_iA^j\rangle = -ie\,\langle\vec{\nabla}\times\vec{A}\rangle_k = -ie\,B_k \end{array} \tag{3.341}$$

where we have used the ordinary definition of the magnetic field $\vec{B}$:

$$\vec{B} = \vec{\nabla} \times \vec{A}\,. \tag{3.342}$$

Thus, we have

$$(\vec{\sigma}\cdot\vec{D})^2 = \vec{D}^2 + e\vec{\sigma}\cdot\vec{B}\,, \tag{3.343}$$

and (3.337) becomes

$$i\,\dot{\phi} = \left[-\frac{(\vec{\nabla} - ie\vec{A})^2}{2m} - \frac{e}{2m}\vec{\sigma}\cdot\vec{B} + e\Phi\right]\phi\,. \tag{3.344}$$

The first term is the coupling of orbital motion to the photon field, the last term is the potential energy due to the Φ field, and these are familiar terms from non-relativistic extension of the Schrödinger equation to include electromagnetic interaction. The second term, however, is a new term and it represents the potential energy of the electron magnetic moment in a magnetic field $\vec{B}$. This can be seen as follows: suppose that $\vec{B} = (0, 0, B)$ and that ϕ represents a state with the spin in z-direction; namely, $\sigma_z\phi = \phi$. Then,

$$-\frac{e}{2m}\vec{\sigma}\cdot\vec{B} = -\frac{e}{2m}\sigma_z B \quad \rightarrow \quad -\frac{e}{2m}B \tag{3.345}$$

Comparing this to the potential energy of a magnetic moment parallel to the magnetic field $-\mu B$, we identify the magnetic moment to be

$$\mu = \frac{e}{2m}\,. \tag{3.346}$$

The gyro-magnetic ratio g is defined by

$$\mu = gs\mu_o\,, \qquad \mu_0 \equiv \frac{e}{2m}:\ \text{Bohr magneton} \tag{3.347}$$

where s is the absolute spin of the particle and the Bohr magneton μ_0 is the magnetic moment of a classical particle with charge e and one unit ($\hbar$) of orbital angular momentum for which the distributions of charge and mass are such that charge to mass ratio is the same everywhere. Since $s = 1/2$ for an electron, we have

$$g = 2\,. \tag{3.348}$$

The current experimental value is

$$g = 2 \times 1.001159652193\,(10)\,, \tag{3.349}$$

$$\uparrow$$

uncertainty

which is quite close to the value we have just obtained. The theoretical calculation including higher-order effects has a comparable accuracy and is consistent with the experiment as we will see in a later chapter (7.222).

Does the derivation above mean that all spin-1/2 particles should have $g \sim 2.0$? Apparently not, since we know that the magnetic moment of proton is $g = 2.79$ and that of neutron is $g = -1.91$ (even though the above derivation indicates that a neutral particle should not have a magnetic moment since it does not couple to A_μ to begin with). The key assumption in the derivation above was the minimal substitution (3.325) which does not in general work for particles which are not pointlike, and proton and neutron are known to be made of quarks and have size of order 1 fm (femto meter $= 10^{-15}$ m, also called 'fermi').

3.10 High energy limit - massless case

We now examine the behavior of free Dirac fields at high energy, or equivalently, in the massless limit. We will see that the helicity + and the helicity − components decouple in such cases, satisfying separate equations of motion, where the *helicity* is defined to be the spin component in the direction of the momentum.

Define the 'right-handed' and 'left-handed' components by

$$\psi_R \stackrel{\text{def}}{\equiv} P_R\psi\,, \qquad \psi_L \stackrel{\text{def}}{\equiv} P_L\psi\,, \tag{3.350}$$

with

$$\boxed{P_R \stackrel{\text{def}}{\equiv} \frac{1+\gamma_5}{2}\,, \quad P_L \stackrel{\text{def}}{\equiv} \frac{1-\gamma_5}{2}\,.} \tag{3.351}$$

One can readily verify that they satisfy the properties of projection operators:

$$\begin{gathered} P_{R,L}^2 = P_{R,L}\,, \quad P_R + P_L = 1\,, \\ P_R P_L = P_L P_R = 0\,. \end{gathered} \tag{3.352}$$

In particular, $P_R + P_L = 1$, gives

$$\psi = \psi_R + \psi_L\,. \tag{3.353}$$

Also, note that $\{\gamma^\mu, \gamma_5\} = 0$ (3.134) and $\overline{\gamma_5} = -\gamma_5$ (3.285) lead to

$$P_R\gamma^\mu = \gamma^\mu P_L\,, \quad P_L\gamma^\mu = \gamma^\mu P_R\,, \tag{3.354}$$

$$\overline{P_R} = P_L\,, \quad \overline{P_L} = P_R\,. \tag{3.355}$$

Now, set the mass to zero in the Dirac equation to get

$$i\,\gamma^\mu\partial_\mu\psi = 0 \quad (m = 0)\,. \tag{3.356}$$

Then, we see that ψ_R also satisfies the massless Dirac equation:

$$i\gamma^\mu \underbrace{\partial_\mu \psi_R}_{P_R \partial_\mu \psi} = i \underbrace{\gamma^\mu P_R}_{P_L \gamma^\mu} \partial_\mu \psi = P_L \underbrace{i\gamma^\mu \partial_\mu \psi}_{0} = 0 \,. \tag{3.357}$$

Simply exchanging L and R everywhere in the above, we also obtain $i\gamma^\mu \partial_\mu \psi_L = 0$. Thus, we see that ψ_R and ψ_L separately satisfy the massless Dirac equation:

$$i\partial\!\!\!/ \psi_R = 0 \,, \quad i\partial\!\!\!/ \psi_L = 0 \quad (m = 0) \,. \tag{3.358}$$

If the mass is non-zero, then they cannot be separated:

$$(i\gamma^\mu \partial_\mu - m)\psi_R = \underbrace{i\gamma^\mu \partial_\mu \psi_R}_{P_L i\gamma^\mu \partial_\mu \psi} - m \underbrace{\psi_R}_{P_R \psi} \neq 0 \,. \tag{3.359}$$

One important property of $\psi_{R,L}$ is that they do not mix under a proper and orthochronous transformation S. In fact, since $[\gamma_5, S] = 0$ (3.137), we have

$$[P_{R,L}, S] = \Big[\frac{1 \pm \gamma_5}{2}, S\Big] = 0 \,. \tag{3.360}$$

Then, using $\psi'(x') = S\psi(x)$, we find

$$\psi'_R(x') \equiv P_R \psi'(x') = P_R S \psi(x) = S P_R \psi(x) = S\psi_R(x) \,, \tag{3.361}$$

and similarly for ψ_L. Thus, ψ_R and ψ_L transform separately under a proper and orthochronous transformation S:

$$\psi'_R(x') = S\psi_R(x) \,, \quad \psi'_L(x') = S\psi_L(x) \,. \tag{3.362}$$

This holds even if the mass is non-zero.

The above discussion is independent of representation. The situation, however, becomes simpler in the Weyl representation (3.155) in which $P_{R,L}$ are written as

$$\gamma_5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad \to \quad P_R = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} , \quad P_L = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \quad (\text{Weyl}) \,, \tag{3.363}$$

which means that P_R filters out the top two components and P_L filters out the bottom two components. Namely, in the Weyl representation the top half and the bottom half of the spinor transform independently under S. Or equivalently, the matrix S becomes block diagonal:

$$S = \left(\begin{array}{cc|cc} \cdot & \cdot & 0 & \\ \cdot & \cdot & & \\ \hline & & \cdot & \cdot \\ 0 & & \cdot & \cdot \end{array}\right) \quad : \text{proper and orthochronous (Weyl)} \,. \tag{3.364}$$

Therefore, in order to represent all *proper and orthochronous* transformations, that is, to reflect the product rule of the space-time Lorentz transformations, one needs only 2×2 matrices. In fact, one could use the 2×2 matrices that act on, say, the top half of the spinor as the representation. In this sense, the 4×4 representation of proper and orthochronous Lorentz transformations is said to be *reducible*. When the parity transformation $S_P = \gamma^0$ is included, however, the 2×2 representation is not enough. This can be easily seen by the expression of γ^0 in the Weyl representation (3.155) which mixes the top half and the bottom half of the spinor.

We called ψ_R and ψ_L as the right-handed and left-handed components because in the massless limit the fermion have helicity $+$ and $-$, respectively. Now, we will show that it is indeed the case. We take the spin quantization axis $\vec{s}$ to be the direction of the boost under which a rest mass m will acquire a momentum $\vec{p}$:

$$\vec{s} = \hat{p} \equiv \frac{\vec{p}}{|\vec{p}|} \,. \tag{3.365}$$

Noting that in the rest frame, we have $s^0 = 0$ and $s_\parallel = 1$, where $s_\parallel$ is the component of $\vec{s}$ along the direction of boost, the spin 4-vector s^μ in the boosted frame is given by

$$\begin{pmatrix} s^0 \\ s_\parallel \end{pmatrix} = \begin{pmatrix} \gamma & \eta \\ \eta & \gamma \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \eta \\ \gamma \end{pmatrix} = \begin{pmatrix} |\vec{p}|/m \\ E/m \end{pmatrix} , \quad \vec{s}_\perp = 0 \,, \tag{3.366}$$

where γ, η are related to $E, \vec{p}$ as before (3.230), and E and $\vec{p}$ are the energy and momentum of an electron or a positron (with $E \geq 0$).

In the massless limit (or high energy limit) we have $E = |\vec{p}|$, and s^μ becomes proportional to p^μ:

$$s^\mu = \Big(\frac{|\vec{p}|}{m}, \frac{E}{m}\hat{p}\Big) \quad \overset{E\to\infty}{\to} \quad s^\mu = \Big(\frac{E}{m}, \frac{|\vec{p}|}{m}\hat{p}\Big) = \frac{p^\mu}{m} \,. \tag{3.367}$$

Since $\not{p}/m$ is $+1$ or -1 depending on whether the state is an electron or a positron (3.253), γ_5 is then equal to the spin operator $\gamma_5 \not{s}$ (with $\vec{s} = \hat{p}$) up to a sign:

$$\gamma_5 \not{s} = \gamma_5 \frac{\not{p}}{m} = \begin{cases} \gamma_5 & \text{(electron)} \\ -\gamma_5 & \text{(positron)} \end{cases} \qquad (\vec{s} = \hat{p} \,,\ m \to 0) \,. \tag{3.368}$$

Since the eigenvalue of $\gamma_5 \not{s}$ correctly represents the spin component along $\hat{p}$ for both electron and positron solutions, we have

$$\gamma_5 : \pm 1 \leftrightarrow \begin{cases} \text{helicity } \pm \text{ for } e^- \ (u\text{'s}) \\ \text{helicity } \mp \text{ for } e^+ \ (v\text{'s}) \end{cases} \qquad (m \to 0) \,, \tag{3.369}$$

or equivalently,

$$\boxed{P_{R,L} \equiv \frac{1 \pm \gamma_5}{2} : \begin{cases} \text{helicity } \pm \text{ projection operator for } e^- \ (u\text{'s}) \\ \text{helicity } \mp \text{ projection operator for } e^+ \ (v\text{'s}) \end{cases} \qquad (m \to 0)} \,. \tag{3.370}$$

Since γ_5 and any proper and orthochronous transformation S commute, once a state is an eigenstate of γ_5, one cannot change the eigenvalue of γ_5 by boosting it or rotating it:

$$\gamma_5\psi = \pm\psi \quad \rightarrow \quad \gamma_5(S\psi) = S\gamma_5\psi = \pm(S\psi)\,. \tag{3.371}$$

Namely, in the massless limit, one cannot change the value of helicity by boosting or rotating. This can be understood intuitively. For a rotation, it is plausible since the spin and the direction of motion rotate together, and thus the component of spin along the motion would stay the same. For a boost, a classical picture also works just fine. The only way to reverse the spin component along the motion is for the observer to move faster than the particle and overtake it. Then, the direction of momentum viewed by the observer will flip while the spin will stay the same, and thus the helicity will change its sign. In the massless limit, the particle will be moving at the speed of light, and thus it is impossible to overtake it.

As we have seen, it does not require the mass to be zero in order for ψ_R and ψ_L to transform independently under proper and orthochronous transformations. If the mass is non-zero, however, ψ_R and ψ_L do not correspond to helicity $+1$ and -1, respectively. The distinction between ψ_R and ψ_L, or what we called 'handedness', is usually referred to as the *chirality* regardless of the mass, while helicity is defined as the spin component along the direction of motion. Helicity $+(-)$ and right-handedness (left-handedness), however, are often used synonymously.

Problems

3.1 *Property of* γ_5.
(*a*) *The matrix* γ_5 *is defined by* $\gamma_5 \equiv i\,\gamma^0\gamma^1\gamma^2\gamma^3$. *Show that it can also be written as*

$$\gamma_5 = \frac{i}{24}\epsilon_{\mu\nu\sigma\rho}\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho\,,$$

where $\epsilon_{\mu\nu\sigma\rho}$ *is the totally antisymmetric 4-th rank tensor and sum is implied over the Lorentz indexes* $\mu, \nu, \sigma,$ *and* ρ.
(*b*) *Use the definition of determinant for a* 4×4 *matrix* Λ

$$\det\Lambda = \epsilon_{\mu\nu\sigma\rho}\Lambda^\mu{}_0\Lambda^\nu{}_1\Lambda^\sigma{}_2\Lambda^\rho{}_3$$

or equivalently

$$\epsilon_{\alpha\beta\gamma\delta}\det\Lambda = \epsilon_{\mu\nu\sigma\rho}\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta\Lambda^\sigma{}_\gamma\Lambda^\rho{}_\delta$$

to show that the quantity $\bar\psi(x)\gamma_5\psi(x)$ *transforms under a general Lorentz transformation as*

$$\bar\psi'(x')\gamma_5\psi'(x') = (\det\Lambda)\bar\psi(x)\gamma_5\psi(x)$$

where $\psi(x)$ *is a 4-component spinor field. Namely, a pseudoscalar current changes the sign under an improper Lorentz transformation.*

3.2 *Helicity states.*
(*a*) *Show that the product of the spin and the momentum, which are 3-dimentional vectors, is conserved for free fields; namely,*

$$\left[\vec\Sigma\cdot\vec p\,, H\right] = 0\,,$$

with $\Sigma_i = i\gamma_j\gamma_k$ $(i,j,k:$ *cyclic*$)$ *and* $H = \vec\alpha\cdot\vec p + \beta m$ $(\vec p = -i\vec\nabla)$. *The spin matrix* Σ_i *can also be written as* $\Sigma_i = \frac{i}{2}\epsilon_{ijk}\gamma_j\gamma_k$. *Note the factor of 2, why? (comment: This means that the spin component along the direction of motion, called 'helicity', is conserved - for free fields, at least.)*
(*b*) *We now consider the solutions of the Dirac equation where the spin quantization axis* $\vec s$ *is taken in the direction of the momentum. By directly applying* $\vec\Sigma\cdot\vec p$ *in the Dirac representation, verify that the eigen spinors are given by (up to a constant)*

$$\begin{pmatrix}\chi_+\\ \dfrac{\vec q\cdot\vec\sigma}{E+m}\chi_+\end{pmatrix}e^{-iq\cdot x}\,,\quad\begin{pmatrix}\dfrac{\vec q\cdot\vec\sigma}{E+m}\chi_-\\ \chi_-\end{pmatrix}e^{iq\cdot x}\ :\ \textit{with eigen value}\ +|\vec q|\,,$$

$$\begin{pmatrix}\chi_-\\ \dfrac{\vec q\cdot\vec\sigma}{E+m}\chi_-\end{pmatrix}e^{-iq\cdot x}\,,\quad\begin{pmatrix}\dfrac{\vec q\cdot\vec\sigma}{E+m}\chi_+\\ \chi_+\end{pmatrix}e^{iq\cdot x}\ :\ \textit{with eigen value}\ -|\vec q|\,,$$

where $(\hat{q}\cdot\vec{\sigma})\chi_\pm = \pm\chi_\pm$ *Note that* $\vec{p}$ *is the differential operator* $-i\vec{\nabla}$, $q^\mu = (q^0, \vec{q})$ *is a real-number 4-vector with* $q^0 = E$ *and* $\hat{q}$ *is the unit vector in the direction of* $\vec{q}$.

3.3 *Plane-wave solutions of the Dirac equation.*
(*a*) *Let* $p^\mu = (E, \vec{p})$ *be a 4-vector. Show that in the Dirac representation of gamma matrixes, the* 4×4 *matrix* $\not{p}$ *can be written as*

$$\not{p} = \begin{pmatrix} E & -\vec{p}\cdot\vec{\sigma} \\ \vec{p}\cdot\vec{\sigma} & -E \end{pmatrix}.$$

(*b*) *Using the result above, explicitly verify that the spinors* u *and* v *given by*

$$u = \sqrt{E+m}\begin{pmatrix} \chi \\ \dfrac{\vec{p}\cdot\vec{\sigma}}{E+m}\chi \end{pmatrix}, \quad v = \sqrt{E+m}\begin{pmatrix} \dfrac{\vec{p}\cdot\vec{\sigma}}{E+m}\chi \\ \chi \end{pmatrix}$$

satisfy the Dirac equations in momentum space for positive and negative energy solutions $(\not{p} - m)u = 0$ *and* $(\not{p} + m)v = 0$, *where* χ *is any arbitrary 2-component column vector with normalization* $\chi^\dagger\chi = 1$, *and* $p^0 \equiv E \equiv \sqrt{\vec{p}^2 + m^2}$. (*hint: Note that* $(\vec{p}\cdot\vec{\sigma})^2 = \vec{p}^2$.)

3.4 *Spinor representation of* T *- a naive attempt.*
When a spinor field $\psi(x)$ *satisfies the Dirac equation in a Lorentz frame, then in a frame transformed by* Λ *(i.e.* $x' = \Lambda x$*),* $\psi'(x')$ *will also satisfy the Dirac equation in that frame provided that* $\psi'(x') = S\psi(x)$ *with* $S^{-1}\gamma^\mu S = \Lambda^\mu{}_\nu\gamma^\nu$, *where* S *is a* 4×4 *matrix in the spinor space corresponding to the particular Lorentz transformation* Λ *which, of course, is in the space-time 4 dimentional space. This was true for any Lorentz transformation including time inversion given by* $T : T^0{}_0 = -1, T^i{}_i = 1$ $(i = 1,2,3)$, *all other components* $= 0$.
(*a*) *Show that* $S(T) = \gamma^1\gamma^2\gamma^3$ *satisfies the relation* $S^{-1}\gamma^\mu S = \Lambda^\mu{}_\nu\gamma^\nu$ *for the time reversal Lorentz transformation* T.
(*b*) *Write down the explicit* 4×4 *matrix* $S(T)$ *in the Dirac representation.*
(*c*) *Apply* $S(T)$ *to a solution of the Dirac equation corresponding to an electron (not positron) with momentum* $\vec{p}$ *and spin* $+\vec{s}$. *What are the physical momentum and spin of the resulting wave function? Follow the rule* $\psi'(x') = S\psi(x)$ *strictly, and express the resulting function in terms of* x'. *Use the Dirac representation.* (*comment: As you see, this transformation changes an electron into a positron or positive energy to negative energy and vice versa. That is not exactly what we want for the usual sense of* T *inversion which keeps an electron as an electron and a positron as a positron and simply reverses its momentum and spin.*)

3.5 *Weyl representation.*

(*a*) *Find a matrix that transforms the Dirac representation to the Weyl representation; namely, find a* 4×4 *unitary matrix* V *that satisfies*

$$\gamma^{\mu\prime} = V\gamma^{\mu}V^{\dagger} \quad (\mu = 0, 1, 2, 3)\,,$$

where $\gamma^{\mu\prime}$ *is in the Weyl representation and* γ^{μ} *is in the Dirac representation.*
b) Use the matrix V *found above to transform the boost matrix* S *given in the Dirac representation as*

$$S = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} 1 & \dfrac{\vec{p}\cdot\vec{\sigma}}{E+m} \\ \dfrac{\vec{p}\cdot\vec{\sigma}}{E+m} & 1 \end{pmatrix}$$

to the Weyl representation.
(*c*) *Use the Weyl representation of the* γ *matrixes in*

$$S = e^{\frac{1}{2}a_{\alpha\beta}B^{\alpha\beta}} \quad \text{with} \quad B^{\alpha\beta} = \frac{1}{4}\left[\gamma^{\alpha}, \gamma^{\beta}\right]$$

to find two 2×2 *representations of the proper and orthochronous transformations corresponding to* ψ_R *and* ψ_L*, respectively. Namely, for each Lorentz transformation expressed as*

$$\Lambda = e^{\xi_i K_i + \theta_i L_i}\,,$$

with $\xi_i = \alpha_{0i}$, $\theta_i = \alpha_{jk}$ $(i, j, k : \text{cyclic})$, *find a* 2×2 *matrix that preserves the product rule. Express the result as*

$$S_{2\times 2} = e^{\xi_i G_i + \theta_i H_i}$$

where G_i, H_i *are* 2×2 *matrixes (namely, find the* 2×2 *generators* G_i, H_i*).*
(*d*) *Show that in the Weyl representation, the Dirac equation in the massless limit decouples to a pair of equations for two 2-component spinors* ϕ_R *and* ϕ_L *as*

$$\begin{cases} i(\partial_0 + \vec{\sigma}\cdot\vec{\nabla})\phi_R &= 0 \\ i(\partial_0 - \vec{\sigma}\cdot\vec{\nabla})\phi_L &= 0 \end{cases}$$

where ϕ_R *and* ϕ_L *are the top half and the bottom half of the 4-component spinor* ψ*, respectively. This pair of equations are known as Weyl equations.*

Chapter 4

Quantization of Fields

Take some uniform medium that vibrates and in which waves can propagate, which could for example be a lattice of molecules. In such system, each normal mode of the vibration can be treated as an independent harmonic oscillator. Now suppose that it is small enough that quantum effects become important; then, the quantum mechanical treatment results in each oscillator containing an integer number of quanta corresponding to the wave length and oscillation frequency of the normal mode. These quanta behave in many ways as if they are particles, each carrying a definite energy and a momentum. The system is classically well-defined, and the procedure to obtain the quantum mechanical system is straightforward as we will discuss shortly. Quantum field theory of elementary particles takes such system as a *model* to describe particles where the wave length is interpreted as the momentum of each particle and the oscillation frequency as its energy. There are some important advantages to this approach. First, since it is based on a classical model, the energy of the system is well-defined by the total Hamiltonian of the system which is likely to be positive since the energy of the classical system would be positive even after quantized. Second, this will allow the theory to deal with multiple particles in a consistent way. So far, wave functions for the Klein-Gordon and the Dirac equations described a single particle; even in the hole theory, a pair creation was described as a transition of a single negative energy electron to a positive energy state. This resulted from the fact that there is only one pair of canonical observables $(\vec{x}, \vec{p})$ for such theories. In quantum field theory, there are infinite number of degrees of freedom corresponding to the position and momentum of each point of the field (or each *molecule*), and this allows the theory to describe infinite number of particles.

Let's have a quick look at how one quantizes a vibrating medium. Imagine a one-dimensional string that can vibrate, where there will be a number $\phi(x)$ attached to each point x on the string. It could be the transverse or longitudinal displacement from the natural position, or anything else that can result in a wave. Classically, one would have a certain equation of motion that describes such wave, or equivalently,

the Lagrangian of the system. If the system is very small, the required quantum mechanical description of the system can be obtained from the Lagrangian of the classical system by the prescription called the *canonical quantization* which proceeds as follows. First, one chooses a set of generalized coordinates that uniquely describes the configuration of the string, $\underline{q} \equiv q_i$ $(i = 1, \ldots, n)$, and write down the Lagrangian of the system $L(\underline{q}, \underline{\dot{q}})$:

$$L(\underline{q}, \underline{\dot{q}}) = T - V \tag{4.1}$$

where T is the kinetic energy and V is the potential energy of the system. Then, the canonical momentum p_i conjugate to q_i is defined by

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i} \tag{4.2}$$

and the Hamiltonian is formed as

$$H(\underline{p}, \underline{q}) \equiv \sum_i p_i \dot{q}_i - L \tag{4.3}$$

where the result is expressed as a function of $\underline{p}$ and $\underline{q}$. At this point, $q_i(t)$ and $p_i(t)$ are considered as hermitian operators (i.e. observables) in the Heisenberg picture where state vectors stay constant and all time dependences are carried by operators. Then, a set of *equal-time* commutation relations are introduced as

$$[q_i(t), p_j(t)] = i\delta_{ij}\,, \quad [q_i(t), q_j(t)] = [p_i(t), p_j(t)] = 0\,. \tag{4.4}$$

The time evolution of any observable $O(t)$ in the Heisenberg picture is given by Heisenberg's equation of motion:

$$-i\dot{O} = [H, O]\,, \tag{4.5}$$

and the matrix element of $O(t)$ at any given time for states $|\phi_{1,2}\rangle$ is then given by $\langle\phi_1|O(t)|\phi_2\rangle$, which in principle allows the theory to be compared with measurements.

Since changing the order of q_i and p_i could result in a different quantum mechanical H, this procedure is not unique; namely, there are in general more than one quantum mechanical system that has the same classical counterpart. When there is ambiguity, the choice should be made based on comparison with experiments and theoretical consistency. In practice, however, the choice is usually either quite obvious or ordering changes simply result in constant offsets that do not affect observable effects.

For the one-dimensional string, one obvious choice for the general coordinates is, for example, the transverse displacement $\phi(x)$ of each point labeled by x, or equivalently that of i-th molecule ϕ_i. An alternative choice is to describe the configuration of the string at a given time t by a superposition of normal modes:

$$\phi(x) = \sum_p c_p e^{ipx} + \text{complex conjugate} \tag{4.6}$$

which is nothing but the Fourier transform of $\phi(x)$. Each normal mode is a harmonic oscillator labeled by its wave length $2\pi/p$, and the oscillators are decoupled from each other; namely, each oscillator oscillates with a frequency uniquely given by the equation of motion without being perturbed by other oscillators. Thus, one can take the amplitudes of the normal modes as the general coordinates and proceed to quantize the system. As we will see later, the resulting quantized system becomes equivalent to the system quantized using the displacement $\phi(x)$ as the general coordinates.

So quantum field theory takes such quantized vibrating medium as a model to describe multiple particles where a creation of a particle with momentum p and energy E corresponds to an excitation of the normal mode with wave number p and angular frequency E by one quanta. It is just about the simplest way to describe multiple particles that can be created or annihilated, and it is quite amazing that such simple prescription actually works for a complicated system that consists of variety of elementary particles. Actually, in order for the model to work for the elementary particles, we need to address a few non-trivial questions:

1. Is the theory independent of the frame in which the quantization is performed?

2. Is the causality respected? Namely, are the measurements of the field at two points separated by a time-like distance independent of each other?

3. Electrons are known to obey Pauli's exclusion principle; how can it be incorporated into the theory?

The classical model described above is clearly inconsistent with special relativity since there is a spacial frame where the material is at rest. The move to a relativistic theory is similar to the case of the electromagnetic wave where the concept of the *ether* which transmits light was simply discarded and the electromagnetic fields and their transformation properties survived. After we make such transition, we will see that the answers to the item 1 and 2 are miraculously yes. Then, we will see that the third question is solved by using anticommutators rather than commutators to quantize the Dirac field which will limit the number of particles that occupy a single normal mode to one. Even after these issues are resolved, we will encounter further problems: how to formulate massless spin-1 particles (for example, photons), how to handle the infinities when calculating higher-order effects, etc. These topics will be dicussed in later chapters.

We will now start by briefly reviewing the quantum mechanics of a harmonic oscillator, and see how Pauli's exclusion principle can be incorporated using anticommutators in the quantization procedure.

4.1 Harmonic oscillator

Classical mechanics
Take a unit mass $m = 1$ attached to a spring with a spring constant ω^2. It is placed in a gravitational field and we take the vertical displacement upward from its natural position to be q. The Lagrangian, which is a function of q and $\dot{q}$, is

$$L(q, \dot{q}) = T - V = \frac{1}{2}(\dot{q}^2 - \omega^2 q^2)\,, \tag{4.7}$$

where the potential energy V is measured relative to the natural position. The Lagrange's equation of motion is then

$$\frac{d}{dt}\Big(\underbrace{\frac{\partial L}{\partial \dot{q}}}_{\dot{q}}\Big) = \underbrace{\frac{\partial L}{\partial q}}_{-\omega^2 q} \quad \to \quad \ddot{q} = -\omega^2 q \quad \text{(Lagrangian form)}\,, \tag{4.8}$$

which means $q(t) \propto e^{\pm i\omega t}$, and the general solution is (requiring that q be real)

$$q(t) = c\, e^{-i\omega t} + c^* e^{i\omega t} \tag{4.9}$$

where c is an arbitrary complex constant.

The canonical momentum is defined as

$$p \overset{\text{def}}{\equiv} \frac{\partial L}{\partial \dot{q}} = \dot{q}\,, \tag{4.10}$$

and the Hamiltonian is obtained by writing $p\dot{q} - L$ in terms of q and p only:

$$H(p, q) \overset{\text{def}}{\equiv} p\dot{q} - L = \dot{q}^2 - \frac{1}{2}(\dot{q}^2 - \omega^2 q^2) = \frac{1}{2}(\dot{q}^2 + \omega^2 q^2) \tag{4.11}$$

$$\to \quad H(p, q) = \frac{1}{2}(p^2 + \omega^2 q^2)\,. \tag{4.12}$$

Then Hamilton's equations of motions are

$$\begin{cases} \dot{q} \;=\; \dfrac{\partial H}{\partial p} \\ \dot{p} \;=\; -\dfrac{\partial H}{\partial q} \end{cases} \quad \to \quad \begin{cases} \dot{q} \;=\; p \\ \dot{p} \;=\; -\omega^2 q \end{cases} \quad \text{(Hamiltonian form)}\,, \tag{4.13}$$

where the first equation simply recovers the definition of p, which together with the second reproduces the equation of motion (4.8).

Quantization

When the system of the spring is very small then we have to use a quantum mechanical description of the system to correctly take into account quantum effects such as the quantization of energy levels. We will now follow the procedure of canonical quantization: namely, we regard $q(t)$ and $p(t)$ as hermitian operators in the Heisenberg picture, and impose the equal-time commutation relation

$$[q(t), p(t)] = i\,. \tag{4.14}$$

Then, the equations of motion are obtained by applying Heisenberg's equation of motion $-i\dot{O} = [H, O]$ (4.5) to $q(t)$ and $p(t)$. Using the relation $[AB, C] = A[B, C] + [A, C]B$,

$$\begin{aligned} -i\dot{q} &= [H, q\,] = \frac{1}{2}[p^2 + \omega^2 q^2, q\,] = \frac{1}{2}[p^2, q\,] \\ &= \frac{1}{2}\Big(p\underbrace{[p, q\,]}_{-i} + \underbrace{[p, q\,]}_{-i}p\Big) \\ &= -ip \qquad\qquad\qquad \to \quad \dot{q} = p\,, \end{aligned} \tag{4.15}$$

$$\begin{aligned} -i\dot{p} &= [H, p\,] = \frac{1}{2}[p^2 + \omega^2 q^2, p\,] = \frac{1}{2}[\omega^2 q^2, p\,] \\ &= \frac{\omega^2}{2}\Big(q\underbrace{[q, p\,]}_{i} + \underbrace{[q, p\,]}_{i}q\Big) \\ &= i\,\omega^2 q \qquad\qquad\qquad \to \quad \dot{p} = -\omega^2 q\,, \end{aligned} \tag{4.16}$$

thus reproducing the classical results which are now expressed as relations among operators.

A new feature created by moving to the quantized system is that the energy level, or the eigenvalue of H, can take only discrete values. The eigenvalue problem can be studied by introducing

$$\boxed{a(t) \stackrel{\text{def}}{\equiv} \frac{1}{\sqrt{2}}\Big(\sqrt{\omega}q + i\,\frac{p}{\sqrt{\omega}}\Big) \quad \to \quad a^\dagger(t) = \frac{1}{\sqrt{2}}\Big(\sqrt{\omega}q - i\,\frac{p}{\sqrt{\omega}}\Big)}\,. \tag{4.17}$$

Note that $a(t)$ is not hermitian, and thus it is not an observable. The commutator of $a(t)$ and $a^\dagger(t)$ is then

$$\begin{aligned} [a(t), a^\dagger(t)] &= \frac{1}{2}\Big[\sqrt{\omega}q + i\,\frac{p}{\sqrt{\omega}}, \sqrt{\omega}q - i\,\frac{p}{\sqrt{\omega}}\Big] \\ &= \frac{1}{2}\Big(i\underbrace{[p, q]}_{-i} - i\underbrace{[q, p]}_{i}\Big) = 1\,. \end{aligned} \tag{4.18}$$

Solving (4.17) for q and p,

$$q(t) = \frac{1}{\sqrt{2\omega}}\Big(a(t) + a^\dagger(t)\Big)\,, \quad p(t) = -i\sqrt{\frac{\omega}{2}}\Big(a(t) - a^\dagger(t)\Big)\,. \tag{4.19}$$

Using this, one sees that the commutation relation of $a(t)$ and $a^\dagger(t)$ (4.18) leads to that of q and p (4.14):

$$\begin{aligned}[q(t), p(t)] &= -\frac{i}{2}\,[\,a(t) + a^\dagger(t)\,,\, a(t) - a^\dagger(t)\,] \\ &= -\frac{i}{2}\Big(\underbrace{[a^\dagger(t), a(t)]}_{-1} - \underbrace{[a(t), a^\dagger(t)]}_{1}\Big) = i\,. \end{aligned} \tag{4.20}$$

Thus, the two commutation relations are equivalent.

The Hamiltonian can be written in terms of $a(t)$ and $a^\dagger(t)$ as follows:

$$\begin{aligned} a(t)a^\dagger(t) + a^\dagger(t)a(t) &= \frac{1}{2}\Big\{\Big(\sqrt{\omega}q + i\,\frac{p}{\sqrt{\omega}}\Big)\Big(\sqrt{\omega}q - i\,\frac{p}{\sqrt{\omega}}\Big) \\ &\qquad + \Big(\sqrt{\omega}q - i\,\frac{p}{\sqrt{\omega}}\Big)\Big(\sqrt{\omega}q + i\,\frac{p}{\sqrt{\omega}}\Big)\Big\} \\ &= \frac{1}{2}\Big(2\omega q^2 + 2\frac{p^2}{\omega}\Big) = \frac{1}{\omega}\,\underbrace{\Big(\omega^2 q^2 + p^2\Big)}_{2H}\,. \end{aligned} \tag{4.21}$$

Thus,

$$H = \frac{\omega}{2}\Big(\underbrace{a(t)a^\dagger(t)}_{1 + a^\dagger(t)a(t)\text{ by (4.18)}} + a^\dagger(t)a(t)\Big) = \omega\,\Big(a^\dagger(t)a(t) + \frac{1}{2}\Big)\,. \tag{4.22}$$

The time dependence of $a(t)$ is then given by

$$\begin{aligned} -i\dot{a}(t) &= [H, a(t)] = \omega[a^\dagger(t)a(t), a(t)] = \omega\,\underbrace{[a^\dagger(t), a(t)]}_{-1}\,a(t) \\ &\to\ \dot{a}(t) = -i\omega a(t)\,, \end{aligned} \tag{4.23}$$

which has the general solution

$$a(t) = ae^{-i\omega t} \quad \text{with} \quad a \stackrel{\text{def}}{\equiv} a(0)\,. \tag{4.24}$$

The a operator without explicit time dependence '(t)' is understood hereafter to be the value at $t = 0$. Then,

$$a(t)a^\dagger(t) = aa^\dagger\,, \quad a^\dagger(t)a(t) = a^\dagger a\,, \tag{4.25}$$

and thus,

$$\boxed{[a, a^\dagger] = 1}\,, \tag{4.26}$$

and

$$\boxed{H = \omega\Big(N + \frac{1}{2}\Big)\,, \quad N \stackrel{\text{def}}{\equiv} a^\dagger a}\,, \tag{4.27}$$

where the operator N is called the *number operator* for the reason we will see below.

We first evaluate $[a^\dagger, N]$ and $[a, N]$:

$$\begin{aligned} \underbrace{[a^\dagger, N]}_{a^\dagger N - N a^\dagger} &= [a^\dagger, a^\dagger a] = a^\dagger \underbrace{[a^\dagger, a]}_{-1} = -a^\dagger \;\;\rightarrow\;\; N a^\dagger = a^\dagger (N+1)\,, \\ \underbrace{[a, N]}_{aN - Na} &= [a, a^\dagger a] = \underbrace{[a, a^\dagger]}_{1} a = a \qquad\rightarrow\quad N a = a(N-1)\,. \end{aligned} \tag{4.28}$$

Let $|n\rangle$ be an eigenstate of N with an eigenvalue n:

$$N|n\rangle = n|n\rangle\,. \tag{4.29}$$

where n is real since N is hermitian: $N^\dagger = (a^\dagger a)^\dagger = a^\dagger a = N$. Applying (4.28) to $|n\rangle$,

$$\begin{aligned} N a^\dagger |n\rangle &= a^\dagger \underbrace{(N+1)}_{n+1} |n\rangle \;\;\rightarrow\;\; N\Big(a^\dagger|n\rangle\Big) = (n+1)\Big(a^\dagger|n\rangle\Big) \\ N a |n\rangle &= a \underbrace{(N-1)}_{n-1} |n\rangle \quad\rightarrow\quad N\Big(a|n\rangle\Big) = (n-1)\Big(a|n\rangle\Big)\,. \end{aligned} \tag{4.30}$$

Namely, $a^\dagger$ raises the eigenvalue by 1 and a lowers eigenvalue by 1, which then translates to raising or lowering the energy, or the eigenvalue of H written as (4.27), by one unit of ω. Accordingly, $a^\dagger$ is called the *creation operator*, and a the *annihilation operator*.

That $n \geq 0$ can be seen as follows:

$$\begin{aligned} \underbrace{\langle n|N|n\rangle}_{n\underbrace{\langle n|n\rangle}_{>0}} &= \underbrace{\langle n|a^\dagger a|n\rangle}_{(\text{norm of } a|n\rangle) \geq 0} \quad\rightarrow\quad n \geq 0\,. \end{aligned} \tag{4.31}$$

If n is not an integer, then the state $a^k|n\rangle$ with $k > n$ will have an eigenvalue $n-k < 0$ which contradicts the fact that n be non-negative. This can be avoided if at some point applying a results in the null state (the state with zero norm). If n is an integer, then indeed such is the case since $a|0\rangle$ is a state with zero norm as can be seen by setting $n = 0$ in (4.31):

$$\langle 0|a^\dagger a|0\rangle = 0 \quad\rightarrow\quad a|0\rangle = 0\,. \tag{4.32}$$

Thus, $n = 0$ corresponds to the state with lowest energy which we normalize as

$$\langle 0|0\rangle = 1\,. \tag{4.33}$$

All eigenstates of N can then be obtained by applying $a^\dagger$ to the ground state $|0\rangle$:

$$|n\rangle \stackrel{\text{def}}{\equiv} \frac{1}{\sqrt{n\,!}} a^{\dagger\,n}|0\rangle\,, \tag{4.34}$$

where $1/n\,!$ is a normalization factor. It can be readily shown that this state is indeed properly normalized:

$$\langle n|n\rangle = 1\,, \tag{4.35}$$

and that the operations of $a^\dagger$ and a on $|n\rangle$ are given by

$$\left\{ \begin{array}{rcl} a^\dagger|n\rangle & = & \sqrt{n+1}\,|n+1\rangle \\ a\,|n\rangle & = & \sqrt{n}\,|n-1\rangle \end{array} \right. . \tag{4.36}$$

Exercise 4.1 *Eigenstates of harmonic oscilator.*
(*a*) *Show that the state* $|n\rangle$ *as defined in (4.34) is normalized to unity; i.e.* $\langle n|n\rangle = 1$.
(*b*) *Derive (4.36).*

From (4.19) and the time dependence of $a(t)$ (4.24), $q(t)$ can be written as

$$\boxed{q(t) = \frac{1}{\sqrt{2\omega}}\Big(a e^{-i\omega t} + a^\dagger e^{i\omega t}\Big)}\,. \tag{4.37}$$

Comparing this with the classical solution (4.9), one can see that the same quantum mechanical system could be obtained by the following procedure: First, write down the general classical solution as (4.9), identify the coefficient of $e^{-i\omega t}$ as $a/\sqrt{2\omega}$ and then introduce the commutation relation $[a, a^\dagger] = 1$. How did the factor $1/\sqrt{2\omega}$ in (4.37) come about? It originated from the definition (4.17). If $a(t)$ is defined as $(cq + ip/c)/\sqrt{2}$ where c is an arbitrary real constant, then the same derivation as (4.18) shows that $[a(t), a^\dagger(t)] = 1$ still results from $[q, p] = i$, or equivalently, $a^\dagger(t)a(t)$ is still the number operator with integer eigenvalues. However, the requirement that the Hamiltonian be a function only of $N = a^\dagger a$ and without terms such as $a^{\dagger 2}$ or a^2 leads to $c^2 = \omega$, which then uniquely fixes the definition (4.17) up to an overall sign.

It would be worthwhile at this point to emphasize an important feature of the Heisenberg picture. We have a set of basis states $|i\rangle$ $(i = 0, 1, 2, \ldots)$ which forms a complete set in the Hilbert space that represents a single oscillator. In the Heisenberg picture, these states do not vary with time. The time evolution corresponding to any of these states, or any superpositions thereof, are contained in the set of time-dependent observables such as $q(t)$, $p(t)$, etc. It is important to note that the same set

of operators with the same time dependences takes care of any states. For example, one can form a wave packet by a linear combination of the basis states where the probability to find the mass is localized as a bump near a certain position at a given time. Such a bump would follow a motion approximating a classical oscillation. In the Heisenberg picture, such motion is representated by a stationary state in the Hilbert space, and different motions (e.g., different amplitudes of the oscillation) are represented by different static states. The operators and their time dependences are identical for all these states representing different motions; $q(t)$, for example, is always given by (4.37).

Fermionic oscillator

The quantized harmonic oscillator we have just studied will serve as a normal mode of a field corresponding to a given momentum where the number of quanta of the oscillator is identified as the number of particles with that momentum. It works fine for particles with an integer spin (spin 0, 1, ...: called *bosons*) for which a given state can be occupied by any number of such particles. However, particles with a half-integer spin (1/2, 3/2, ...: called *fermions*) cannot occupy an already occupied state, and some modification is needed to limit the number of quanta for a given oscillator to one. This can be accomplished by replacing the commutators among a and $a^\dagger$ by anticommutators:

$$\begin{aligned} & [a, a^\dagger] = 1, \quad [a, a] = [a^\dagger, a^\dagger] = 0 \\ \rightarrow \quad & \boxed{\{a, a^\dagger\} = 1, \quad \{a, a\} = \{a^\dagger, a^\dagger\} = 0} \,. \end{aligned} \tag{4.38}$$

The anticommutation relations $\{a, a\} = \{a^\dagger, a^\dagger\} = 0$ mean that the square of a or $a^\dagger$ is zero:

$$a^2 = 0\,, \quad a^{\dagger 2} = 0\,. \tag{4.39}$$

Define the number operator as before: $N \equiv a^\dagger a$, then,

$$\begin{aligned} N^2 &= a^\dagger a \underbrace{a^\dagger a}_{1 - a a^\dagger} = a^\dagger a - a^\dagger \underbrace{a a}_{0} a^\dagger = N \\ &\rightarrow \; N(N-1) = 0 \end{aligned} \tag{4.40}$$

which means that N has eigenvalues 0 and 1 as promised. Let $|0\rangle$ be the eigenstate with eigenvalue 0

$$N|0\rangle = 0 \tag{4.41}$$

which is assumed to be normalized as

$$\langle 0|0\rangle = 1\,. \tag{4.42}$$

Using the anticommutation relation (4.38), we have

$$Na^\dagger = \underbrace{a^\dagger a}_{1-aa^\dagger}\, a^\dagger = a^\dagger - a\underbrace{a^{\dagger 2}}_{0} = a^\dagger \,. \tag{4.43}$$

Applying this to $|0\rangle$, we get

$$Na^\dagger|0\rangle = a^\dagger|0\rangle \,, \tag{4.44}$$

which indicates the identification,

$$|1\rangle = a^\dagger|0\rangle \,, \tag{4.45}$$

if it is not a null state. In fact, its norm is unity:

$$\langle 1|1\rangle = \langle 0|\underbrace{a\,a^\dagger}_{1-a^\dagger a}|0\rangle = \langle 0|(1-\underbrace{N}_{0})|0\rangle = \langle 0|0\rangle = 1 \,. \tag{4.46}$$

Applying a to (4.45),

$$\begin{aligned} a\,|1\rangle &= \underbrace{aa^\dagger}_{1-a^\dagger a}\,|0\rangle = (1-\underbrace{N}_{0})|0\rangle \\ &\rightarrow\ |0\rangle = a\,|1\rangle \,. \end{aligned} \tag{4.47}$$

Equations (4.45) and (4.47) show that a and $a^\dagger$ are acting as an annihilation operator and a creation operator, respectively. If we try to lower the eigenvalue of $|0\rangle$ or raise the eigenvalue of $|1\rangle$, then the state vanishes:

$$\begin{cases} a^\dagger|1\rangle = a^{\dagger 2}|0\rangle = 0 \\ a\,|0\rangle = a^2|1\rangle = 0 \end{cases} . \tag{4.48}$$

The Hamiltonian should be proportional to the number operator up to a constant offset which may be ignored:

$$H = \omega a^\dagger a \,, \tag{4.49}$$

where the proportionality coefficient ω is identified as the energy of one quanta. Heisenberg's equation of motion (using a commutator) is assumed to be still valid; then, the time dependence of a is given by

$$\begin{aligned} -i\dot{a} &= [H,a\,] = \omega[a^\dagger a, a] = \omega(a^\dagger \underbrace{a\,a}_{0} - \underbrace{aa^\dagger}_{1-a^\dagger a}\, a) = \omega(-a + a^\dagger \underbrace{a\,a}_{0}) = -\omega a \\ &\rightarrow \dot{a} = -i\omega a \,. \end{aligned} \tag{4.50}$$

Thus, a has the same time dependence as in the boson case:

$$a(t) = ae^{-i\omega t} \,, \tag{4.51}$$

where a without explicit time dependence '(t)', in this equation and the rest of this section, is simply understood to be the value of the operator at $t = 0$.

Thus, an oscillator that satisfies Pauli's exclusion principle can be constructed using anticommutators for the annihilation and creation operators a and $a^\dagger$. Such an oscillator is sometimes called a fermionic oscillator. Before leaving the topic, we note that the above formulation of the fermionic oscillator is symmetric under the simultaneous exchanges $|0\rangle \leftrightarrow |1\rangle$ and $a \leftrightarrow a^\dagger$. Namely, if we define

$$\begin{cases} a' \equiv a^\dagger \\ a'^\dagger = a \end{cases} , \quad \begin{cases} |0\rangle' \equiv |1\rangle \\ |1\rangle' \equiv |0\rangle \end{cases} , \tag{4.52}$$

then, a' and $a'^\dagger$ satisfy the same anticommutation relations (4.38) and the raising (4.45) and lowering (4.47) relations:

$$\{a', a'^\dagger\} = 1, \quad \{a', a'\} = \{a'^\dagger, a'^\dagger\} = 0\,, \tag{4.53}$$

$$a'^\dagger|0\rangle' = |1\rangle'\,, \quad a'|1\rangle' = |0\rangle'\,. \tag{4.54}$$

If we define the new number operator by

$$N' \equiv a'^\dagger a' = aa^\dagger\,, \tag{4.55}$$

the anticommutation relation $\{a, a^\dagger\} = 1$ reads

$$N' + N = 1\,. \tag{4.56}$$

Then, the relabeled eigenstates are correctly labeled by the eigenvalues of N':

$$N'|0\rangle' = (1 - \underbrace{N}_{1})|1\rangle = 0\,, \quad N'|1\rangle' = (1 - \underbrace{N}_{0})|0\rangle = |1\rangle'\,. \tag{4.57}$$

Thus, it is arbitrary which is the occupied state and which is the empty state at this point. Later, we will define the state with lower energy to be $|0\rangle$.

Exercise 4.2 *Matrix representation of fermionic oscillator.*
In the space spanned by $|0\rangle$ and $|1\rangle$, a general state $|\psi\rangle = a|0\rangle + b|1\rangle$ can be expressed as a column vector as

$$|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix} .$$

Use the raising and lowering relations

$$a|1\rangle = |0\rangle\ , \quad a^\dagger|0\rangle = |1\rangle\ ,$$

and

$$a^\dagger|1\rangle = 0;\,, \quad a|0\rangle = 0\ ,$$

together with the normalizations

$$\langle 1|1\rangle = \langle 0|0\rangle = 1 \ ,$$

express the operators a and $a^\dagger$ as 2 by 2 matrices, and explicitly verify the anti-commutation relations

$$\{a, a^\dagger\} = 1, \quad \{a, a\} = \{a^\dagger, a^\dagger\} = 0 \ .$$

4.2 Lagrangian formulation of classical wave

As stated earlier, the first step toward quantizing a field is to find the Lagrangian of the system, then we can proceed to derive the canonical momenta from it, form the Hamiltonian, and then introduce commutation relations between the coordinates and momenta to quantize the system. Thus, let's first find the Lagrangian of a simple vibrating material. At this stage, it has nothing to do with quantum mechanics; it is just old classical mechanics.

Lagrangian density and the Euler-Lagrange equation
Consider a string of masses (m each) as shown in Figure 4.1. Each mass is attached to a spring with a spring constant k. Each mass is constrained to move vertically without friction, and the displacement of i-th mass from the natural position is denoted as ϕ_i, which can also be labeled by the position along the string

$$\phi_i \equiv \phi(x_i) \ . \tag{4.58}$$

In order for waves to propagate, there should be some kind of coupling between the masses which is provided by a rubber band stretched with a constant tension τ. The tension is assumed to be independent of the vertical displacements or the slope of the rubber band which are assumed to be small. All these assumptions seem rather artificial; the only essential elements, however, are that the string has some uniform

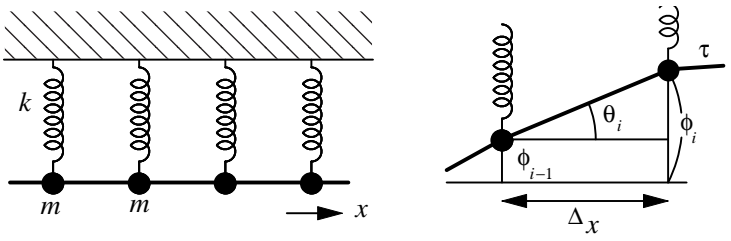


Figure 4.1: A string of springs, each with mass m and spring constant k. The masses are connected by a rubber band of tension τ.

mass and that the potential has one term that is proportional to ϕ^2 and another term that is proportional to $(\partial\phi/\partial x)^2$, which can lead to the Klein-Gordon equation by adjusting parameters. It is the case for our system as we will see shortly. When measured relative to the minimum potential (the natural state), these terms are the lowest order terms when the potential is expanded in terms of ϕ and $\partial\phi/\partial x$. The higher-order terms can be separately treated as self interactions of the field, and ignored at this point.

Assuming that the angles θ_i's (see Figure 4.1) are small, the total potential energy of the rubber band relative to the natural state is

$$\begin{aligned} V_{\text{rubber}} &= \sum_i \tau \underbrace{\Delta x\Big(\overbrace{\frac{1}{\cos\theta_i} - 1}^{\theta_i^2/2}\Big)}_{\text{rubber stretch length}} = \sum_i \tau\Delta x\frac{\theta_i^2}{2} \\ &= \sum_i \tau\Delta x\frac{1}{2}\Big(\frac{\phi_i - \phi_{i-1}}{\Delta x}\Big)^2 , \end{aligned} \tag{4.59}$$

and that of the springs is

$$V_{\text{spring}} = \sum_i \frac{1}{2}k\phi_i^2 . \tag{4.60}$$

The Lagrangian of the system is then

$$\begin{aligned} L(\underline{\phi}, \dot{\underline{\phi}}) &= T - V_{\text{spring}} - V_{\text{rubber}} \\ &= \sum_i \Big[\frac{1}{2}m\dot{\phi}_i^2 - \frac{1}{2}k\phi_i^2 - \frac{1}{2}\tau\Delta x\Big(\frac{\phi_i - \phi_{i-1}}{\Delta x}\Big)^2\Big] \end{aligned} \tag{4.61}$$

with

$$\underline{\phi} \equiv (\phi_1, \ldots, \phi_n) . \tag{4.62}$$

In terms of the mass per unit length μ and the spring constant per unit length κ, m and k can be written as

$$m = \mu\Delta x , \quad k = \kappa\Delta x . \tag{4.63}$$

The Lagrangian is then

$$L(\underline{\phi}, \dot{\underline{\phi}}) = \sum_i \Delta x\Big[\frac{\mu}{2}\dot{\phi}_i^2 - \frac{\kappa}{2}\phi_i^2 - \frac{\tau}{2}\Big(\frac{\phi_i - \phi_{i-1}}{\Delta x}\Big)^2\Big] . \tag{4.64}$$

Lagrange's equation of motion is

$$\begin{aligned} \frac{d}{dt}\Big(\frac{\partial L}{\partial\dot{\phi}_i}\Big) &= \frac{\partial L}{\partial\phi_i} \quad (i = 1, \ldots, n) \\ \rightarrow \quad \mu\ddot{\phi}_i &= -\kappa\phi_i - \tau\frac{(\phi_i - \phi_{i-1}) - (\phi_{i+1} - \phi_i)}{(\Delta x)^2} , \end{aligned} \tag{4.65}$$

where the tension term in $\partial L/\partial\phi_i$ resulted from two terms corresponding to i and $i+1$. This is a set of n differential equations that are coupled. Let's take the limit $n \to \infty$ and $\Delta x \to 0$ while keeping $n\Delta x$, μ and κ constant. Noting that in this limit we have

$$\frac{(\phi_i - \phi_{i-1})}{\Delta x} = \frac{\partial\phi}{\partial x}(x_i) \tag{4.66}$$

the tension term in (4.65) can be written as

$$\begin{aligned}\frac{(\phi_i - \phi_{i-1}) - (\phi_{i+1} - \phi_i)}{(\Delta x)^2} &= \frac{\dfrac{(\phi_i - \phi_{i-1})}{\Delta x} - \dfrac{(\phi_{i+1} - \phi_i)}{\Delta x}}{\Delta x} \\ &= \frac{\dfrac{\partial\phi}{\partial x}(x_i) - \dfrac{\partial\phi}{\partial x}(x_{i+1})}{\Delta x} = -\frac{\partial^2\phi}{\partial x^2}(x_i)\,,\end{aligned} \tag{4.67}$$

and the equation of motion is now written in a differential form:

$$\mu\ddot{\phi}(x) = -\kappa\phi(x) + \tau\frac{\partial^2\phi}{\partial x^2}(x)\,. \tag{4.68}$$

To find the solution, let us try the plane wave form

$$\phi = \phi_0 e^{i(\omega t + kx)} \quad (\phi_0 : \text{constant}) \tag{4.69}$$

in the above differential equation. One obtain,

$$\mu\omega^2 - \tau k^2 = \kappa\,. \tag{4.70}$$

Namely, the solution is a plane wave whose oscillation frequency ω and wave number k satisfies the above relation.

In the limit $n \to \infty$, the Lagrangian (4.64) can be written as an integral over x. Using (4.66),

$$L(\underline{\phi}, \underline{\dot{\phi}}) = \int dx\, \mathcal{L}\Big(\phi, \dot{\phi}, \frac{\partial\phi}{\partial x}\Big) \tag{4.71}$$

with

$$\mathcal{L}\Big(\phi, \dot{\phi}, \frac{\partial\phi}{\partial x}\Big) = \frac{\mu}{2}\dot{\phi}^2 - \frac{\kappa}{2}\phi^2 - \frac{\tau}{2}\Big(\frac{\partial\phi}{\partial x}\Big)^2\,. \tag{4.72}$$

The range of integration could be either $(-\infty, \infty)$ or $(-L/2, L/2)$ where L is some large length. If we take the range to be infinity, then we assume that the function $\phi(x)$ vanishes at sufficiently large distances from origin. If we take the range to be finite, we impose periodicity condition:

$$\phi(x + L) = \phi(x)\,. \tag{4.73}$$

These conditions are required so that the boundary values do not contribute in the partial integrations that occur in what follows. The integrand $\mathcal{L}(\phi, \dot{\phi}, \partial\phi/\partial x)$ is a function of the field value $\phi(x)$, its time derivative $\dot{\phi}(x)$ and of the spacial derivative $\partial\phi/\partial x(x)$, and called the *Lagrangian density*.

The equation of motion (4.68) was obtained by taking the limit $\Delta x \to 0$ of the equation of motion for the discrete positions. It is convenient to establish a formula which gives us the continuous equation of motion directly from the Lagrangian density. To do so, we start from the action principle itself. The action S is the time integral of the Lagrangian from time t_1 to t_2 which are the start and end of the motion of interest:

$$S \equiv \int_{t_1}^{t_2} L\, dt = \int_{t_1}^{t_2} dt \int dx\, \mathcal{L}\Big(\phi, \dot{\phi}, \frac{\partial\phi}{\partial x}\Big)\,. \tag{4.74}$$

Suppose we fix the shape of the string at t_1 and t_2; namely, $\phi(t_1, x)$ and $\phi(t_2, x)$ are given. Between t_1 and t_2, the function $\phi(t, x)$ specifies the motion of the string which may or may not be a true realizable motion, and for each motion, there is *a real number* S associated with it as defined by (4.74). The action S is thus a mapping of a function $\phi(t, x)$ to a number; such mapping is called a *functional*. The action principle then tells us that the true motion is the one that has the smallest S. Or equivalently, when we vary the function slightly around the true motion,

$$\phi'(t, x) = \phi(t, x) + \delta\phi(t, x) \tag{4.75}$$

with the variation set to zero at the start and end of the time window,

$$\delta\phi(t_1, x) = \delta\phi(t_2, x) = 0\,, \tag{4.76}$$

then the change in S should be zero to the first order in $\delta\phi$:

$$\delta S \equiv S' - S = 0\,, \tag{4.77}$$

where

$$S = \int dtdx \mathcal{L}\Big(\phi, \dot{\phi}, \frac{\partial\phi}{\partial x}\Big) \quad \text{and} \quad S' = \int dtdx \mathcal{L}\Big(\phi', \dot{\phi}', \frac{\partial\phi'}{\partial x}\Big)\,. \tag{4.78}$$

The range of time integration is understood to be (t_1, t_2).

We can write δS as

$$\delta S = \int dtdx\, \delta\mathcal{L}(t, x) = 0 \quad \text{with} \quad \delta\mathcal{L} = \mathcal{L}\Big(\phi', \dot{\phi}', \frac{\partial\phi'}{\partial x}\Big) - \mathcal{L}\Big(\phi, \dot{\phi}, \frac{\partial\phi}{\partial x}\Big)\,. \tag{4.79}$$

On the other hand, the change in $\mathcal{L}$ at given time and position (t, x) occurs through the changes in ϕ, $\dot{\phi}$, and $\partial\phi/\partial x$, each evaluated at the point (t, x):

$$\delta\mathcal{L}(t, x) = \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial\dot{\phi}}\delta(\dot{\phi}) + \frac{\partial\mathcal{L}}{\partial\big(\frac{\partial\phi}{\partial x}\big)}\delta\Big(\frac{\partial\phi}{\partial x}\Big)\,. \tag{4.80}$$

Since $\phi' = \phi + \delta\phi$,

$$\delta(\dot{\phi}) \equiv \dot{\phi}' - \dot{\phi} = (\dot{\delta\phi}), \qquad \delta\Big(\frac{\partial\phi}{\partial x}\Big) \equiv \frac{\partial\phi'}{\partial x} - \frac{\partial\phi}{\partial x} = \frac{\partial}{\partial x}(\delta\phi)\,, \tag{4.81}$$

then using the chain rule $A(\frac{\partial}{\partial s}B) = \frac{\partial}{\partial s}(AB) - (\frac{\partial}{\partial s}A)B$ with $s = t$ or x,

$$\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\underbrace{\delta(\dot{\phi})}_{(\dot{\delta\phi})} = \underbrace{\frac{\partial}{\partial t}\Big(\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\delta\phi\Big)}_{\to 0} - \Big(\frac{\partial}{\partial t}\Big(\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\Big)\Big)\delta\phi \tag{4.82}$$

$$\frac{\partial\mathcal{L}}{\partial\Big(\frac{\partial\phi}{\partial x}\Big)}\underbrace{\delta\Big(\frac{\partial\phi}{\partial x}\Big)}_{\frac{\partial}{\partial x}(\delta\phi)} = \underbrace{\frac{\partial}{\partial x}\Big(\frac{\partial\mathcal{L}}{\partial\Big(\frac{\partial\phi}{\partial x}\Big)}\delta\phi\Big)}_{\to 0} - \Big(\frac{\partial}{\partial x}\Big(\frac{\partial\mathcal{L}}{\partial\Big(\frac{\partial\phi}{\partial x}\Big)}\Big)\Big)\delta\phi \tag{4.83}$$

where the first term in the right hand side of (4.82) or (4.83) vanishes upon integration over t or x, respectively, due to the boundary conditions (4.76) and (4.73). Thus, δS can now be written as

$$\delta S = \int dtdx\Big[\frac{\partial\mathcal{L}}{\partial\phi} - \frac{\partial}{\partial t}\Big(\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\Big) - \frac{\partial}{\partial x}\Big(\frac{\partial\mathcal{L}}{\partial\Big(\frac{\partial\phi}{\partial x}\Big)}\Big)\Big]\delta\phi = 0\,. \tag{4.84}$$

Since δS should vanish for any variation $\delta\phi$, the integrand $[\cdots]$ should be zero at all (t, x):

$$\frac{\partial}{\partial t}\Big(\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\Big) + \frac{\partial}{\partial x}\Big(\frac{\partial\mathcal{L}}{\partial\Big(\frac{\partial\phi}{\partial x}\Big)}\Big) = \frac{\partial\mathcal{L}}{\partial\phi}\,, \tag{4.85}$$

which is called the *Euler-Lagrange equation* for the Lagrangian density. Applying this to the Lagrangian density (4.72), we obtain an equation of motion

$$\frac{\partial}{\partial t}\Big(\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\Big) = \mu\ddot{\phi}\,, \quad \frac{\partial}{\partial x}\Big(\frac{\partial\mathcal{L}}{\partial\Big(\frac{\partial\phi}{\partial x}\Big)}\Big) = -\tau\frac{\partial^2\phi}{\partial x^2}\,, \quad \frac{\partial\mathcal{L}}{\partial\phi} = -\kappa\phi$$

$$\to \quad \mu\ddot{\phi} - \tau\frac{\partial^2\phi}{\partial x^2} = -\kappa\phi\,, \tag{4.86}$$

which is exactly the same equation as (4.68) which was obtained directly from $L(\underline{\phi}, \underline{\dot{\phi}})$ in the limit $\Delta x \to 0$.

When the field is labeled by a continuous parameter as in $\phi(x)$, the total Lagrangian is not convenient to deal with. This is because Lagrange's equation of motion for the total Lagrangian is written in terms of each discrete coordinate ϕ_i and not in terms of $\phi(x)$ [see (4.65)]. In the continuous limit, a natural object is the Lagrangian density which allows one to derive the relevant equation of motion directly.

It is easy to extend the Euler-Lagrange equation (4.85) to three dimensions. Now the field is a function of $\vec{x}$, and $\vec{\nabla}\phi$ replaces $\partial\phi/\partial x$ in the argument of the Lagrangian density:

$$\mathcal{L}(\phi, \dot{\phi}, \vec{\nabla}\phi) = \mathcal{L}(\phi, \partial_\mu \phi)\,.$$

Then in (4.80), $\delta\mathcal{L}$ needs to be expanded in terms of $\delta(\partial\phi/\partial y)$ and $\delta(\partial\phi/\partial z)$ in addition to $\delta(\partial\phi/\partial x)$, and following through the same derivation results in

$$\frac{\partial}{\partial t}\Big(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\Big) + \frac{\partial}{\partial x}\Big(\frac{\partial \mathcal{L}}{\partial\big(\frac{\partial\phi}{\partial x}\big)}\Big) + \frac{\partial}{\partial y}\Big(\frac{\partial \mathcal{L}}{\partial\big(\frac{\partial\phi}{\partial y}\big)}\Big) + \frac{\partial}{\partial z}\Big(\frac{\partial \mathcal{L}}{\partial\big(\frac{\partial\phi}{\partial z}\big)}\Big) = \frac{\partial \mathcal{L}}{\partial \phi}$$

or

$$\partial_\mu\Big(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\Big) = \frac{\partial \mathcal{L}}{\partial \phi}\,, \tag{4.87}$$

which is the Euler-Lagrange equation now written in a space-time symmetric form.

When there are more than one fields, $\phi_1(x), \ldots, \phi_N(x)$, the Lagrangian density becomes a function of these fields and their space time derivatives:

$$\boxed{\mathcal{L}\big(\tilde{\phi}, \partial_\mu\tilde{\phi}\big)\,, \qquad \tilde{\phi}(x) \equiv (\phi_1(x), \ldots, \phi_N(x))\,,\ x^\mu = (t, \vec{x})}\,. \tag{4.88}$$

Then the Lagrangian and the action are given by

$$L = \int d^3x \mathcal{L}\,, \qquad S = \int d^4x \mathcal{L}\,, \tag{4.89}$$

where the integration range over time is (t_1, t_2), and the space integral is over the entire space (infinite), or within a cube defined by $-L/2 < x_i < L/2$ $(i = 1, 2, 3)$. If the range is infinite, then the fields are assumed to vanish at sufficiently large distances from the origin:[1]

$$\tilde{\phi}(t, \vec{x}) = 0 \quad (|\vec{x}| \to \infty)\,, \tag{4.90}$$

or if the range is finite, then the periodicity is imposed:

$$\tilde{\phi}(t, x+L, y, z) = \tilde{\phi}(t, x, y+L, z) = \tilde{\phi}(t, x, y, z+L) = \tilde{\phi}(t, x, y, z)\,. \tag{4.91}$$

The corresponding Euler-Lagrange equation is obtained by taking the variation with respect to only one field ϕ_k in (4.80) while keeping other fields fixed. Then the same derivation leads to

$$\boxed{\partial_\mu\Big(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi_k)}\Big) = \frac{\partial \mathcal{L}}{\partial \phi_k} \quad (k = 1, \ldots, N)}\,. \tag{4.92}$$

[1]Strictly speaking, what is required of how quickly $\phi(t, \vec{x})$ vanishes as $|\vec{x}| \to \infty$ is that surface terms such as $\int d^3x \vec{\nabla}\phi$ vanish when integrated over all space. It is usually sufficient if ϕ vanishes faster than $1/\vec{x}^2$.

This form indicates that if the Lagrangian density $\mathcal{L}$ is a Lorentz scalar, then the resulting equation of motion is also Lorentz-invariant assuming that ϕ_k transforms in a well-defined way on both sides of the equation (scalar field, vector field, etc.).

Hamiltonian density
Let's start from the one-dimensional discrete system whose total Lagrangian was given by $L(\underline{\phi}, \underline{\dot{\phi}})$ (4.64) with $\underline{\phi} = \phi_i$ $(i = 1, \ldots, n)$. One should not confuse the index i in ϕ_i, which labels the position along the string, and the index k in $\phi_k(t, \vec{x})$ which labels the field. Here, we have only one field. The canonical momentum corresponding to the i-th general coordinate ϕ_i is

$$\pi_i \stackrel{\text{def}}{\equiv} \frac{\partial L}{\partial \dot{\phi}_i} = \Delta x \mu \dot{\phi}_i \,, \tag{4.93}$$

and the Hamiltonian is by definition

$$\begin{aligned} H &\equiv \sum_i \pi_i \dot{\phi}_i - L \\ &= \sum_i \Delta x \mu \dot{\phi}_i^2 - \sum_i \Delta x \Big[\frac{\mu}{2} \dot{\phi}_i^2 - \frac{\kappa}{2} \phi_i^2 - \frac{\tau}{2} \Big(\frac{\phi_i - \phi_{i-1}}{\Delta x} \Big)^2 \Big] \\ &= \sum_i \Delta x \Big[\frac{\mu}{2} \dot{\phi}_i^2 + \frac{\kappa}{2} \phi_i^2 + \frac{\tau}{2} \Big(\frac{\phi_i - \phi_{i-1}}{\Delta x} \Big)^2 \Big] \,. \end{aligned} \tag{4.94}$$

This is kinetic energy + potential energy, and thus it is the total energy of the system. In the continuous limit, it becomes

$$H = \int dx \Big[\frac{\mu}{2} \dot{\phi}^2 + \frac{\kappa}{2} \phi^2 + \frac{\tau}{2} \Big(\frac{\partial \phi}{\partial x} \Big)^2 \Big] \,. \tag{4.95}$$

The same result can be obtained by the following general procedure: first, the canonical field conjugate to ϕ is defined using the Lagrangian density $\mathcal{L}(\phi, \dot{\phi}, \partial\phi/\partial x)$ as

$$\pi(x) \stackrel{\text{def}}{\equiv} \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \mu \dot{\phi}(x) \,, \tag{4.96}$$

and the *Hamiltonian density* $\mathcal{H}$ is defined by

$$\begin{aligned} \mathcal{H}\Big(\pi, \phi, \frac{\partial \phi}{\partial x} \Big) &\stackrel{\text{def}}{\equiv} \pi \dot{\phi} - \mathcal{L}\Big(\phi, \dot{\phi}, \frac{\partial \phi}{\partial x} \Big) \\ &= \mu \dot{\phi}^2 - \Big[\frac{\mu}{2} \dot{\phi}^2 - \frac{\kappa}{2} \phi^2 - \frac{\tau}{2} \Big(\frac{\partial \phi}{\partial x} \Big)^2 \Big] \\ &= \frac{\mu}{2} \dot{\phi}^2 + \frac{\kappa}{2} \phi^2 + \frac{\tau}{2} \Big(\frac{\partial \phi}{\partial x} \Big)^2 \end{aligned} \tag{4.97}$$

The total Hamiltonian is then its space integral:

$$H = \int dx \mathcal{H}\,, \tag{4.98}$$

which gives the same Hamiltonian as (4.95).

Extension to multiple fields in 3-dimension is straightforward. For a set of fields $\tilde{\phi} = (\phi_1(x), \ldots, \phi_N(x))$, the conjugate fields are defined by

$$\boxed{\pi_k(x) \stackrel{\text{def}}{\equiv} \frac{\partial \mathcal{L}\left(\tilde{\phi}, \partial_\mu \tilde{\phi}\right)}{\partial \dot{\phi}_k} \qquad (k = 1, \ldots, N)}\,; \tag{4.99}$$

then the total Hamiltonian is then given by

$$\boxed{H = \int d^3x \mathcal{H}}\,, \tag{4.100}$$

with the Hamiltonian density defined by

$$\boxed{\mathcal{H}\left(\tilde{\pi}, \tilde{\phi}, \vec{\nabla}\tilde{\phi}\right) \stackrel{\text{def}}{\equiv} \sum_k \pi_k \dot{\phi}_k - \mathcal{L}}\,. \tag{4.101}$$

Noether currents

One advantage of Lagrangian formulation is that symmetry of the system it describes is evident in the form of the Lagrangian. If certain transformation leaves the action invariant, then the resulting equations of motion will be the same before and after the transformation and thus the law of physics it describes will stay the same. In general, transformations that leave the action invariant are easy to spot in Lagrangian.

There exists a general theorem, called *Noether's theorem*, that states that for each transformation that leaves the action invariant, there exists a conserved quantity. We now discuss such conserved quantities corresponding to the symmetry under the space-time translations, which turn out to be the total energy and momentum of the system.

That the equations of motion (or the laws of physics) is the same anytime anywhere means that the Lagrangian does not depend on space-time explicitly. That is actually what we have been assuming since there is no explicit x^μ in the argument of $\mathcal{L}(\phi, \partial_\mu \phi)$. Suppose a function $\phi(x)$ represents a true motion. For a given such motion $\phi(x)$, the Lagrangian density $\mathcal{L}$ is a function of space-time though $\phi(x)$ and $\partial_\mu \phi(x)$. Applying the chain rule of differentiation,

$$\begin{array}{ccc} \underbrace{\partial_\nu \mathcal{L}} & = & \underbrace{\frac{\partial \mathcal{L}}{\partial \phi}}\, \partial_\nu \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \underbrace{\partial_\nu(\partial_\mu \phi)} \\ g^\mu{}_\nu \partial_\mu \mathcal{L} & & \partial_\mu\Big(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\Big) \quad \text{by (4.87)} \end{array}$$

$$= \partial_\mu\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Big)\partial_\nu\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu(\partial_\nu\phi) = \partial_\mu\Big[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi\Big]\,. \tag{4.102}$$

$$\to \quad \partial_\mu\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi - \mathcal{L}\,g^\mu{}_\nu\Big) = 0\,. \tag{4.103}$$

Note that this holds only for a true motion $\phi(x)$ since we have used the Euler-Lagrange equation. We now have four conserved currents:

$$\partial_\mu J^{\mu\nu} = 0 \quad (\nu = 0,1,2,3)\,, \tag{4.104}$$

with

$$J^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial^\nu\phi - \mathcal{L}\,g^{\mu\nu}\,. \tag{4.105}$$

These conserved currents are called the *Noether currents* of the space-time translations. Then, the space integral of the time component of each current is conserved:

$$\partial_0 J^{0\nu} = -\partial_i J^{i\nu}$$

$$\to \quad \frac{\partial}{\partial t}\Big(\int d^3x J^{0\nu}\Big) = \int d^3x\,\partial_0 J^{0\nu} = -\int d^3x\,\partial_i J^{i\nu} = -\int_A da_i\,J^{i\nu} = 0 \tag{4.106}$$

where A is the boundary surface, da_i is the i-th component of the area element and the surface integral vanishes due to the boundary condition (4.90) or (4.91). The conserved quantities are thus

$$\boxed{P^\nu \overset{\text{def}}{\equiv} \int d^3x J^{0\nu} \quad (\nu = 0,1,2,3)}\,. \tag{4.107}$$

We can see that P^0 is nothing but the Hamiltonian (namely, the total energy):

$$P^0 \equiv \int d^3x J^{00} = \int d^3x \underbrace{\Big(\overbrace{\frac{\partial\mathcal{L}}{\partial\dot\phi}}^{\pi}\dot\phi - \mathcal{L}\Big)}_{\mathcal{H}} = H\,. \tag{4.108}$$

Then, it is natural to identify the space component $\vec{P}$ given by

$$P^i \equiv \int d^3x J^{0i} = \int d^3x\Big(\frac{\partial\mathcal{L}}{\partial\dot\phi}\,\partial^i\phi\Big) = \int d^3x\,\pi\,\partial^i\phi \tag{4.109}$$

as the total momentum of the system. In fact, this identification is required if we can prove that P^ν is a 4-vector, which, as it turns out, is not trivial. The main problem is that P^ν's are defined as space integrals in a given frame. In the following, we

will assume that $\mathcal{L}$ is a Lorentz scalar and show that P^ν transforms as a 4-vector under an infinitesimal Lorentz transformation which is sufficient for all proper and orthochronous transformations.

The following proof does not require that ϕ is a scalar, as long as fields are combined in $\mathcal{L}$ to make it a scalar quantity. That $\mathcal{L}$ is a Lorentz scalar means

$$\mathcal{L}'(x') = \mathcal{L}(x)\,, \quad x' = \Lambda x\,. \tag{4.110}$$

Then, since the quantity $\delta\mathcal{L}$ in (4.102) is a Lorentz scalar (it is the difference of the values of $\mathcal{L}$ at two event points, x and $x+\epsilon$) and ∂_μ and ϵ^ν are 4-vectors, the quantity inside the square bracket should transform as a tensor. Together with the fact that the metric tensor is itself a tensor (1.65), we see that $J^{\mu\nu}$ defined by (4.105) transforms as a tensor:

$$J'^{\mu\nu}(x') = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta J^{\alpha\beta}(x)\,. \tag{4.111}$$

Since P'^ν and P^ν are constants of motion, we choose to evaluate them at $t'=0$ and $t=0$, respectively:

$$P'^\nu \equiv \int d^3x' J'^{0\nu}(0,\vec{x}')\,, \quad P^\nu \equiv \int d^3x J^{0\nu}(0,\vec{x})\,. \tag{4.112}$$

For a proper transformation, we have $\det\Lambda = 1$ and thus

$$d^4x' = (\det\Lambda) d^4x = d^4x\,. \tag{4.113}$$

Also, using the property of the delta function

$$\delta(f(s)) = \sum_i \frac{1}{|f'(s_i)|}\delta(s-s_i)\,, \quad (s_i\text{: solutions of } f(s)=0)\,, \tag{4.114}$$

we have

$$\delta(t') = \delta(\Lambda^0{}_\rho\, x^\rho) = \frac{1}{\Lambda^0{}_0}\delta\Big(t + \frac{\Lambda^0{}_i x^i}{\Lambda^0{}_0}\Big)\,. \tag{4.115}$$

Then, P'^ν can be written as

$$\begin{aligned} P'^\nu &\equiv \int d^3x' J'^{0\nu}(0,\vec{x}') = \int \overbrace{d^4x'}^{d^4x}\, \underbrace{\delta(t')}_{\frac{1}{\Lambda^0{}_0}\delta\left(t+\frac{\Lambda^0{}_i x^i}{\Lambda^0{}_0}\right)}\, \overbrace{J'^{0\nu}(t',\vec{x}')}^{\Lambda^0{}_\alpha\Lambda^\nu{}_\beta\, J^{\alpha\beta}(t,\vec{x})} \\ &= \frac{1}{\Lambda^0{}_0}\int dt d^3x\, \delta\Big(t+\frac{\Lambda^0{}_i x^i}{\Lambda^0{}_0}\Big)\,\Lambda^0{}_\alpha\Lambda^\nu{}_\beta\, J^{\alpha\beta}(t,\vec{x}) \\ &= \frac{1}{\Lambda^0{}_0}\int d^3x\, \Lambda^0{}_\alpha\Lambda^\nu{}_\beta\, J^{\alpha\beta}\Big(-\tfrac{\Lambda^0{}_i x^i}{\Lambda^0{}_0}, \vec{x}\Big)\,. \end{aligned} \tag{4.116}$$

If Λ is an infinitesimal transformation, we can write

$$\Lambda^{\mu}{}_{\nu} = g^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu}\,, \tag{4.117}$$

where ω is a small parameter. In particular, we have [see (1.83)]

$$\omega^{0}{}_{0} = 0\,, \quad \rightarrow \quad \Lambda^{0}{}_{0} = 1\,, \tag{4.118}$$

which means

$$-\frac{\Lambda^{0}{}_{i}\,x^{i}}{\Lambda^{0}{}_{0}} = -(g^{0}{}_{i} + \omega^{0}{}_{i})\,x^{i} = -\omega^{0}{}_{i}\,x^{i}\,. \tag{4.119}$$

Then, to the first order in ω and expanding $J^{\alpha\beta}(t,\vec{x})$ about $t=0$, P'^{ν} becomes

$$\begin{aligned}
P'^{\nu} &= \int d^3x\,(g^{0}{}_{\alpha} + \omega^{0}{}_{\alpha})(g^{\nu}{}_{\beta} + \omega^{\nu}{}_{\beta})\,J^{\alpha\beta}(-(\omega^{0}{}_{i}\,x^{i}),\vec{x}) \\
&= \int d^3x\,(g^{0}{}_{\alpha}g^{\nu}{}_{\beta} + g^{0}{}_{\alpha}\omega^{\nu}{}_{\beta} + \omega^{0}{}_{\alpha}g^{\nu}{}_{\beta})\Big[J^{\alpha\beta}(0,\vec{x}) - (\partial_0 J^{\alpha\beta})(\omega^{0}{}_{i}\,x^{i})\Big] \\
&= \int d^3x\,\Big[J^{0\nu} + \omega^{\nu}{}_{\beta}J^{0\beta} + \underbrace{\omega^{0}{}_{\alpha}J^{\alpha\nu}}_{\omega^{0}{}_{i}J^{i\nu}} - \underbrace{(\partial_0 J^{0\nu})}_{-\partial_j J^{j\nu}}(\omega^{0}{}_{i}\,x^{i})\Big] \quad \text{by (4.104)} \\
&= \underbrace{\int d^3x J^{0\nu}}_{P^{\nu}} + \omega^{\nu}{}_{\beta}\underbrace{\int d^3x J^{0\beta}}_{P^{\beta}} + \int d^3x\,\omega^{0}{}_{i}J^{i\nu} + \int d^3x\,\omega^{0}{}_{i}\,\underbrace{(\partial_j J^{j\nu})x^{i}}_{\underbrace{\partial_j(J^{j\nu}x^{i})}_{\to 0} - J^{j\nu}\underbrace{\partial_j x^{i}}_{\delta_{ij}}} \\
&= P^{\nu} + \omega^{\nu}{}_{\beta}P^{\beta} + \int \cancel{d^3x\,\omega^{0}{}_{i}}J^{i\nu} - \int \cancel{d^3x\,\omega^{0}{}_{i}}J^{i\nu} \\
&= (g^{\nu}{}_{\beta} + \omega^{\nu}{}_{\beta})P^{\beta} \\
&= \Lambda^{\nu}{}_{\beta}P^{\beta}\,,
\end{aligned} \tag{4.120}$$

where all $J^{\mu\nu}$'s are understood to be evaluated at $(0,\vec{x})$ unless otherwise indicated. Thus, we have shown that P^{ν} transforms as a 4-vector under an infinitesimal Lorentz transformation. The proof can be easily extended to a finite transformation by dividing the finite tranformation into a large number of infinitesimal transformations. The four conserved quantities P^{ν} therefore form a 4-vector, and since the time component has been shown to be the total energy, we identify P^{ν} as the total energy-momentum 4-vector. The tensor quantity $J^{\mu\nu}$ is called the *energy-momentum tensor*.

Extension to multiple fields is again straightforward: for a set of *real* fields $\phi_k(x)$ $(k = 1,\ldots,N)$, the energy momentum tensor is given by

$$\boxed{J^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_k)}\partial^{\nu}\phi_k - \mathcal{L}\,g^{\mu\nu}\,.}\,, \tag{4.121}$$

and the total energy and momentum given by (4.107) are (using $\partial^i = -\nabla_i$)

$$\boxed{P^0 = H\,, \quad \vec{P} = -\int d^3x\, \pi_k \vec{\nabla}\phi_k}\,, \tag{4.122}$$

where π_k and H are given by (4.99) through (4.101), and summations over k are implied.

Note that P^0 and $\vec{P}$ defined above are conserved even when the Lagrangian density is not a Lorentz scalar. All needed for the proof of conservation was that the Lagrangian density does not contain x explicitly. If the Lagrangian density is a Lorentz scalar, then $P^\nu \equiv (P^0, \vec{P})$ transforms as a 4-vector.

Transition to Lorentz invariance

We are jumping ahead too fast, since we formed our Lagrangian density for a classical system which is at rest in certain frame and yet assumed that the resulting Lagrangian density is a Lorentz scaler. Let's go back to the equation of motion of a string (4.68), and discuss how we can make the system consistent with special relativity. If we set $\mu = \tau = 1$ in the equation of motion, we obtain

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} + \kappa\phi = 0\,. \tag{4.123}$$

Then, we see that its three-dimensional extension is nothing but the Klein-Gordon equation $(\partial^2 + m^2)\phi = 0$ with the identification $\kappa = m^2$. Here, the critical condition is $\mu = \tau$ which allows the time and space derivatives to be combined into ∂^2. Then, the equation of motion becomes Lorentz-invariant *provided* that we define the field $\phi(x)$ to be a scalar field:

$$\phi'(x') = \phi(x) \quad (x' = \Lambda x)\,. \tag{4.124}$$

Just as we discarded the idea of *ether* that transmits light and took the electromagnetic fields and their transformation properties as the reality, we forget the classical string at this point and take the field $\phi(x)$ and its transformation property (4.124) as reality. As we will see shortly in the next section, the Lagrangian density indeed becomes a Lorentz scaler with this transformation property of ϕ and the choice of parameters $\mu = \tau = 1$.

4.3 Quantization of the Klein-Gordon field

Canonical quantization

We are now ready to quantize the Klein-Gordon field following the standard canonical

quantization procedure. First, the Lagrangian density of the Klein-Gordon field is obtained by setting $\mu = \tau = 1$ and $k = m^2$ in (4.72) and extending it to 3-dimension

$$\mathcal{L}(\phi, \partial_\mu \phi) = \frac{1}{2}\Big(\dot{\phi}^2 - (\vec{\nabla}\phi)^2 - m^2\phi^2\Big) \tag{4.125}$$

or

$$\boxed{\mathcal{L}(\phi, \partial_\mu \phi) = \frac{1}{2}\Big(\partial_\mu \phi\, \partial^\mu \phi - m^2\phi^2\Big)}\,. \tag{4.126}$$

If the above extension to the three-dimensional case is somewhat unclear, it is justified by the fact that this Lagrangian density indeed leads to the Klein-Gordon equation: the Euler-Lagrange equation (4.92) gives

$$\underbrace{\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right)}_{\partial^\mu \phi\ :\ \text{note the factor of 2}} - \overbrace{\frac{\partial \mathcal{L}}{\partial \phi}}^{-m^2\phi} = (\partial^2 + m^2)\phi = 0\,. \tag{4.127}$$

From the definition (4.99), the conjugate field is

$$\boxed{\pi(x) \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}\,,} \tag{4.128}$$

and the Hamiltonian density defined by (4.101) is then

$$\boxed{\mathcal{H}(\pi, \phi, \vec{\nabla}\phi) = \frac{1}{2}\Big(\pi^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2\Big)}\,. \tag{4.129}$$

Note that the field $\phi(x)$ is a real number as in the case of the one-dimensional string. So far, this is just classical mechanics. The only difference from the string is that the choice of parameters as well as the assumption that $\phi(x)$ is a scalar has made the Lagrangian density a Lorentz scaler; namely, when $\mathcal{L}$ is regarded as a function of x through $\phi(x)$, the value of $\mathcal{L}$ evaluated at x is the same as the value of $\mathcal{L}$ in the trassformed frame at $x' = \Lambda x$: Using $\phi'(x') = \phi(x)$ and $\partial'_\mu = \Lambda_\mu{}^\alpha \partial_\alpha$,

$$\begin{aligned}
\mathcal{L}'(x') &\equiv \frac{1}{2}\Big(\underbrace{\partial'_\mu \phi'(x')\, \partial^{\mu'} \phi'(x')}_{\underbrace{\Lambda_\mu{}^\alpha \Lambda^\mu{}_\beta}_{g^\alpha{}_\beta}\, \partial_\alpha \phi(x) \partial^\beta \phi(x)} - m^2 \phi'^2(x')\Big) \\
&= \frac{1}{2}\Big(\partial_\mu \phi(x)\, \partial^\mu \phi(x) - m^2 \phi^2(x)\Big) = \mathcal{L}(x)\,.
\end{aligned} \tag{4.130}$$

Now we will turn the system into a quantum mechanical system by regarding $\phi(t,\vec{x})$ and $\pi(t,\vec{x})$ as operators in the Heisenberg picture, and introducing commutation relations among them. Here, ϕ's and π's at different positions are considered independent observables.

For clarity and simplicity, let's start from the discrete one-dimensional case. The total Lagrangian is (4.64) with $\mu=\tau=1$ and $\kappa=m^2$:

$$L(\underline{\phi},\underline{\dot{\phi}}) = \sum_i \Delta x \frac{1}{2}\Big[\dot{\phi}_i^2 - m^2\phi_i^2 - \Big(\frac{\phi_i-\phi_{i-1}}{\Delta x}\Big)^2\Big]\,. \tag{4.131}$$

The operator conjugate to ϕ_i is then

$$\pi_i(t) \equiv \frac{\partial L}{\partial\dot{\phi}_i} = \Delta x\dot{\phi}_i(t)\,. \tag{4.132}$$

The equal-time commutation relations are then well-defined; namely, the commutator of ϕ_i and π_i is i if they are conjugate to each other (i.e. have same index i) and zero otherwise:

$$[\phi_i(t),\pi_j(t)] = i\delta_{ij}\,, \tag{4.133}$$

$$[\phi_i(t),\phi_j(t)] = [\pi_i(t),\pi_j(t)] = 0\,. \tag{4.134}$$

The continuous limit of the conjugate field is defined directly from the lagrangian density (4.72) with $\mu=\tau=1$ and $\kappa=m^2$:

$$\mathcal{L}\Big(\phi,\dot{\phi},\frac{\partial\phi}{\partial x}\Big) = \frac{1}{2}\Big[\dot{\phi}^2 - m^2\phi^2 - \Big(\frac{\partial\phi}{\partial x}\Big)^2\Big]\,. \tag{4.135}$$

and given by

$$\pi(t,x) \equiv \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi}(t,x) \tag{4.136}$$

Together with the definition $\phi_i(t)\equiv\phi(t,x_i)$ (4.58), the relations connecting the continuous and discrete operators are:

$$\phi_i(t) = \phi(t,x_i)\,,\quad \pi_i(t) = \Delta x\,\pi(t,x_i)\,. \tag{4.137}$$

Thus, the commutation relation (4.133) can be written as

$$[\phi(t,x_i),\pi(t,x_j)] = i\frac{\delta_{ij}}{\Delta x}\,. \tag{4.138}$$

For a finite Δx, $\phi(t,x)$ is considered to have the value $\phi_i(t)$ for $x_i-\Delta x/2 < x < x_i+\Delta x/2$, and similarly for $\pi(t,x)$ (Figure 4.2). Then, for a given i, the value of $[\phi(t,x),\pi(x_i)]$ will be $[\phi_i(t),\pi_i(t)] = i/\Delta x$ for $x_i-\Delta x/2 < x < x_i+\Delta x/2$.

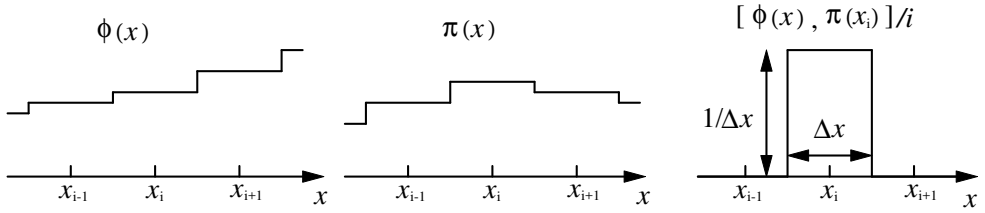


Figure 4.2: The operator functions $\phi(t,x)$ and $\pi(t,x)$ are shown symbolically for a finite Δx. The corresponding $[\phi(t,x),\pi(t,x_i)]/i$ is shown on the right.

In the limit of small Δx, $[\phi(t,x),\pi(t,x_i)]$ is then identified as $i\delta(x-x_i)$. Thus, in the continuous limit we have

$$\begin{aligned}[\phi(t,x),\pi(t,x')] &= i\delta(x-x')\,,\\ [\phi(t,x),\phi(t,x')] &= [\pi(t,x),\pi(t,x')]=0\,.\end{aligned}$$

Extension to the three-dimensional case is straightforward and given by

$$\boxed{\begin{aligned}[\phi(t,\vec{x}),\pi(t,\vec{x}')] &= i\delta^3(\vec{x}-\vec{x}')\,,\\ [\phi(t,\vec{x}),\phi(t,\vec{x}')] &= [\pi(t,\vec{x}),\pi(t,\vec{x}')]=0\,.\end{aligned}}\,, \tag{4.139}$$

with

$$\delta^3(\vec{x}-\vec{x}') \stackrel{\text{def}}{\equiv} \delta(x-x')\delta(y-y')\delta(z-z')\,. \tag{4.140}$$

Note that we are using the Heisenberg picture where all time dependences are contained in the operators, and that the commutation relations above are defined at a given time t.

Heisenberg's equations of motion

We are now in the realm of quantum mechanics. We have infinite number of operators $\phi(t,\vec{x})$ representing the field values at different positions in space, and their conjugate operators $\pi(t,\vec{x})$. Then, the time dependence of any operator which is a function of ϕ and π is given by Heisenberg's equation of motion and the commutation relations (4.139). One natural question is then what kind of equation of motion does ϕ satisfy, is it the Klein-Gordon equation? As we will see now, the answer is yes. Heisenberg's equations of motion for ϕ and π are

$$-i\dot{\phi}(t,\vec{x}) = [H,\phi(t,\vec{x})]\,, \tag{4.141}$$

$$-i\dot{\pi}(t,\vec{x}) = [H,\pi(t,\vec{x})]\,, \tag{4.142}$$

where all times are understood to be the same. Note that what appears in the commutators is the total Hamiltonian and not the Hamiltonian density. Using the Hamiltonian density (4.129) and $H = \int d^3x \mathcal{H}$, the above equations are written as (dropping 't' for simplicity)

$$-i\dot{\phi}(\vec{x}) = \Big[\int d^3x' \frac{1}{2}\Big(\pi^2(\vec{x}') + (\vec{\nabla}'\phi(\vec{x}'))^2 + m^2\phi^2(\vec{x}')\Big), \phi(\vec{x})\Big], \tag{4.143}$$

$$-i\dot{\pi}(\vec{x}) = \Big[\int d^3x' \frac{1}{2}\Big(\pi^2(\vec{x}') + (\vec{\nabla}'\phi(\vec{x}'))^2 + m^2\phi^2(\vec{x}')\Big), \pi(\vec{x})\Big]. \tag{4.144}$$

Some mathematical preparations are in order. For the operator function $\phi(\vec{x})$, the definition of $\nabla_i\phi(x)$ is

$$\frac{\partial\phi}{\partial x^i}(\vec{x}) \stackrel{\text{def}}{\equiv} \frac{\phi(\vec{x} + \epsilon\,\hat{e}_i) - \phi(\vec{x})}{\epsilon} \tag{4.145}$$

where ϵ is a small real number and $\hat{e}_i$ is the unit vector in the direction of the i-th axis. We note that $\partial_i\phi$ is nothing but the difference between neighboring $\phi(\vec{x})$'s. Since $\phi(\vec{x})$ and $\phi(\vec{x}')$ commute at equal time for any $\vec{x}$ and $\vec{x}'$, we then see that $\partial_i\phi(\vec{x}')$ commutes with $\phi(\vec{x})$ (at a given time t):

$$[\vec{\nabla}'\phi(\vec{x}'), \phi(\vec{x})] = 0\,. \tag{4.146}$$

Note that the same argument cannot be applied to the time derivative $\dot{\phi}$ since the commutation relation $[\phi(t,\vec{x}), \phi(t,\vec{x}')] = 0$ is for a given time t; namely, commutators such as $[\phi(t+dt,\vec{x}), \phi(t,\vec{x}')]$ are not defined at this point, and in fact it is not in general zero as we will see later.

Next, if $f(\vec{x})$ is an arbitrary operator that is a function of $\vec{x}$, then from the definition of $\partial_i\phi$ (4.145),

$$\begin{aligned}
\int d^3x' f(\vec{x}')\Big[\frac{\partial\phi}{\partial x^{i\prime}}(\vec{x}'), \pi(\vec{x})\Big] &= \int d^3x' f(\vec{x}')\Big[\frac{\phi(\vec{x}' + \epsilon\,\hat{e}_i) - \phi(\vec{x}')}{\epsilon}, \pi(\vec{x})\Big] \\
&= \frac{1}{\epsilon}\int d^3x' f(\vec{x}')\Big\{\underbrace{[\phi(\vec{x}' + \epsilon\,\hat{e}_i), \pi(\vec{x})]}_{i\delta^3(\vec{x}' + \epsilon\,\hat{e}_i - \vec{x})} - \underbrace{[\phi(\vec{x}'), \pi(\vec{x})]}_{i\delta^3(\vec{x}' - \vec{x})}\Big\} \\
&= \frac{i}{\epsilon}\Big(f(\vec{x} - \epsilon\,\hat{e}_i) - f(\vec{x})\Big) \\
&= -i\,\frac{\partial f}{\partial x^i}(\vec{x})\,.
\end{aligned} \tag{4.147}$$

In the above, note that the same result is obtained when the operator $f(\vec{x}')$ is placed after the commutator.

Then, (4.143) becomes

$$-i\dot{\phi}(\vec{x}) = \frac{1}{2}\int d^3x' \Big[\pi^2(\vec{x}') + \underbrace{(\vec{\nabla}'\phi(\vec{x}'))^2}_{\text{by (4.146)}} + m^2\phi^2(\vec{x}'), \phi(\vec{x})\Big]$$

$$
\begin{aligned}
&= \frac{1}{2}\int d^3x' \; \underbrace{[\pi^2(\vec{x}'),\phi(\vec{x})]}_{\pi(\vec{x}')\,\underbrace{[\pi(\vec{x}'),\phi(\vec{x})]}_{-i\delta^3(\vec{x}-\vec{x}')} + \underbrace{[\pi(\vec{x}'),\phi(\vec{x})]}_{-i\delta^3(\vec{x}-\vec{x}')}\,\pi(\vec{x}')} \\
&= -i\pi(\vec{x})\,; \qquad (4.148)
\end{aligned}
$$

namely,

$$
\boxed{\dot{\phi}(t,\vec{x}) = \pi(t,\vec{x})}\,, \qquad (4.149)
$$

reproducing (4.136), this time derived from Heisenberg's equation of motion. The second equation of motion (4.144) is now,

$$
\begin{aligned}
-i\dot{\pi}(\vec{x}) &= \frac{1}{2}\int d^3x' \Big[\cancel{\pi^2}(\vec{x}') + (\vec{\nabla}'\phi(\vec{x}'))^2 + m^2\phi^2(\vec{x}')\,,\pi(\vec{x})\Big] \\
&= \underbrace{\frac{1}{2}\int d^3x' \Big[(\vec{\nabla}'\phi(\vec{x}'))^2,\pi(\vec{x})\Big]}_{\equiv I_1} + \underbrace{\frac{m^2}{2}\int d^3x' \Big[\phi^2(\vec{x}'),\pi(\vec{x})\Big]}_{\equiv I_2}\,. \qquad (4.150)
\end{aligned}
$$

Using (4.147), the first term becomes

$$
\begin{aligned}
I_1 &= \frac{1}{2}\sum_i \int d^3x' \Big[(\partial'_i\phi(\vec{x}'))^2,\pi(\vec{x})\Big] \\
&= \frac{1}{2}\sum_i \Big\{ \underbrace{\int d^3x' \Big\{\partial'_i\phi(\vec{x}')\,[\partial'_i\phi(\vec{x}'),\pi(\vec{x})]}_{-i\partial_i\partial_i\phi(\vec{x})} + \underbrace{[\partial'_i\phi(\vec{x}'),\pi(\vec{x})]\,\partial'_i\phi(\vec{x}')}_{\to\; -i\partial_i\partial_i\phi(\vec{x})} \Big\} \\
&= -i\nabla^2\phi(\vec{x})\,, \qquad (4.151)
\end{aligned}
$$

while the second term is simpler:

$$
\begin{aligned}
I_2 &= \frac{m^2}{2}\int d^3x' \Big\{\phi(\vec{x}')\,\underbrace{[\phi(\vec{x}'),\pi(\vec{x})]}_{i\delta(\vec{x}'-\vec{x})} + \underbrace{[\phi(\vec{x}'),\pi(\vec{x})]}_{i\delta(\vec{x}'-\vec{x})}\,\phi(\vec{x}')\Big\} \\
&= im^2\phi(\vec{x})\,. \qquad (4.152)
\end{aligned}
$$

Putting the pieces together, we obtain

$$
\dot{\pi}(t,\vec{x}) = \nabla^2\phi(t,\vec{x}) - m^2\phi(t,\vec{x})\,, \qquad (4.153)
$$

which, combined with $\pi = \dot{\phi}$ (4.149), leads to

$$
\ddot{\phi}(t,\vec{x}) = \nabla^2\phi(t,\vec{x}) - m^2\phi(t,\vec{x})\,,
$$

$$\rightarrow \quad \boxed{(\partial^2 + m^2)\phi(t,\vec{x}) = 0}\,, \tag{4.154}$$

which is nothing but the Klein-Gordon equation. Thus, we have shown that Heisenberg's equations of motion for $\phi(x)$ and $\pi(x)$ together with the commutation relations among $\phi(x)$ and $\pi(x)$ lead to the Klein-Gordon equation which is now satisfied by the operator field $\phi(t,\vec{x})$ in the Heisenberg picture.

Momentum expansion

We will now express the operator field $\phi(t,\vec{x})$ in terms of normal modes and introduce creation and annihilation operators associated with each mode. In performing the expansion, it is convenient to impose the periodicity (4.91), and focus our attention on the (large) box $-L/2 < x_i < L/2$ $(i = 1,2,3)$. Then at a given time t, any operator function $\phi(t,\vec{x})$ can be expanded as

$$\phi(t,\vec{x}) = \sum_{\vec{p}} C_{\vec{p}}(t) e^{i\vec{p}\cdot\vec{x}}\,, \tag{4.155}$$

where $C_{\vec{p}}$ are operators and $\vec{p}$ takes discrete values that satisfy

$$\begin{gathered}\vec{p} = (p_x, p_y, p_z) = \left(\frac{2\pi n_x}{L}, \frac{2\pi n_y}{L}, \frac{2\pi n_z}{L}\right) \\ (-\infty < n_x, n_y, n_z < \infty : \text{integers})\,.\end{gathered} \tag{4.156}$$

As we will show below, the set of functions $\{e^{i\vec{p}\cdot\vec{x}}\}$ forms a complete orthonormal set in the box:

$$\begin{aligned}\int d^3x (e^{i\vec{p}\cdot\vec{x}})^* e^{i\vec{p}'\cdot\vec{x}} &= V\delta_{\vec{p},\vec{p}'} \\ \sum_{\vec{p}} (e^{i\vec{p}\cdot\vec{x}})^* e^{i\vec{p}\cdot\vec{x}'} &= V\delta^3(\vec{x}-\vec{x}')\,,\end{aligned} \tag{4.157}$$

where $\delta_{\vec{p},\vec{p}'}$ is defined as

$$\delta_{\vec{p},\vec{p}'} \stackrel{\text{def}}{\equiv} \begin{cases} 1 & \text{if } \vec{p} = \vec{p}' \\ 0 & \text{otherwise} \end{cases}\,. \tag{4.158}$$

In the above and hereafter, the space integral is understood to be within the box of volume $V \equiv L^3$ unless otherwise stated. The first of (4.157) is easy to verify: since if $\vec{p} = \vec{p}'$ then integrand is 1 and the integral is V, while if $\vec{p} \neq \vec{p}'$ then the integrand oscillates and the integral vanishes. The second of (4.157) can be proven by converting the sum over $\vec{p}$ to an integral using

$$\sum_{\vec{p}} = \frac{V}{(2\pi)^3}\int d^3p \qquad (L \to \infty)\,. \tag{4.159}$$

This can be seen as follows: the possible values of $\vec{p} = (2\pi n_x/L, 2\pi n_y/L, 2\pi n_z/L)$ are on a cubic grid of cell size $2\pi/L$. Each cell of volume $dv = (2\pi/L)^3$ contains exactly

one grid point, say, at its center. The integral of an arbitrary function $f(\vec{p})$ over $\vec{p}$ can be written in the limit $2\pi/L \to 0$ as

$$\int d^3p f(\vec{p}) = \sum_{\vec{p}} \underbrace{dv}_{(2\pi/L)^3} f(\vec{p}) = \frac{(2\pi)^3}{V} \sum_{\vec{p}} f(\vec{p}) \,, \tag{4.160}$$

which establishes the correspondence (4.159). Then, we have

$$\sum_{\vec{p}} (e^{i\vec{p}\cdot\vec{x}})^* e^{i\vec{p}\cdot\vec{x}'} = \frac{V}{(2\pi)^3} \underbrace{\int d^3p\, e^{i\vec{p}\cdot(\vec{x}'-\vec{x})}}_{(2\pi)^3\delta^3(\vec{x}-\vec{x}')} = V\delta^3(\vec{x}-\vec{x}') \,, \tag{4.161}$$

which proves the second orthogonality of (4.157)

The expansion (4.155) is unique since the coefficient $C_{\vec{p}}$ is uniquely given by

$$\int d^3x (e^{i\vec{p}\cdot\vec{x}})^* \phi(t,\vec{x}) = \sum_{\vec{p}'} C_{\vec{p}'}(t) \underbrace{\int d^3x (e^{i\vec{p}\cdot\vec{x}})^* e^{i\vec{p}'\cdot\vec{x}}}_{V\delta_{\vec{p},\vec{p}'}} = VC_{\vec{p}} \,,$$

$$\to \quad C_{\vec{p}} = \frac{1}{V} \int d^3x (e^{i\vec{p}\cdot\vec{x}})^* \phi(t,\vec{x}) \,, \tag{4.162}$$

and it is complete since the original function ϕ is recovered as

$$\begin{aligned} \sum_{\vec{p}} C_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}} &= \frac{1}{V} \sum_{\vec{p}} \int d^3x' (e^{i\vec{p}\cdot\vec{x}'})^* \phi(t,\vec{x}') e^{i\vec{p}\cdot\vec{x}} \\ &= \frac{1}{V} \int d^3x' \underbrace{\sum_{\vec{p}} (e^{i\vec{p}\cdot\vec{x}'})^* e^{i\vec{p}\cdot\vec{x}}}_{V\delta^3(\vec{x}-\vec{x}')} \phi(t,\vec{x}') = \phi(t,\vec{x}) \end{aligned} \tag{4.163}$$

Since ϕ is an observable, it is hermitian, and thus

$$\phi^\dagger = \phi \quad \to \quad \underbrace{\sum_{\vec{p}} C^\dagger_{\vec{p}}\, e^{-i\vec{p}\cdot\vec{x}}}_{\sum_{\vec{p}} C^\dagger_{-\vec{p}} e^{i\vec{p}\cdot\vec{x}}} = \sum_{\vec{p}} C_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}} \,. \quad (\text{relabeled } \vec{p} \to -\vec{p}) \tag{4.164}$$

Equating the coefficients of the orthonormal functions, the hermiticity condition is

$$C^\dagger_{-\vec{p}}(t) = C_{\vec{p}}(t) \tag{4.165}$$

Thus, any hermitian field can be expanded at any given time as (4.155) with the hermiticity condition (4.165). The time dependence of the field, past and future, is

then given as follows if $\phi(t,\vec{x})$ is to satisfy the Klein-Gordon equation. Substituting (4.155) in $\ddot{\phi} = \nabla^2\phi - m^2\phi$,

$$\sum_{\vec{p}} \ddot{C}_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}} = \sum_{\vec{p}} \Big[(i\vec{p})^2 - m^2\Big] C_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}}$$

$$\rightarrow \quad \ddot{C}_{\vec{p}} = -{p^0}^2 C_{\vec{p}}\,, \tag{4.166}$$

where p^0 is defined as a positive number given by

$$p^0 \equiv \sqrt{\vec{p}^{\,2} + m^2} \geq 0\,. \tag{4.167}$$

The general solution is then

$$C_{\vec{p}}(t) = A_{\vec{p}}\, e^{-ip^0 t} + B_{\vec{p}}\, e^{ip^0 t}\,, \tag{4.168}$$

where $A_{\vec{p}}$ and $B_{\vec{p}}$ are constant operators. The hermitian condition (4.165) is now

$$A^\dagger_{-\vec{p}}\, e^{ip^0 t} + B^\dagger_{-\vec{p}}\, e^{-ip^0 t} = A_{\vec{p}}\, e^{-ip^0 t} + B_{\vec{p}}\, e^{ip^0 t} \tag{4.169}$$

which should hold for all t; namely,

$$A^\dagger_{-\vec{p}} = B_{\vec{p}}\,, \quad B^\dagger_{-\vec{p}} = A_{\vec{p}} \tag{4.170}$$

where the second condition is actually the same as the first. We then obtain the general solution of the Klein-Gordon equation by substituting (4.168) in the expansion (4.155) and using $A^\dagger_{-\vec{p}} = B_{\vec{p}}$:

$$\begin{aligned}
\phi(t,\vec{x}) &= \sum_{\vec{p}} \Big(A_{\vec{p}}\, e^{-ip^0 t} + A^\dagger_{-\vec{p}} e^{ip^0 t}) e^{i\vec{p}\cdot\vec{x}} \\
&= \sum_{\vec{p}} \Big(A_{\vec{p}}\, e^{-ip^0 t + i\vec{p}\cdot\vec{x}} + \underbrace{A^\dagger_{-\vec{p}} e^{ip^0 t + i\vec{p}\cdot\vec{x}}}_{\text{relabel } \vec{p} \rightarrow -\vec{p}}\Big) \\
&= \sum_{\vec{p}} \Big(A_{\vec{p}}\, e^{-ip\cdot x} + A^\dagger_{\vec{p}} e^{ip\cdot x}\Big)
\end{aligned} \tag{4.171}$$

where we have used the 4-vector notation $p\cdot x = p^0 t - \vec{p}\cdot\vec{x}$. Note that in relabeling $\vec{p} \rightarrow -\vec{p}$, the value of p^0 did not change because of the definition (4.167).

In analogy to the single harmonic oscillator case (4.37), we define the operator $a_{\vec{p}}$ as

$$A_{\vec{p}} = \frac{1}{\sqrt{2p^0 V}} a_{\vec{p}}\,. \tag{4.172}$$

We will see below that $a_{\vec{p}}$ and $a^\dagger_{\vec{p}}$ act as the annihilation and the creation operator for a particle with momentum $\vec{p}$. The normalization factor $1/\sqrt{2p^0 V}$ is such that

the commutation relations among ϕ and π (4.139) leads to the commutation relation $[a_{\vec{p}}, a^\dagger_{\vec{p}}] = \delta_{\vec{p},\vec{p}'}$ while satisfying $\pi = \dot{\phi}$.

Putting (4.171) and (4.172) together, the expansion can be written as

$$\boxed{\phi(x) = \sum_{\vec{p}} \Big(a_{\vec{p}}\, e_{\vec{p}}(x) + a^\dagger_{\vec{p}}\, e^*_{\vec{p}}(x) \Big)} \tag{4.173}$$

with the *normal-mode functions* defined by

$$\boxed{e_{\vec{p}}(x) \stackrel{\text{def}}{\equiv} \frac{e^{-ip\cdot x}}{\sqrt{2p^0 V}} \quad (x \equiv (t,\vec{x}),\ p^0 = \sqrt{\vec{p}^{\,2} + m^2} \geq 0)} \tag{4.174}$$

This expansion is general; that is, you give me any solution of the Klein-Gordon equation, then I will look at its spacial dependence at a fixed time t and expand it uniquely as (4.155). Then the time dependence for the past and future is given as in (4.168) in order to satisfy the Klein-Gordon equation, and together with the hermitian condition (4.170), it results in the expansion above. [2] Note that p^0 is a function of $\vec{p}$, $e_{\vec{p}}(x)$ is just a number (namely, not an operator), and $a_{\vec{p}}$ is an operator that does not depend on t. The conjugate field can be obtained by (4.149):

$$\pi(x) = \dot{\phi}(x) = \sum_{\vec{p}} (-ip^0) \Big(a_{\vec{p}}\, e_{\vec{p}}(x) - a^\dagger_{\vec{p}}\, e^*_{\vec{p}}(x) \Big)\,, \tag{4.175}$$

which is also hermitian.

In order to show that $a_{\vec{p}}$ and $a^\dagger_{\vec{p}}$ indeed are creation and annihilation operators, we need the orthonormality relations of the normal-mode functions:

$$\begin{aligned} \int d^3x\, e^*_{\vec{p}}(x) e_{\vec{p}'}(x) &= \frac{1}{2V\sqrt{p^0 p^{0\prime}}} \int d^3x\, e^{ip\cdot x} e^{-ip'\cdot x} \\ &= \frac{e^{i(p^0 - p^{0\prime})t}}{2V\sqrt{p^0 p^{0\prime}}} \underbrace{\int d^3x (e^{i\vec{p}\cdot\vec{x}})^* e^{i\vec{p}'\cdot\vec{x}}}_{V\delta_{\vec{p},\vec{p}'}} \\ &= \frac{\delta_{\vec{p},\vec{p}'}}{2p^0}\,, \end{aligned} \tag{4.176}$$

where in the last step we have set $p^0 = p^{0\prime}$ which is allowed because of the delta function $\delta_{\vec{p},\vec{p}'}$. The integral of $e_{\vec{p}}(x) e_{\vec{p}'}(x)$ can be obtained similarly:

$$\int d^3x\, e_{\vec{p}}(x) e_{\vec{p}'}(x) = e^{-2ip^0 t}\, \frac{\delta_{\vec{p},-\vec{p}'}}{2p^0}\,. \tag{4.177}$$

[2] Because of the hermitian condition $C^\dagger_{-\vec{p}} = C_{\vec{p}}$, the number of independent $A_{\vec{p}}$ is twice that of $C_{\vec{p}}$. Thus, $\phi(x)$ at given time alone cannot uniquely fix all $A_{\vec{p}}$'s. Additional information on $\dot{\phi}(x)$ at given time will do. This is because the Klein-Gordon equation is second order in time derivative.

This is a little awkward; it would be much better if the above gives zero instead of the delta function with a phase factor. This can be accomplished by using the differential operator $\overset{\leftrightarrow}{\partial}_0$ introduced in (2.46):

$$a \overset{\leftrightarrow}{\partial}_0 b \equiv a(\partial_0 b) - (\partial_0 a) b \,, \tag{4.178}$$

where a and b are arbitrary functions of t. We then have

$$\int d^3x\, e_{\vec{p}}(x) i \overset{\leftrightarrow}{\partial}_0 e_{\vec{p}'}(x) = (p^{0\prime} - p^0) \underbrace{\int d^3x\, e_{\vec{p}}(x) e_{\vec{p}'}(x)}_{e^{-2ip^0 t}\delta_{\vec{p},-\vec{p}'}/2p^0} = 0 \tag{4.179}$$

where the delta function forced $p^{0\prime} = p^0$. With $\overset{\leftrightarrow}{\partial}_0$, the orthonormality (4.176) also becomes simpler:

$$\int d^3x\, e^*_{\vec{p}}(x) i \overset{\leftrightarrow}{\partial}_0 e_{\vec{p}'}(x) = (p^{0\prime} + p^0) \underbrace{\int d^3x\, e^*_{\vec{p}}(x) e_{\vec{p}'}(x)}_{\delta_{\vec{p},\vec{p}'}/2p^0} = \delta_{\vec{p},\vec{p}'} . \tag{4.180}$$

Taking complex conjugate of these relations, we have a set of orthonormality relations given by

$$\boxed{\begin{array}{lll} \int d^3x\, e^*_{\vec{p}}(x) i \overset{\leftrightarrow}{\partial}_0 e_{\vec{p}'}(x) = \delta_{\vec{p},\vec{p}'} \,, & \int d^3x\, e_{\vec{p}}(x) i \overset{\leftrightarrow}{\partial}_0 e^*_{\vec{p}'}(x) = -\delta_{\vec{p},\vec{p}'} \,, \\ \int d^3x\, e_{\vec{p}}(x) i \overset{\leftrightarrow}{\partial}_0 e_{\vec{p}'}(x) = 0 \,, & \int d^3x\, e^*_{\vec{p}}(x) i \overset{\leftrightarrow}{\partial}_0 e^*_{\vec{p}'}(x) = 0 \,. \end{array}} \tag{4.181}$$

In the classical (i.e. non-quantized) Klein-Gordon theory, the probability density is given by $j_0 = \phi^* i \overset{\leftrightarrow}{\partial}_0 \phi$ (2.45). Thus, when interpreted as a c-number solution of the Klein-Gordon equation, the normal-mode functions $e_{\vec{p}}(x)$ are normalized to be unit probability in the volume V.

These relations allow us to express $a_{\vec{p}}$ and $a^\dagger_{\vec{p}}$ in terms of ϕ and π: using the expansion (4.173),

$$\int d^3x\, e^*_{\vec{p}}(x) i \overset{\leftrightarrow}{\partial}_0 \phi(x) = \sum_{\vec{p}'} \Big[a_{\vec{p}'} \underbrace{\int d^3x\, e^*_{\vec{p}} i \overset{\leftrightarrow}{\partial}_0 e_{\vec{p}'}}_{\delta_{\vec{p},\vec{p}'}} + a^\dagger_{\vec{p}'} \underbrace{\int d^3x\, e^*_{\vec{p}} i \overset{\leftrightarrow}{\partial}_0 e^*_{\vec{p}'}}_{0} \Big] = a_{\vec{p}} \,, \tag{4.182}$$

where we have used the linearity of the operator $\overset{\leftrightarrow}{\partial}_0$ to take the sum over momentum out of the integral. Writing out the left-hand side of the above explicitly

$$a_{\vec{p}} = \int d^3x\, e^*_{\vec{p}}(x) \Big[i\dot{\phi} + p^0 \phi(x) \Big] = \int d^3x\, e^*_{\vec{p}}(x) \Big[p^0 \phi(x) + i\pi(x) \Big] \,. \tag{4.183}$$

Taking the hermitian conjugate of this relation, and noting that ϕ and π are hermitian, we get

$$a^\dagger_{\vec{p}} = \int d^3x\, e_{\vec{p}}(x)\Big[p^0\phi(x) - i\pi(x)\Big]\,. \tag{4.184}$$

Note that the relations (4.183) and (4.184) are valid for any t.

Then, the commutation relations among $a_{\vec{p}}$ and $a^\dagger_{\vec{p}}$ follow directly from those among ϕ and π (4.139): with the notation

$$x \equiv (t,\vec{x}), \quad x' \equiv (t,\vec{x}') \qquad \text{(equal time)} \tag{4.185}$$

we have

$$\begin{aligned}
[a_{\vec{p}}, a^\dagger_{\vec{p}'}] &= \Big[\int d^3x\, e^*_{\vec{p}}(x)\Big(p^0\phi(x) + i\pi(x)\Big)\,, \int d^3x' e_{\vec{p}'}(x')\Big(p^{0\prime}\phi(x') - i\pi(x')\Big)\Big] \\
&= \int d^3x \int d^3x' e^*_{\vec{p}}(x) e_{\vec{p}'}(x') \underbrace{\Big[p^0\phi(x) + i\pi(x)\,, p^{0\prime}\phi(x') - i\pi(x')\Big]}_{\begin{array}{c} ip^{0\prime}\ \underbrace{[\pi(x),\phi(x')]}_{-i\delta^3(\vec{x}-\vec{x}')} \end{array} \ -ip^0\ \underbrace{[\phi(x),\pi(x')]}_{i\delta^3(\vec{x}-\vec{x}')}} \\
&= \underbrace{\int d^3x\, e^*_{\vec{p}}(x) e_{\vec{p}'}(x)}_{\delta_{\vec{p},\vec{p}'}/2p^0}\, p^0 + \underbrace{\int d^3x\, e^*_{\vec{p}}(x) e_{\vec{p}'}(x)}_{\delta_{\vec{p},\vec{p}'}/2p^0 = \delta_{\vec{p},\vec{p}'}/2p^{0\prime}}\, p^{0\prime} \\
&= \delta_{\vec{p},\vec{p}'}\,.
\end{aligned} \tag{4.186}$$

Similarly, we can see that $[a_{\vec{p}}, a_{\vec{p}'}]$ vanishes, which results from the two terms with delta functions in (4.186) cancelling each other. Then, its hermitian conjugate $[a^\dagger_{\vec{p}}, a^\dagger_{\vec{p}'}]$ also vanishes. Thus, the commutation relations of ϕ and π resulted in those of $a_{\vec{p}}$ and $a^\dagger_{\vec{p}}$ given by

$$\boxed{\begin{array}{c} [a_{\vec{p}}, a^\dagger_{\vec{p}'}] = \delta_{\vec{p},\vec{p}'} \\ [a_{\vec{p}}, a_{\vec{p}'}] = [a^\dagger_{\vec{p}}, a^\dagger_{\vec{p}'}] = 0 \end{array}}\,. \tag{4.187}$$

which are exactly the commutation relations for the annihilation and creation operators for a set of independent harmonic oscillators where each harmonic oscillator is labeled by $\vec{p}$; namely, for a given $\vec{p}$ we have $[a, a^\dagger] = 1$, and operators belonging to different harmonic oscillators commute. Then by the same argument as in the case of a single harmonic oscillator (4.28), this commutation relation between a and $a^\dagger$ singlehandedly leads to the fact that $a^\dagger$ raises the eigenvalue of the number operator $N = a^\dagger a$ by one and a lowers by one, thus justifying the interpretation that they are indeed creation and annihilation operators. The commutation relations among ϕ and π (4.139) can be re-derived from those of $a_{\vec{p}}$ and $a^\dagger_{\vec{p}}$, and it is left as an exercise.

Exercise 4.3 *Quantization conditions of hermitian spin-0 field.*
Start from the commutation relations among creation and annihilation operators (4.187) and derive those among fields (4.139) using the momentum expansions.

Total energy and momentum

We have seen that a hermitian field $\phi(t,\vec{x})$ that satisfies the Klein-Gordon equation can be regarded as a set of harmonic oscillators each labeled by $\vec{p}$ and with associated pair of annihilation and creation operators $a_{\vec{p}}$ and $a^\dagger_{\vec{p}}$. Then each harmonic oscillator or normal mode can have an integer number of quanta each of which we regard as a particle with momentum $\vec{p}$ and energy $p^0=\sqrt{\vec{p}^{\,2}+m^2}$. If this picture is correct, then the total energy and momentum should be simply the sum of energies and momenta of the particles in each normal modes summed over all normal modes. We will now see that it is indeed the case.

First, let's examine in what kind of space the operators such as $a_{\vec{p}}$, ϕ, π, etc. are acting. This is in fact the Hilbert space we are dealing with. Since each normal mode is an independent harmonic oscillator, there is a number operator for each $\vec{p}$:

$$N_{\vec{p}} \equiv a^\dagger_{\vec{p}} a_{\vec{p}}\,, \tag{4.188}$$

and the corresponding set of eigenstates denoted as $|n_{\vec{p}}\rangle_{\vec{p}}$:

$$N_{\vec{p}}|n_{\vec{p}}\rangle_{\vec{p}} = n_{\vec{p}}|n_{\vec{p}}\rangle_{\vec{p}}\,. \tag{4.189}$$

Namely, $|n_{\vec{p}}\rangle_{\vec{p}}$ is a state of the harmonic oscillator labeled by $\vec{p}$ that contains $n_{\vec{p}}$ quanta. Then just as in the case of the single harmonic oscillator (4.34), the state $|n_{\vec{p}}\rangle_{\vec{p}}$ can be constructed by repeatedly applying $a^\dagger_{\vec{p}}$ on the ground state of the oscillator $|0\rangle_{\vec{p}}$:

$$|n_{\vec{p}}\rangle_{\vec{p}} = \frac{(a^\dagger_{\vec{p}})^{n_{\vec{p}}}}{\sqrt{n_{\vec{p}}!}}|0\rangle_{\vec{p}} \tag{4.190}$$

Let $|\{n_{\vec{p}}\}\rangle$ be the overall state with $n_{\vec{p}}$ quanta in the harmonic oscillator labeled by $\vec{p}$ where $\vec{p}$ runs over all possible values. It is the direct product of the states of all harmonic oscillators:

$$|\{n_{\vec{p}}\}\rangle = \prod_{\vec{p}} |n_{\vec{p}}\rangle_{\vec{p}} = \left[\prod_{\vec{p}} \frac{(a^\dagger_{\vec{p}})^{n_{\vec{p}}}}{\sqrt{n_{\vec{p}}!}}\right]|0\rangle\,, \tag{4.191}$$

where the vacuum state $|0\rangle$ is the product of all ground states:

$$|0\rangle \stackrel{\text{def}}{\equiv} \prod_{\vec{p}} |0\rangle_{\vec{p}}\,. \tag{4.192}$$

Since each $|0\rangle_{\vec{p}}$ is normalized, the over-all vacuum is also normalized:

$$\langle 0|0\rangle = 1\,. \tag{4.193}$$

The Hilbert space is then spanned by the states $|\{n_{\vec{p}}\}\rangle$ with all possible combinations of $\{n_{\vec{p}}\}$. It is a very large space. When there are more than one types of particles, then the space should be extended even larger: the basis states in this case are $|\{n_{r,\vec{p}}\}\rangle$, where $n_{r,\vec{p}}$ is the number of quanta in the normal mode corresponding to the particle type r and momentum $\vec{p}$. These states with definite numbers of quanta in each particle type and momentum are called the *Fock states*. For now, we assume that there is only one type of particle.

What we would like to show is then, for the operator P^μ that represents the total energy-momentum, its action on a Fock state $|\{n_{\vec{p}}\}\rangle$ is

$$P^\mu|\{n_{\vec{p}}\}\rangle = \Big(\sum_{\vec{p}} n_{\vec{p}}\, p^\mu\Big)|\{n_{\vec{p}}\}\rangle\,, \tag{4.194}$$

where $p^\mu \equiv (p^0, \vec{p})$ with $p^0 = \sqrt{\vec{p}^{\,2} + m^2}$; namely, the total energy-momentum 4-vector becomes the sum of those for all the existing particles in the universe. This would be accomplished if the total energy-momentum operator can be written as

$$P^\mu = \sum_{\vec{p}} p^\mu N_{\vec{p}}\,, \tag{4.195}$$

where $N_{\vec{p}}$ is the number operator given by (4.188). Now what should we take for the energy-momentum operator P^μ to start with? In the case of classical field, we had the conserved 4-vector (4.107) obtained from the Noether current of space-time translation, which is a good candidate to use after regarding the fields in the expressions as operators. In fact, the Hamiltonian of the quantized Klein-Gordon field (4.129) was constructed exactly in such manner and it was shown that such Hamiltonian indeed leads to the Klein-Gordon equation through Heisenberg's equations of motion. Thus, we will start from the Hamiltonian density (4.129), regard the fields as operators, and express the total Hamiltonian $H = \int d^3\mathcal{H}$ in terms of $a_{\vec{p}}$'s and $a^\dagger_{\vec{p}}$'s.

Let's first use a partial integration and the Klein-Gordon equation to simplify the expression (4.129) for H:

$$\begin{aligned} H &= \int d^3x \frac{1}{2}\Big(\pi^2 + \underbrace{(\vec{\nabla}\phi)^2}_{\vec{\nabla}\cdot(\phi\vec{\nabla}\phi) - \phi\nabla^2\phi} + m^2\phi^2\Big) = \frac{1}{2}\int d^3x \Big(\underbrace{\pi^2}_{\dot\phi^2} - \phi\ddot\phi\Big) \\ &\qquad \underbrace{\vec{\nabla}\cdot(\phi\vec{\nabla}\phi)}_{\to 0} - \phi\underbrace{\nabla^2\phi}_{\ddot\phi + m^2\phi\ :\ \text{by the K-G eq.}} \\ &= \frac{1}{2}\int d^3x \phi\, i\overleftrightarrow{\partial}_0 (i\partial_0\phi)\,, \end{aligned} \tag{4.196}$$

where we have used the fact that the surface term vanishes when integrated over all space:

$$\int d^3x \vec{\nabla}\cdot(\phi\vec{\nabla}\phi) = \int_{\text{boundary}} d\vec{a}\cdot(\phi\vec{\nabla}\phi) = 0\,. \tag{4.197}$$

Then, substituting the expansions (4.173) and using the orthonormality relations (4.181), we have

$$\begin{aligned}
H &= \frac{1}{2}\int d^3x \Big[\sum_{\vec{p}}(a_{\vec{p}}\,e_{\vec{p}} + a^\dagger_{\vec{p}}\,e^*_{\vec{p}})\Big] i\overset{\leftrightarrow}{\partial}_0 \Big[\sum_{\vec{p}'} p^{0\prime}(a_{\vec{p}'}\,e_{\vec{p}'} - a^\dagger_{\vec{p}'}\,e^*_{\vec{p}'})\Big] \\
&= \frac{1}{2}\int d^3x \sum_{\vec{p},\vec{p}'} p^{0\prime}\Big(a_{\vec{p}}\,a_{\vec{p}'}\underbrace{e_{\vec{p}}\,i\overset{\leftrightarrow}{\partial}_0 e_{\vec{p}'}}_{\to\,0} - a_{\vec{p}}\,a^\dagger_{\vec{p}'}\underbrace{e_{\vec{p}}\,i\overset{\leftrightarrow}{\partial}_0 e^*_{\vec{p}'}}_{\to\,-\delta_{\vec{p},\vec{p}'}} \\
&\qquad + a^\dagger_{\vec{p}}\,a_{\vec{p}'}\underbrace{e^*_{\vec{p}}\,i\overset{\leftrightarrow}{\partial}_0 e_{\vec{p}'}}_{\to\,\delta_{\vec{p},\vec{p}'}} - a^\dagger_{\vec{p}}\,a^\dagger_{\vec{p}'}\underbrace{e^*_{\vec{p}}\,i\overset{\leftrightarrow}{\partial}_0 e^*_{\vec{p}'}}_{\to\,0}\Big) \\
&= \frac{1}{2}\sum_{\vec{p}} p^0(\underbrace{a_{\vec{p}}\,a^\dagger_{\vec{p}}}_{a^\dagger_{\vec{p}}a_{\vec{p}}+1} + a^\dagger_{\vec{p}}\,a_{\vec{p}}) \\
&= \sum_{\vec{p}} p^0\Big(a^\dagger_{\vec{p}}\,a_{\vec{p}} + \frac{1}{2}\Big)\,.
\end{aligned} \tag{4.198}$$

Thus, the total Hamiltonian is now a simple function of the number operators $N_{\vec{p}} = a^\dagger_{\vec{p}}a_{\vec{p}}$. This is almost exactly what we wanted except the term $\sum_{\vec{p}} p^0/2$ which apparently is infinity. It is the energy of the state with $n_{\vec{p}} = 0$ for all $\vec{p}$, namely, the energy of the vacuum $|0\rangle$. If the object we are dealing with is a lattice of atoms, then there is a maximum value for $|\vec{p}|$ where the wavelength becomes smaller than the lattice spacing and waves stop propagating. In that case, this ground state energy is finite. In our case of the Klein-Gordon field, theory may break down at very high energy, and there may be a natural cutoff that would make the ground state energy finite. In any case, such a constant offset of energy does not have observable effects on the phenomena we are interested in, and thus we choose to simply discard it. Then, our Hamiltonian is

$$H = \sum_{\vec{p}} p^0 a^\dagger_{\vec{p}}\,a_{\vec{p}} \tag{4.199}$$

which gives the time component of (4.195).

We will encounter similar situations where we want to discard constant offsets corresponding to vacuum expectation values; thus, it is convenient to introduce a procedure that forces the vacuum expectation value to be zero. This can be accomplished by the so-called *normal ordering* denoted by ‘: :’. Whatever is inside the colons is a polynomial of annihilation and creation operators, and the normal ordering

reorders each term such that all annihilation operators (a's) are to the right of all creation operators ($a^\dagger$'s). For example,

$$\begin{aligned} :a_{\vec{p}}a^\dagger_{\vec{p}'}: &= a^\dagger_{\vec{p}'}a_{\vec{p}}\,, \qquad :a^\dagger_{\vec{p}'}a_{\vec{p}}: = a^\dagger_{\vec{p}'}a_{\vec{p}}\,, \\ :a_{\vec{p}_1}a^\dagger_{\vec{p}_2}a^\dagger_{\vec{p}_3}: &= a^\dagger_{\vec{p}_2}a^\dagger_{\vec{p}_3}a_{\vec{p}_1}\,, \qquad \text{etc.} \end{aligned} \tag{4.200}$$

Recalling that $a_{\vec{p}}$ annihilates the vacuum state [see (4.32)], we have

$$a_{\vec{p}}|0\rangle = 0 \quad \to \quad \langle 0|a^\dagger_{\vec{p}} = 0\,, \tag{4.201}$$

which means that the vacuum expectation value of any normal-ordered product of $a_{\vec{p}}$'s and $a^\dagger_{\vec{p}}$'s is zero:

$$\langle 0|:\cdots:|0\rangle = 0\,, \tag{4.202}$$

because there is at least one $a_{\vec{p}}$ directly to the left of $|0\rangle$ or at least one $a^\dagger_{\vec{p}}$ directly to the right of $\langle 0|$. Since $a_{\vec{p}}$'s commute among themselves as do $a^\dagger_{\vec{p}}$'s, the normal ordering uniquely defines the resulting operator. One has to be careful, however, not to use commutation relations *before* the normal ordering is performed since it defeats the very purpose of the normal ordering, namely to discard the constant term arising from the commutator:

$$:a_{\vec{p}}a^\dagger_{\vec{p}'}: = :\delta_{\vec{p},\vec{p}'} + a^\dagger_{\vec{p}'}a_{\vec{p}}: = \delta_{\vec{p},\vec{p}'} + a^\dagger_{\vec{p}'}a_{\vec{p}} \qquad \text{(don't do this)}\,. \tag{4.203}$$

Normal ordering is understood to be a simple reordering of a's and $a^\dagger$'s after products are expanded if there are any. Then, the total Hamiltonian can be written using the normal ordering symbol as

$$H = :\int d^3x \frac{1}{2}\Big(\pi^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2\Big): = \sum_{\vec{p}} p^0\, N_{\vec{p}}\,. \tag{4.204}$$

From now on, when we deal with terms in a Hamiltonian or Lagrangian, normal ordering is implicitly assumed.

For the total momentum, we start from $\vec{P}$ given by (4.109), regard the fields as operators, and use the momentum expansion to write it in terms of a's and $a^\dagger$'s. In doing so, it is convenient to write it using $\overset{\leftrightarrow}{\partial}_0$. First, we note that the implicit normal ordering means

$$:(\vec{\nabla}\pi)\phi: = :\phi(\vec{\nabla}\pi):\ . \tag{4.205}$$

Then, the total momentum can be expressed as

$$\vec{P} \;=\; -\int d^3x\, \pi\vec{\nabla}\phi = -\frac{1}{2}\int d^3x(\pi\vec{\nabla}\phi + \underbrace{\pi\vec{\nabla}\phi}_{\vec{\nabla}(\pi\phi) - (\vec{\nabla}\pi)\phi}) \qquad \begin{matrix} \\ \\ \text{surface term } \to 0 \end{matrix} \qquad \underbrace{\phantom{\pi\vec{\nabla}\phi}}_{\phi(\vec{\nabla}\pi)}$$

$$= -\frac{1}{2}\int d^3x(\dot{\phi}\vec{\nabla}\phi - \phi\vec{\nabla}\dot{\phi})$$
$$= \frac{1}{2}\int d^3x\,\phi\, i\overset{\leftrightarrow}{\partial}_0(-i\vec{\nabla}\phi)\,. \tag{4.206}$$

At this point, we notice the pattern: both H and $\vec{P}$ are written in the form

$$\frac{1}{2}\int d^3x\phi i\overset{\leftrightarrow}{\partial}_0\mathcal{O}\phi\,, \tag{4.207}$$

where $\mathcal{O} = i\partial_0$ for H, and $\mathcal{O} = -i\vec{\nabla}$ for $\vec{P}$. Once written in this form, the rest is nearly identical to the Hamiltonian case (4.198), and the result is

$$\vec{P} = -:\int d^3x\,\pi\vec{\nabla}\phi: = \sum_{\vec{p}}\vec{p}\,N_{\vec{p}}\,. \tag{4.208}$$

Exercise 4.4 *Complete the derivation above.*

Note that in the case of classical fields there is no difference between $\pi\vec{\nabla}\phi$ and $(\nabla\phi)\pi$, but for the quantized fields they differ since π and ϕ do not commute. The normal ordering, however, resolves such ambiguity.

We have seen in the classical case that the quantities P^ν given by (4.107) are conserved. Now we are dealing with quantized fields, and a natural question is if the operators P^ν are indeed constants of motion. Time variations of operators are given by Heisenberg's equation of motion, and thus immediately we see that $P^0 = H$ is conserved since H commutes with itself. It is easy to see that $\vec{P}$ is also conserved:

$$-i\dot{\vec{P}} = [H,\vec{P}] = \Big[\sum_{\vec{p}}p^0\,N_{\vec{p}}\,,\sum_{\vec{p}}\vec{p}\,N_{\vec{p}}\Big] = 0\,, \tag{4.209}$$

since $N_{\vec{p}}$ commutes with all other $N_{\vec{p}}$'s including itself. Similarly, the components of total momentum $\vec{P}$ commute among themselves:

$$\Big[P^i,P^j\Big] = 0 \quad (i,j=1,2,3)\,. \tag{4.210}$$

Thus, all the four components of P^μ commute among themselves, and thus can simultaneously good quantum numbers of a system.

Space-time translation

This is a good place to show that the operators P^ν given in general by (4.122) are the generators of space-time translations; namely, if $F(\phi(x),\pi(x))$ is an arbitrary function (namely, polynomial) of ϕ and $\pi\equiv\dot{\phi}$, then

$$e^{iP\cdot a}F(x)e^{-iP\cdot a} = F(x+a) \tag{4.211}$$

provided that ϕ satisfies the equation of motion, where the operator $F(x)$ is a function of $x = (t, \vec{x})$ through $\phi(x)$ and $\pi(x)$ and a^μ is a real 4-vector constant. We will assume the number of fields to be one, but the derivation can be readily extended to multiple fields.

We first evaluate the commutator of $\vec{P}$ and ϕ, π (all at equal time):

$$\begin{aligned}
[\vec{P}, \phi(x)] &= -\int d^3x' [\pi(\vec{x}') \overbrace{\vec{\nabla}'\phi(\vec{x}'), \phi(\vec{x})}^{\text{commute by (4.146)}}] \\
&= -\int d^3x' \underbrace{[\pi(\vec{x}'), \phi(\vec{x})]}_{-i\delta^3(\vec{x}' - \vec{x})} \vec{\nabla}'\phi(\vec{x}') \\
&= i\vec{\nabla}\phi(\vec{x}) \qquad (4.212)
\end{aligned}$$

and

$$\begin{aligned}
[\vec{P}, \pi(x)] &= -\int d^3x' \, [\pi(\vec{x}')\vec{\nabla}'\phi(\vec{x}'), \pi(\vec{x})] \\
&= -\int d^3x' \, \pi(\vec{x}') \, [\vec{\nabla}'\phi(\vec{x}'), \pi(\vec{x})] \\
&= i\vec{\nabla}\pi(\vec{x}) \quad \text{by (4.147)}. \qquad (4.213)
\end{aligned}$$

Together with Heisenberg's equations of motion $-i\dot{\phi} = [P^0, \phi]$ and $-i\dot{\pi} = [P^0, \pi]$, we have

$$\begin{aligned}
[P^\mu, \phi(x)] &= -i\partial^\mu\phi(x) \\
[P^\mu, \pi(x)] &= -i\partial^\mu\pi(x)
\end{aligned} \quad . \qquad (4.214)$$

Then, it is easy to show that, for an arbitrary polynomial of ϕ and π, $F(\phi, \pi)$,

$$\boxed{[P^\mu, F(\phi, \pi)] = -i\partial^\mu F(\phi, \pi)}. \qquad (4.215)$$

Exercise 4.5 *Space-time translation operators.*
Use the property of the energy-momentum operator P^μ (4.214) and show that (4.215) holds for any $F(\phi(x), \pi(x))$ that can be expanded as

$$F(\phi, \pi) = \sum_{n,m=0}^{\infty} c_{n,m}\phi^n\pi^m \qquad (4.216)$$

where $c_{n,m}$ are c-numbers.
(*hint: Prove and use*

$$[A, B^n] = B^{n-1}[A, B] + B^{n-2}[A, B]B + ... + [A, B]B^{n-1}.$$

Comment: note that the above can be extended to any number of different fields.)

For a small translation $a^\mu = \epsilon^\mu$, we have (to the first order in ϵ),

$$\begin{aligned} e^{iP\cdot\epsilon}F(x)e^{-iP\cdot\epsilon} &= (1+iP\cdot\epsilon)F(1-iP\cdot\epsilon) \\ &= F + i(P^\mu\epsilon_\mu F - FP^\mu\epsilon_\mu) \\ &= F + i\underbrace{[P^\mu, F]}_{-i\partial^\mu F}\epsilon_\mu \\ &= F(x) + (\partial^\mu F)\epsilon_\mu = F(x+\epsilon)\,. \end{aligned} \tag{4.217}$$

Applying again $e^{iP\cdot\epsilon}$ from the left and $e^{-iP\cdot\epsilon}$ from the right, and repeating n times total, we obtain

$$e^{iP\cdot n\epsilon}F(x)e^{-iP\cdot n\epsilon} = F(x+n\epsilon)\,. \tag{4.218}$$

Taking the limit $n\to\infty$ while keeping $a = n\epsilon$, we obtain (4.211).

Thus, we have shown that the total energy-momentum operator P^μ indeed generates space-time translation where the time translation is nothing but the time evolution according to Heisenberg's equation of motion. Note that we started from the definition of P^μ for a general field and used only the commutation relations between ϕ and π which happens to be valid for all bosons. We will see later that (4.211) also holds for spin-1/2 fields; it is in fact a general feature of quantum field theory.

4.4 Two scalar fields of the same mass

In the case of a hermitian scalar field, we had one pair of creation and annihilation operators, $a^\dagger_{\vec{p}}$ and $a_{\vec{p}}$, for a given momentum $\vec{p}$. Thus, it cannot have more than one degree of freedom for a given $\vec{p}$; namely, it cannot represent a particle with spin greater than zero, or a system of a particle and its antiparticle. We will now discuss a framework that can describe a spin-0 particle and its antiparticle that are distinct. It is accomplished by combining two hermitian Klein-Gordon fields with the same mass into one non-hermitian field. The fact that the two fields have same mass introduces a symmetry to the combined Lagrangian, and the conserved quantity associated with the symmetry will turn out to be the 'charge' of the particles where a particle and an antiparticle have charges with opposite signs and same magnitude. A hermitian or non-hermitian field in quantum field theory corresponds respectively to a real or complex field in classical field theory. These terminologies are often used interchangeably.

Lagrangian

Consider two real fields with same mass m, $\phi_1(x)$ and $\phi_2(x)$, within the framework of classical field theory. The Lagrangian density for each is given by (4.126):

$$\mathcal{L}_k(\phi_k, \partial_\mu\phi_k) = \frac{1}{2}(\partial_\mu\phi_k\partial^\mu\phi_k - m^2\phi_k^2) \qquad (k=1,2;\ \text{no sum over } k)\,. \tag{4.219}$$

Assuming there is no interaction, the Lagrangian of the system is just the sum of the two:

$$\begin{aligned}\mathcal{L}(\tilde{\phi}, \partial_\mu \tilde{\phi}) &= \mathcal{L}_1(\phi_1, \partial_\mu \phi_1) + \mathcal{L}_2(\phi_2, \partial_\mu \phi_2) \\ &= \frac{1}{2}\Big[\partial_\mu \phi_1 \partial^\mu \phi_1 + \partial_\mu \phi_2 \partial^\mu \phi_2 - m^2(\phi_1^2 + \phi_2^2)\Big] \end{aligned} \tag{4.220}$$

where $\tilde{\phi} = (\phi_1, \phi_2)$. The Euler-Lagrange equations of this Lagrangian indeed lead to correct equations of motion:

$$\underbrace{\frac{\partial \mathcal{L}}{\partial \phi_k}}_{-m^2\phi_k} = \partial_\mu \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_k)}}_{\partial^\mu \phi_k} \quad \rightarrow \quad (\partial_\mu \partial^\mu + m^2)\phi_k = 0 \qquad (k = 1, 2)\,. \tag{4.221}$$

The conjugate fields are, by the definition (4.99),

$$\pi_k \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}_k} = \dot{\phi}_k \qquad (k = 1, 2)\,. \tag{4.222}$$

The Hamiltonian then becomes just the sum of those of the two systems: with $\tilde{\pi} = (\pi_1, \pi_2)$, we have

$$\mathcal{H} \equiv \sum_k \pi_k \dot{\phi}_k - \mathcal{L} = \sum_k (\pi_k \dot{\phi}_k - \mathcal{L}_k) = \sum_k \mathcal{H}_k\,, \tag{4.223}$$

and

$$H = H_1 + H_2\,, \quad \text{with} \quad H_k = \int d^3x \mathcal{H}_k \quad (k = 1, 2)\,. \tag{4.224}$$

Define two complex fields ϕ and $\phi^\dagger$ by

$$\boxed{\begin{aligned}\phi &\stackrel{\text{def}}{\equiv} \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \\ \phi^\dagger &\stackrel{\text{def}}{\equiv} \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2)\end{aligned}}\,. \tag{4.225}$$

Then, we have

$$\phi^\dagger \phi = \frac{1}{2}(\phi_1 - i\phi_2)(\phi_1 + i\phi_2) = \frac{1}{2}(\phi_1^2 + \phi_2^2) \tag{4.226}$$

and

$$\partial_\mu \phi^\dagger \partial^\mu \phi = \frac{1}{2}(\partial_\mu \phi_1 - i\partial_\mu \phi_2)(\partial^\mu \phi_1 + i\partial^\mu \phi_2) = \frac{1}{2}(\partial_\mu \phi_1 \partial^\mu \phi_1 + \partial_\mu \phi_2 \partial^\mu \phi_2)\,. \tag{4.227}$$

The Lagrangian of the system (4.220) can then be written as

$$\boxed{\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi}\,. \tag{4.228}$$

Since ϕ_1 and ϕ_2 both satisfy the Klein-Gordon equation, so do ϕ and $\phi^\dagger$:

$$(\partial^2 + m^2)\phi = 0\,, \qquad (\partial^2 + m^2)\phi^\dagger = 0\,, \tag{4.229}$$

which can also be obtained by *regarding ϕ and $\phi^\dagger$ as independent* in (4.228) and applying the Euler-Lagrange equation:

$$\underbrace{\frac{\partial \mathcal{L}}{\partial \phi}}_{-m^2\phi^\dagger} = \partial_\mu \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}}_{\partial^\mu \phi^\dagger} \quad \rightarrow \quad (\partial_\mu \partial^\mu + m^2)\phi^\dagger = 0 \tag{4.230}$$

$$\underbrace{\frac{\partial \mathcal{L}}{\partial \phi^\dagger}}_{-m^2\phi} = \partial_\mu \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^\dagger)}}_{\partial^\mu \phi} \quad \rightarrow \quad (\partial_\mu \partial^\mu + m^2)\phi = 0\,. \tag{4.231}$$

Again regarding ϕ and $\phi^\dagger$ as independent, the corresponding conjugate fields, π and $\pi^\dagger$, are by definition given by

$$\pi \stackrel{\text{def}}{\equiv} \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}^\dagger\,, \qquad \pi^\dagger \stackrel{\text{def}}{\equiv} \frac{\partial \mathcal{L}}{\partial \dot{\phi}^\dagger} = \dot{\phi}\,, \tag{4.232}$$

or using (4.225) and (4.222)

$$\begin{aligned} \pi &= \frac{1}{\sqrt{2}}(\pi_1 - i\pi_2) \\ \pi^\dagger &= \frac{1}{\sqrt{2}}(\pi_1 + i\pi_2) \end{aligned} \tag{4.233}$$

Note that at this point, π and $\pi^\dagger$ are defined as the conjugate fields of ϕ and $\phi^\dagger$, respectively, and considered independent; the relations (4.225) and (4.233), however, are self-consistent when the symbol † is considered as complex conjugation or, in the case of quantized fields, hermitian conjugation. The correct Hamiltonian can be obtained also by regarding ϕ and $\phi^\dagger$ as independent:

$$\begin{aligned} \mathcal{H} &= \pi\dot{\phi} + \pi^\dagger\dot{\phi}^\dagger - \mathcal{L} \\ &= \frac{1}{2}\Big[(\pi_1 - i\pi_2)(\dot{\phi}_1 + i\dot{\phi}_2) + (\pi_1 + i\pi_2)(\dot{\phi}_1 - i\dot{\phi}_2)\Big] - \mathcal{L} \\ &= \sum_k \pi_k\dot{\phi}_k - \mathcal{L}\,. \end{aligned} \tag{4.234}$$

How could ϕ and $\phi^\dagger$ be considered independent without destroying the whole logical structure, since they seem to be clearly related by complex conjugation? Actually,

there is a good reason why such procedure works. If we regard ϕ and $\phi^\dagger$ as independent, and π and $\pi^\dagger$ as independent, then the relations (4.225) and (4.233) formally constitute a canonical transformation generated by

$$F = \frac{1}{\sqrt{2}}\Big[\pi(\phi_1 + i\phi_2) + \pi^\dagger(\phi_1 - i\phi_2)\Big]\,, \tag{4.235}$$

where it can be readily verified that

$$\phi = \frac{\partial F}{\partial \pi}\,, \quad \phi^\dagger = \frac{\partial F}{\partial \pi^\dagger}\,, \quad \text{and} \quad \pi_k = \frac{\partial F}{\partial \phi_k} \quad (k = 1, 2) \tag{4.236}$$

reproduce (4.225) and (4.233). Thus, it is no surprise that the two 'coordinate systems' describe the identical dynamical system using the same $\mathcal{H}$ and $\mathcal{L}$ expressed in terms of appropriate fields.

Quantization

The quantization proceeds as usual. We regard ϕ_k and π_k as operator fields, and introduce the equal-time commutators given by

$$\begin{aligned} [\phi_k(t,\vec{x}), \pi_{k'}(t,\vec{x}')] &= i\delta_{kk'}\delta^3(\vec{x}-\vec{x}') \\ [\phi_k(t,\vec{x}), \phi_{k'}(t,\vec{x}')] &= [\pi_k(t,\vec{x}), \pi_{k'}(t,\vec{x}')] = 0\,; \end{aligned} \tag{4.237}$$

namely, the same as (4.139) except that if ϕ or π belong to different fields, then they always commute. Heisenberg's equation of motion for ϕ_k is

$$-i\dot{\phi}_k(x) = [H, \phi_k(x)] = [H_1 + H_2, \phi_k(x)] = [H_k, \phi_k(x)] \tag{4.238}$$

which, together with the corresponding equation for π_k and following exactly the same derivation as (4.141) through (4.154), leads to the Klein-Gordon equation for ϕ_k:

$$(\partial^2 + m^2)\phi_k = 0 \qquad (k = 1, 2)\,. \tag{4.239}$$

Then, the two fields ϕ_k can be momentum-expanded using the normal-mode functions $e_{\vec{p}}(x)$ (4.174) together with the creation and annihilation operators for each field:

$$\phi_k(x) = \sum_{\vec{p}} \Big(a_{k\vec{p}}\, e_{\vec{p}}(x) + a^\dagger_{k\vec{p}}\, e^*_{\vec{p}}(x)\Big) \qquad (k = 1, 2)\,. \tag{4.240}$$

Note that we can use the same $e_{\vec{p}}(x)$ for both fields since they have the same mass which results in the same value of p^0 for a given momentum $\vec{p}$. Then, as before, the commutation relations (4.237) lead to

$$\begin{aligned} [a_{k\vec{p}}, a^\dagger_{k'\vec{p}'}] &= \delta_{kk'}\delta_{\vec{p},\vec{p}'} \\ [a_{k\vec{p}}, a_{k'\vec{p}'}] &= [a^\dagger_{k\vec{p}}, a^\dagger_{k'\vec{p}'}] = 0\,. \end{aligned} \tag{4.241}$$

The equal-time commutation relations among ϕ_k and π_k (4.237) and the definitions of the non-hermitian fields (4.225) and (4.233) readily lead to

$$\boxed{\begin{array}{c}[\phi(t,\vec{x}),\pi(t,\vec{x}')] = i\delta^3(\vec{x}-\vec{x}') \\ [\phi^\dagger(t,\vec{x}),\pi^\dagger(t,\vec{x}')] = i\delta^3(\vec{x}-\vec{x}') \\ \text{all others} = 0\end{array}}\,. \tag{4.242}$$

Note that this is what one would get if ϕ and $\phi^\dagger$ (and π and $\pi^\dagger$) are regarded as independent fields and the standard quantization procedure is followed.

The momentum expansion of ϕ is obtained simply from those of ϕ_1 and ϕ_2:

$$\begin{aligned}\phi(x) &\equiv \frac{1}{\sqrt{2}}\Big(\phi_1(x)+i\phi_2(x)\Big) \\ &= \frac{1}{\sqrt{2}}\Big[\sum_{\vec{p}}(a_{1\vec{p}}\,e_{\vec{p}}+a_{1\vec{p}}^\dagger e_{\vec{p}}^*)+i\sum_{\vec{p}}(a_{2\vec{p}}\,e_{\vec{p}}+a_{2\vec{p}}^\dagger e_{\vec{p}}^*)\Big] \\ &= \sum_{\vec{p}}\left[\frac{1}{\sqrt{2}}(a_{1\vec{p}}+ia_{2\vec{p}})e_{\vec{p}}+\frac{1}{\sqrt{2}}(a_{1\vec{p}}^\dagger+ia_{2\vec{p}}^\dagger)e_{\vec{p}}^*)\right].\end{aligned} \tag{4.243}$$

Defining a set of new operators by

$$\begin{aligned} a_{\vec{p}} &\stackrel{\text{def}}{\equiv} \frac{1}{\sqrt{2}}(a_{1\vec{p}}+ia_{2\vec{p}})\,, \\ b_{\vec{p}} &\stackrel{\text{def}}{\equiv} \frac{1}{\sqrt{2}}(a_{1\vec{p}}-ia_{2\vec{p}}) \quad\to\quad b_{\vec{p}}^\dagger = \frac{1}{\sqrt{2}}(a_{1\vec{p}}^\dagger+ia_{2\vec{p}}^\dagger)\,, \end{aligned} \tag{4.244}$$

the field ϕ can then be expanded as

$$\boxed{\phi(x)=\sum_{\vec{p}}\Big(a_{\vec{p}}\,e_{\vec{p}}(x)+b_{\vec{p}}^\dagger e_{\vec{p}}^*(x)\Big)}\,. \tag{4.245}$$

The commutation relations among a's and b's easily follow from those among a_k's:

$$\boxed{\begin{array}{c}[a_{\vec{p}},a_{\vec{p}'}^\dagger]=\delta_{\vec{p},\vec{p}'}\,,\quad [b_{\vec{p}},b_{\vec{p}'}^\dagger]=\delta_{\vec{p},\vec{p}'} \\ \text{all others} = 0\end{array}}\,. \tag{4.246}$$

This indicates that $a_{\vec{p}}$ and $b_{\vec{p}}$ act as annihilation operators of some particles. In particular, $a_{\vec{p}}^\dagger a_{\vec{p}}$ would be the number operator of the particle 'a' and $b_{\vec{p}}^\dagger b_{\vec{p}}$ that of the particle 'b'.

Exercise 4.6 *Quantization of charged Klein-Gordon field.*
(a) Derive the commutation relations among a's and b's (4.246) from those among a_1's and a_2's: (4.241).
(b) Derive the equal-time commutators among ϕ and π: (4.242) from those among ϕ_k and π_k (4.237).

Following the same derivation as (4.196) through (4.199), the total Hamiltonian of the two fields, H_1 and H_2, can be written as

$$H_1 = \sum_{\vec{p}} p^0 a^\dagger_{1\vec{p}} a_{1\vec{p}}\,, \quad H_2 = \sum_{\vec{p}} p^0 a^\dagger_{2\vec{p}} a_{2\vec{p}}\,. \tag{4.247}$$

The total Hamiltonian of the system is simply the sum of the two:

$$H = H_1 + H_2 = \sum_{\vec{p}} p^0 (a^\dagger_{1\vec{p}} a_{1\vec{p}} + a^\dagger_{2\vec{p}} a_{2\vec{p}})\,. \tag{4.248}$$

Noting that

$$\begin{aligned} a^\dagger_{\vec{p}} a_{\vec{p}} + b^\dagger_{\vec{p}} b_{\vec{p}} &= \frac{1}{2}(a^\dagger_{1\vec{p}} - i a^\dagger_{2\vec{p}})(a_{1\vec{p}} + i a_{2\vec{p}}) + \frac{1}{2}(a^\dagger_{1\vec{p}} + i a^\dagger_{2\vec{p}})(a_{1\vec{p}} - i a_{2\vec{p}}) \\ &= a^\dagger_{1\vec{p}} a_{1\vec{p}} + a^\dagger_{2\vec{p}} a_{2\vec{p}}\,, \end{aligned} \tag{4.249}$$

we have

$$\rightarrow \quad H = \sum_{\vec{p}} p^0 (a^\dagger_{\vec{p}} a_{\vec{p}} + b^\dagger_{\vec{p}} b_{\vec{p}})\,, \tag{4.250}$$

which shows that the total energy is the sum of the energies of all 'a-particles' and 'b-particles'. Similarly, the total momentum defined by (4.122) is just the sum of those of two fields: using (4.208),

$$\begin{aligned} \vec{P} &= -\int d^3x\, \pi_k \vec{\nabla} \phi_k = \underbrace{-\int d^3x\, \pi_1 \vec{\nabla} \phi_1}_{\sum_{\vec{p}} \vec{p}\, a^\dagger_{1\vec{p}} a_{1\vec{p}}} \underbrace{-\int d^3x\, \pi_2 \vec{\nabla} \phi_2}_{\sum_{\vec{p}} \vec{p}\, a^\dagger_{2\vec{p}} a_{2\vec{p}}} \\ &= \sum_{\vec{p}} \vec{p} (a^\dagger_{1\vec{p}} a_{1\vec{p}} + a^\dagger_{2\vec{p}} a_{2\vec{p}})\,. \end{aligned} \tag{4.251}$$

Then, (4.249) allows us to write it as

$$\vec{P} = \sum_{\vec{p}} \vec{p} (a^\dagger_{\vec{p}} a_{\vec{p}} + b^\dagger_{\vec{p}} b_{\vec{p}})\,, \tag{4.252}$$

which shows that the total momentum is the sum of the momenta of all a-particles and b-particles. Next, we will see that the two types of particles have opposite 'charge', and that the total 'charge' of the universe is conserved.

Noether current of internal symmetry

We have seen that the symmetry under space-time translation led to the conservation of total energy and momentum. Here, we encounter a slightly different kind of conserved quantity arising from an internal symmetry which crept into the system because the two fields ϕ_1 and ϕ_2 have the same mass. We start from non-quantized fields.

We first note that the Lagrangian (4.228) is invariant under the phase rotation

$$\phi'(x) = e^{i\theta}\phi(x) \quad \rightarrow \quad \phi'^{\dagger}(x) = e^{-i\theta}\phi^{\dagger}(x)\,, \tag{4.253}$$

where θ is a real parameter and the space-time points are equal on the both sides of the equalities. In fact,

$$\mathcal{L}' \equiv \partial_\mu \phi'^{\dagger}\partial^\mu\phi' - m^2\phi'^{\dagger}\phi' = \partial_\mu\phi^{\dagger}\partial^\mu\phi - m^2\phi^{\dagger}\phi = \mathcal{L}\,. \tag{4.254}$$

Note that ϕ and $\phi^{\dagger}$ are not treated as independent in this procedure; indeed, it is critical that $\phi^{\dagger}$ is rotated by a phase exactly opposite to that of ϕ. In terms of ϕ_1 and ϕ_2, the phase rotation is written as

$$\begin{aligned}
\phi' &\equiv \frac{1}{\sqrt{2}}(\phi_1' + i\phi_2') \\
&= e^{i\theta}\phi = (\cos\theta + i\sin\theta)\frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \\
&= \frac{1}{\sqrt{2}}\big[(\cos\theta\phi_1 - \sin\theta\phi_2) + i(\sin\theta\phi_1 + \cos\theta\phi_2)\big]\,.
\end{aligned} \tag{4.255}$$

Equating the real parts and the imaginary parts, respectively, we have

$$\begin{pmatrix} \phi_1' \\ \phi_2' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}\,, \tag{4.256}$$

which is a rotation in the space spanned by ϕ_1 and ϕ_2. Such transformations form a group called $SO(2)$ - a group formed by 2×2 *o*rthogonal matrices with det $= 1$ (*s*pecial). This group is then apparently equivalent to the group formed by simple phase rotations $e^{i\theta}$ which are 1×1 *u*nitary matrices - called $U(1)$. Thus, $SO(2)$ and $U(1)$ have the same group structure:

$$SO(2) \sim U(1)\,. \tag{4.257}$$

Let's try Noether's mathematical trick again to extract a conserved current associated with this symmetry. For the case of space-time translation, the Lagrangian density $\mathcal{L}$ changed its value under the transformation since it is a function of x through

$\phi(x)$. This time, the value of $\mathcal{L}$ is unchanged under the transformation. For a small θ, the changes in the fields and their derivatives are

$$\begin{array}{cc} \delta\phi = i\theta\phi\,, & \delta\phi^\dagger = -i\theta\phi^\dagger\,, \\ \delta(\partial_\mu\phi) = i\theta\partial_\mu\phi\,, & \delta(\partial_\mu\phi^\dagger) = -i\theta\partial_\mu\phi^\dagger \end{array} \tag{4.258}$$

Then, the change in $\mathcal{L}(\phi, \phi^\dagger, \partial_\mu\phi, \partial_\mu\phi^\dagger)$, which should be zero, can be written as

$$\begin{aligned} \delta\mathcal{L} &= \underbrace{\frac{\partial\mathcal{L}}{\partial\phi}}\overbrace{\delta\phi}^{i\theta\phi} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\overbrace{\delta(\partial_\mu\phi)}^{i\theta\partial_\mu\phi} + \underbrace{\frac{\partial\mathcal{L}}{\partial\phi^\dagger}}\overbrace{\delta\phi^\dagger}^{-i\theta\phi^\dagger} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^\dagger)}\overbrace{\delta(\partial_\mu\phi^\dagger)}^{-i\theta\partial_\mu\phi^\dagger} \\ &\qquad \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \text{ by (4.230)} \qquad \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^\dagger)} \text{ by (4.231)} \\ &= i\theta\Big[\underbrace{\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu\phi}_{\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\phi\right)} \underbrace{-\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^\dagger)}\phi^\dagger - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^\dagger)}\partial_\mu\phi^\dagger}_{-\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^\dagger)}\phi^\dagger\right)}\Big] \\ &= i\theta\partial_\mu\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\phi - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^\dagger)}\phi^\dagger\Big) = 0\,. \end{aligned} \tag{4.259}$$

Thus, we have a conserved current given by

$$\boxed{\partial_\mu J^\mu = 0\,, \qquad J^\mu = i\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^\dagger)}\phi^\dagger - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\phi\Big)\,,} \tag{4.260}$$

where the factor 'i' is arbitrary at this point. The conserved quantity, which we generically call 'charge', is the space integral of the time component:

$$\boxed{Q = \int d^3x J^0\,, \qquad \dot{Q} = 0}\,. \tag{4.261}$$

So far, we have not assumed any particular functional form of the Lagrangian. For the Klein-Gordon Lagrangian (4.228), we have

$$J^\mu = i\Big[(\partial^\mu\phi)\phi^\dagger - (\partial^\mu\phi^\dagger)\phi\Big] = \phi^\dagger i\overset{\leftrightarrow}{\partial^\mu}\phi\,, \tag{4.262}$$

and

$$Q = \int d^3x J^0 = \int d^3x\,\phi^\dagger i\overset{\leftrightarrow}{\partial^0}\phi\,, \tag{4.263}$$

which is exactly the same conserved current as (2.45) obtained earlier by directly constructing a conserved quantity from the Klein-Gordon equation. This time, the

same conserved current was obtained from the $U(1)$ symmetry of the Lagrangian through Noether's trick.

At this point, we consider the fields that appear in Q as operators, and apply the momentum expansion to write it in terms of creation and annihilation operators. A simple calculation gives (normal ordering is implicit)

$$Q = \int d^3x\, \phi^\dagger i \overset{\leftrightarrow}{\partial}{}^0 \phi = \sum_{\vec{p}} (a^\dagger_{\vec{p}} a_{\vec{p}} - b^\dagger_{\vec{p}} b_{\vec{p}}) \tag{4.264}$$

or

$$Q = N_a - N_b \tag{4.265}$$

where

$$N_a \overset{\text{def}}{\equiv} \sum_{\vec{p}} a^\dagger_{\vec{p}} a_{\vec{p}}\,, \qquad N_b \overset{\text{def}}{\equiv} \sum_{\vec{p}} b^\dagger_{\vec{p}} b_{\vec{p}}\,, \tag{4.266}$$

are the operators that represents the total number of a-particles and b-particles, respectively. Then, the Heisenberg equation of motion tells us that Q is indeed a constant of motion:

$$\begin{aligned} -i\dot{Q} &= [H, Q] \\ &= [\sum_{\vec{p}} p^0 (a^\dagger_{\vec{p}} a_{\vec{p}} + b^\dagger_{\vec{p}} b_{\vec{p}}), \sum_{\vec{p}} (a^\dagger_{\vec{p}} a_{\vec{p}} - b^\dagger_{\vec{p}} b_{\vec{p}})\,] \\ &= 0\,, \end{aligned} \tag{4.267}$$

where we have used the fact that the number operators $a^\dagger_{\vec{p}} a_{\vec{p}}$ and $b^\dagger_{\vec{p}'} b_{\vec{p}'}$ all commute among them.

Thus, the total number of a-particles minus that of b-particles is conserved. The expressions of the total charge (4.266) indicates that an a-particle carries charge $+1$ and a b-particle carries charge -1 regardless of momentum. Together with the fact that the two types of particles have the same mass, a natural interpretation is that they are antiparticles of each other. Which is plus and which is minus is arbitrary at this point, and so is the absolute value of the charge; we could have multiplied any constant to (4.262). This charge is not necessarily the electric charge. Electric charge is something that couples to photons, and we have not introduced any interaction yet. Here, the charge simply refers to the quantum numbers attached to the two types of particles. Then, why cannot ϕ_1 and ϕ_2 be interpreted as antiparticles of each other? This can be seen by writing the conserved charge Q in terms of a_k's and $a^\dagger_k$'s using (4.244). The result is

$$Q = i \sum_{\vec{p}} (a^\dagger_{1\vec{p}} a_{2\vec{p}} - a^\dagger_{2\vec{p}} a_{1\vec{p}}) \tag{4.268}$$

which does not contain number operators of particles 1 and 2. One should keep in mind, however, that the two interpretations - particles a and b, or particles 1 and 2 - are equivalent.

Exercise 4.7 *Conserved quantities of charged spin-0 field.*
(*a*) *Start from the Langrangian density in terms of* ϕ *and* $\phi^\dagger$*:*

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi \,,$$

and treat ϕ *and* $\phi^\dagger$ *as independent variables to show that the hamiltonian is given by*

$$H = \int d^3x \Big(\dot{\phi}^\dagger \dot{\phi} + \vec{\nabla}\phi^\dagger \cdot \vec{\nabla}\phi + m^2 \phi^\dagger \phi \Big) \,.$$

Compare with the result obtained by regarding ϕ_1 *and* ϕ_2 *as independent variables and verify that they are identical. Up to here, regard the fields as non-quantized.*
(*b*) *Express* H *above in terms of creation and annihilation operators* a*'s and* b*'s. Use the momentum expansions in terms of* a*'s and* b*'s and commutators among them. Do not refer to* a_1*'s and* a_2*'s. Normal ordering is implicit.*
c) Similarly, show that the total charge operator Q *given by*

$$Q = \int d^3x J^0 = \int d^3x \phi^\dagger i \overset{\leftrightarrow}{\partial^0} \phi \,,$$

can be written in terms of a*'s and* b*'s as*

$$Q = \sum_{\vec{p}} (a^\dagger_{\vec{p}} a_{\vec{p}} - b^\dagger_{\vec{p}} b_{\vec{p}}) \,.$$

When we studied the Klein-Gordon equation without field quantization, we had two problems: negative energy and negative probability. Now we can see how they are solved by quantizing the field. First, the energy is now representated by the Hamiltonian H and (4.250) shows that both a-particles and b-particles contribute positively. Second, the probability current is now interpreted as the charge current and thus there is no inconsistency in the total charge having a negative value.

For a real Klein-Gordon field, by definition we cannot rotate the phase of the field as done in the derivation of the conserved charge. Nothing stops us, however, from taking the field ϕ in (4.259) as real; then, the entire derivation that leads to the conserved current (4.259) is still valid. Using a real field ϕ in the conserved current, however, gives $J^\mu = 0$. Thus, a conserved charge current does not exist for a hermitian Klein-Gordon field, which is consistent with the interpretation that a hermitian Klein-Gordon field represents a particle that is antiparticle of itself. Then, what happened to the probability current? Can one define the probability to find a particle at a given point x at a given time t in the framework of quantum field theory? This brings us to the next topic: the state $\phi(x)|0\rangle$ and the important related issue of microscopic causality.

4.5 Microscopic causality

The state $\phi(x)|0\rangle$

Let's take a hermitian Klein-Gordon field $\phi(x)$ as the simplest example. In order to define the probability to find a particle at a given space-time point $x = (x^0, \vec{x})$, one needs a state where at time x^0 a particle is localized at position $\vec{x}$. Is $\phi(t, \vec{x})|0\rangle$ such a state? If that is the case, then the inner product of $\phi(t, \vec{x})|0\rangle$ with $\phi(t, \vec{y})|0\rangle$, where times are the same, should be zero unless $\vec{x} = \vec{y}$. We will see that such is not the case. For convenience, we first divide ϕ into the positive-energy part and the negative-energy part:

$$\phi(x) = \phi_a(x) + \phi_a^\dagger(x)\,, \tag{4.269}$$

with

$$\phi_a(x) \stackrel{\text{def}}{\equiv} \sum_{\vec{p}} a_{\vec{p}}\, e_{\vec{p}}(x)\,, \qquad \phi_a^\dagger(x) = \sum_{\vec{p}} a_{\vec{p}}^\dagger\, e_{\vec{p}}^*(x)\,. \tag{4.270}$$

Since $a_{\vec{p}}|0\rangle = 0$ and $\langle 0|a_{\vec{p}}^\dagger = 0$, ϕ_a facing the vacuum on its right or $\phi_a^\dagger$ facing the vacuum on its left annihilates the state. Thus, the inner product of $\phi(x)|0\rangle$ and $\phi(y)|0\rangle$, where x^0 and y^0 are in general not the same, can be written as (using the hermiticity $\phi^\dagger = \phi$)

$$\begin{aligned}
\langle 0|\phi^\dagger(x)\phi(y)|0\rangle &= \langle 0|\phi(x)\phi(y)|0\rangle = \langle 0|\phi_a(x)\phi_a^\dagger(y)|0\rangle \\
&= \langle 0|\sum_{\vec{p}} a_{\vec{p}}\, e_{\vec{p}}(x) \sum_{\vec{p}'} a_{\vec{p}'}^\dagger e_{\vec{p}'}^*(y)|0\rangle \\
&= \langle 0|\sum_{\vec{p},\vec{p}'} \underbrace{a_{\vec{p}}\, a_{\vec{p}'}^\dagger}_{\substack{\delta_{\vec{p},\vec{p}'} + a_{\vec{p}'}^\dagger a_{\vec{p}} \\ \to 0}} e_{\vec{p}}(x) e_{\vec{p}'}^*(y)|0\rangle \\
&= \sum_{\vec{p}} e_{\vec{p}}(x) e_{\vec{p}}^*(y)\, \underbrace{\langle 0|0\rangle}_{1}\,,
\end{aligned} \tag{4.271}$$

The last expression can be written as an integral over $\vec{p}$ using (4.159):

$$\begin{aligned}
\sum_{\vec{p}} e_{\vec{p}}(x) e_{\vec{p}}^*(y) &= \frac{V}{(2\pi)^3} \int d^3p \frac{e^{-ip\cdot x}}{\sqrt{2p^0 V}} \frac{e^{ip\cdot y}}{\sqrt{2p^0 V}} \\
&= \frac{1}{(2\pi)^3} \int \frac{d^3p}{2p^0} e^{-ip\cdot(x-y)} = \Delta_+(x-y)\,,
\end{aligned} \tag{4.272}$$

where we have defined the function $\Delta_+(z)$ to be

$$\Delta_+(z) \stackrel{\text{def}}{\equiv} \frac{1}{(2\pi)^3} \int \frac{d^3p}{2p^0} e^{-ip\cdot z}\,, \tag{4.273}$$

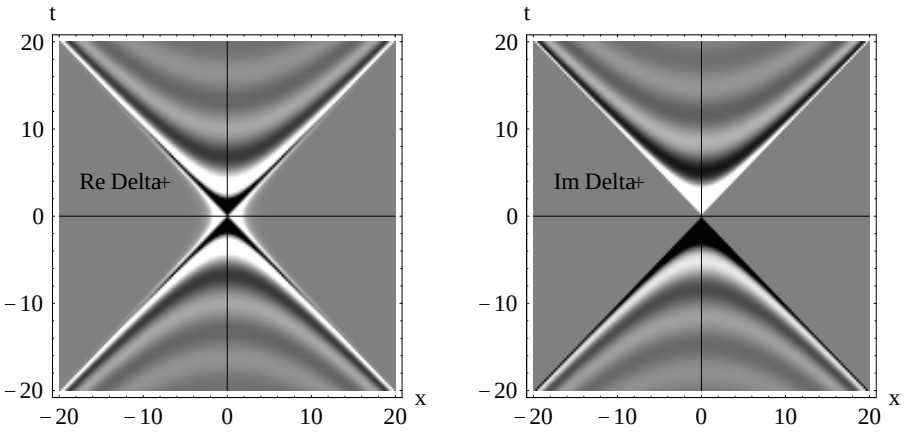


Figure 4.3: The real and imaginary parts of the function $\Delta_+(x)$. The predominant gray tone in the space-like region represents zero, darker is negative and lighter is positive. Units of x and t are $1/m$. The function is real and positive in the space-like region, and in general complex in the time-like region.

where z is a real 4-vector and $p^0 \equiv \sqrt{\vec{p}^2 + m^2}$. Thus, in terms of this function, we have

$$\langle 0|\phi(x)\phi(y)|0\rangle = \Delta_+(x-y)\,. \tag{4.274}$$

Up to this point, x and y are arbitrary. Now let's assume $x^0 = y^0$ (equal-time). Then, the time component of $z \equiv x - y$ becomes zero and

$$\begin{aligned}\Delta_+(0,\vec{z}) &= \frac{1}{(2\pi)^3}\int \frac{d^3p}{2p^0} e^{i\vec{p}\cdot\vec{z}} \\ &= \frac{1}{(2\pi)^3}\int \frac{2\pi p^2 dp\, d\cos\theta}{2\sqrt{p^2+m^2}} e^{ipr\cos\theta}\end{aligned} \tag{4.275}$$

where $p \equiv |\vec{p}|$ and $r \equiv |\vec{z}|$. This looks awful, but actually it is a well-defined real function of r and can be expressed in terms of the modified Bessel function $K_1(z)$ which is a solution of the differential equation $z^2X'' + zX' - (z^2+n^2)X = 0$ with $n = 1$:

$$\Delta_+(0,\vec{z}) = \frac{m}{4\pi^2 r} K_1(mr) \overset{r\to\infty}{\sim} \frac{\sqrt{m}}{(2\pi r)^{3/2}}\, e^{-mr}\,. \tag{4.276}$$

This is a rapidly decreasing function of r, but the important fact is that it is not zero. Thus, the state $\phi(t,\vec{x})|0\rangle$ cannot be interpreted as the state in which a particle is localized at $(t,\vec{x})$; it is sharply peaked at that point, but there is some spill-over. A profile of the function $\Delta_+(x)$ is shown in Figure 4.3. For completeness, the function

$\Delta_+(x)$ in the entire region is given by

$$(x^0>0)\;\; \Delta_+(x) = \begin{cases} \dfrac{m}{8\pi\sqrt{x^2}}\Big[Y_1(m\sqrt{x^2}) + iJ_1(m\sqrt{x^2})\Big] & (x^2>0) \\ \dfrac{m}{4\pi^2\sqrt{-x^2}}K_1(m\sqrt{-x^2}) & (x^2<0) \end{cases} \tag{4.277}$$

$$(x^0<0)\;\; : \text{use} \quad \Delta_+(-x) = \Delta_+^*(x) \quad [\text{follows directly from (4.273)}]\,,$$

where J_1 and Y_1 are the standard Bessel functions that are solutions of $z^2X'' + zX' + (z^2-n^2)X = 0$ with $n=1$. The function $\Delta_+(x)$ is real and positive for $x^2<0$, and complex and oscillatory for $x^2>0$.

Let's step back a little and examine more carefully what we have here. First, we note that the norm of the state $\phi(x)|0\rangle$ is infinity:

$$\langle 0|\phi^\dagger(x)\,\phi(x)|0\rangle = \Delta_+(0) = \int \frac{d^3p}{(2\pi)^3 2p^0} = \infty\,. \tag{4.278}$$

Undaunted, let's proceed. We are in the Heisenberg picture, thus the states are constant over time and the operators carry all time dependences. The state $\phi(t,\vec{x})|0\rangle$ then represents a particle nearly localized at $(t,\vec{x})$, but through the time-varying operators, it also represents the entire history of the evolution from the infinite past to the infinite future. Then what is the inner product of two such states $\phi(x)|0\rangle$ and $\phi(y)|0\rangle$? It may be easier to visualize it if we move to the Schrödinger picture. For simplicity, assume $y^0=0$ and $x^0>y^0$. At $t=y^0=0$, states and operators in the two pictures are taken to be identical. Namely, the operators and states in the Schrödinger picture are defined by

$$\phi_S(\vec{x}) \equiv \phi(0,\vec{x})\,, \quad \phi_S(\vec{y}) \equiv \phi(0,\vec{y})\,, \tag{4.279}$$

$$|\vec{x}\rangle_S \equiv \phi_S(\vec{x})|0\rangle\,, \quad |\vec{y}\rangle_S \equiv \phi_S(\vec{y})|0\rangle\,. \tag{4.280}$$

where the subscript 'S' indicates the Schrödinger picture. The time evolution of the state $|\vec{y}\rangle_S$ is given by

$$|t,\vec{y}\rangle_S = e^{-iHt}|\vec{y}\rangle_S\,. \tag{4.281}$$

Using the space-time translation formula (4.211), $\phi(x^0,\vec{x})$ is related to $\phi(0,\vec{x})$ by

$$\phi(x^0,\vec{x}) = e^{iHx^0}\underbrace{\phi(0,\vec{x})}_{\phi_S(\vec{x})}e^{-iHx^0}\,. \tag{4.282}$$

Then, the inner product $\langle 0|\phi^\dagger(x)\phi(y)|0\rangle$ can be written as (with $y^0=0$)

$$\begin{aligned} \langle 0|\phi^\dagger(x^0,\vec{x})\phi(y^0,\vec{y})|0\rangle &= \underbrace{\langle 0|e^{iHx^0}}_{\langle 0|}\phi_S^\dagger(\vec{x})\underbrace{e^{-iHx^0}\overbrace{\phi(0,\vec{y})|0\rangle}^{|\vec{y}\rangle_S}}_{|x^0,\vec{y}\rangle_S} \\ &= {}_S\langle\vec{x}\,|x^0,\vec{y}\rangle_S\,, \end{aligned} \tag{4.283}$$

where we have used $H = 0$ for the vacuum (H is normal-ordered implicitly), which leads to

$$e^{-iHx^0}|0\rangle_S = |0\rangle_S \,. \tag{4.284}$$

If we interpret the state $|0, \vec{y}\rangle_S$ as a creation of a particle at position $\vec{y}$ at $t = 0$, then the state $|x^0, \vec{y}\rangle_S$ is its time-evolved state at time x^0. The inner product above would then be the amplitude to find the particle at position $\vec{x}$. Thus, $\langle 0|\phi^\dagger(x)\phi(y)|0\rangle$ can be loosely interpreted as the amplitude of a particle created at $(y^0, \vec{y})$ to propagate to $(x^0, \vec{x})$.

Then, what do we make of the fact that the inner product is non-zero even if the separation between x and y are space-like (namely, cannot be reached from one to the other by a velocity less than the speed of light)? This apparent paradox arises since the state $\phi(x)|0\rangle$ is actually not a state a particle is localized at $\vec{x}$ at time t. Thus, the non-zero amplitude outside the light cone does not mean that a particle can actually move faster than speed of light; rather, if we interpret $\phi(x)|0\rangle$ as the state where a particle is created at x, then we have to accept that there is non-zero 'propagation' outside the light cone. It is largely a matter of terminology.

Causality

The 'propagation' outside of the light cone that we have just seen is still alarming. Does it violate the principle of special relativity which states that any physical event at x should not affect another physical event at y if the x and y are outside the light cone of each other? Physical events are detected through observables. A hermitian field $\phi(x)$ can be viewed as an observable; in fact, it was introduced by quantizing the actual string motion. Then, causality requires that two observables $\phi(x)$ and $\phi(y)$ commute if the separation of x and y is space-like; namely,

$$[\phi(x), \phi(y)] = 0 \,, \quad \text{if } (x-y)^2 < 0 \quad (?) \tag{4.285}$$

Note that the quantization condition (4.139) we had before, $[\phi(t, \vec{x}), \phi(t, \vec{y})] = 0$, is an *equal-time* commutator; now, x^0 and y^0 are in general different.

First, divide ϕ into the negative-energy and positive-energy parts as in (4.269). Since $a_{\vec{p}}$'s commute with $a_{\vec{p}}$'s and $a^\dagger_{\vec{p}}$'s commute with $a^\dagger_{\vec{p}}$'s,

$$[\phi_a(x), \phi_a(y)] = [\phi_a^\dagger(x), \phi_a^\dagger(y)] = 0 \,, \tag{4.286}$$

for any x and y. Then, the commutator (4.285) can be written as

$$\begin{aligned} [\phi(x), \phi(y)] &= [\phi_a(x) + \phi_a^\dagger(x) \,, \phi_a(y) + \phi_a^\dagger(y)] \\ &= [\phi_a(x), \phi_a^\dagger(y)] + \underbrace{[\phi_a^\dagger(x), \phi_a(y)]}_{-[\phi_a(y), \phi_a^\dagger(x)]} \\ &= [\phi_a(x), \phi_a^\dagger(y)] - (x \leftrightarrow y) \,. \end{aligned} \tag{4.287}$$

Using the expansion (4.270) and the definition of the function Δ_+ (4.272), we have

$$\begin{aligned}[\phi_a(x), \phi_a^\dagger(y)] &= [\sum_{\vec{p}} a_{\vec{p}}\, e_{\vec{p}}(x)\,, \sum_{\vec{p}'} a_{\vec{p}'}^\dagger e_{\vec{p}'}^*(y)] \\ &= \sum_{\vec{p},\vec{p}'} e_{\vec{p}}(x) e_{\vec{p}'}^*(y) \underbrace{[a_{\vec{p}}\,, a_{\vec{p}'}^\dagger]}_{\delta_{\vec{p},\vec{p}'}} \\ &= \sum_{\vec{p}} e_{\vec{p}}(x) e_{\vec{p}}^*(y) = \Delta_+(x-y)\,. \end{aligned} \tag{4.288}$$

Thus,

$$[\phi(x), \phi(y)] = i\Delta(x-y)\,, \tag{4.289}$$

with

$$i\Delta(z) \stackrel{\text{def}}{\equiv} \Delta_+(z) - \Delta_+(-z)\,. \tag{4.290}$$

Since $\Delta_+(-x) = \Delta_+^*(x)$ (4.277),

$$\Delta_+(z) - \Delta_+(-z) = 2i\mathrm{Im}\Delta_+(z)$$

is purely imaginary, and the factor i is added in the definition to make $\Delta(x)$ real. Our task now is to show that $\Delta(z) = 0$ for $z^2 < 0$ which establishes the causality (4.285). To do so, we will show that $\Delta(z)$ is Lorentz-invariant under proper and orthochronous transformations; or more precisely, it has the same value when one replaces z by Λz where Λ is proper and orthochronous. It should then be a function of the only Lorentz-invariant quantity formed by z: $z^2 \equiv z^{0\,2} - \vec{z}^2$. Then, for a given value of $z^2 < 0$, we take $z' = (0, \vec{z}')$ with $\vec{z}'^2 = -z^2 > 0$ for which $\Delta(z)$ should have the same value; namely

$$\Delta(z) = \Delta(z') \qquad \text{with } z' = (0, \vec{z}') \text{ and } \vec{z}'^2 = -z^2\,. \tag{4.291}$$

With $z' = x - y$, the commutator (4.289) with $z' = (0, \vec{z}')$ becomes the equal-time commutator $[\phi(t, \vec{x}), \phi(t, \vec{y})]$ which we know is zero. Thus, all we need to demonstrate is the Lorentz invariance of Δ (or equivalently, of Δ_+); namely, that the value of the function does not change when the space-time argument is transformed by a proper and orthochronous Lorentz transformation.

The Lorentz invariance of $\Delta_+(x)$ can be shown using the following useful identity:

$$\boxed{\int d^4p\, \delta(p^2 - m^2)\theta(p^0) \cdots = \int \frac{d^3p}{2p^0} \cdots}\,, \tag{4.292}$$

where $d^4p \equiv dp^0 dp^1 dp^2 dp^3$, '$\ldots$' is any function of p^μ, $p^2 \equiv p^{0\,2} - \vec{p}^{\,2}$, and $\theta(p^0)$ is a step function that limits the integration to positive p^0:

$$\theta(p^0) = \begin{cases} 1 & (p^0 \geq 0) \\ 0 & (p^0 < 0) \end{cases}\,. \tag{4.293}$$

In (4.292), p^0 is an integration variable on the left-hand side, while on the right-hand side, p^0 is fixed to $\sqrt{\vec{p}^{\,2}+m^2}$.

Proof of (4.292)
We will apply the property of the delta function (4.114) to $f(x) = x^2 - (\vec{p}^{\,2}+m^2)$. The roots of $f(x)=0$ are

$$x_1 = +\sqrt{\vec{p}^{\,2}+m^2}\,, \quad x_2 = -\sqrt{\vec{p}^{\,2}+m^2}\,. \tag{4.294}$$

Using $f'(x) = 2x$,

$$\begin{aligned}\delta(f(x)) &= \sum_i \frac{1}{|f'(x_i)|}\delta(x-x_i) \\ &= \frac{\delta(x-x_1)}{2|x_1|} + \frac{\delta(x-x_2)}{2|x_2|}\,.\end{aligned} \tag{4.295}$$

Then, for an arbitrary function $F(x)$, we have

$$\begin{aligned}&\int_0^\infty dx\,\delta\Big(x^2-(\vec{p}^{\,2}+m^2)\Big)F(x) \\ &\quad= \int_0^\infty dx\Bigg(\frac{\delta(x-\sqrt{\vec{p}^{\,2}+m^2})}{2\sqrt{\vec{p}^{\,2}+m^2}} + \underbrace{\frac{\delta(x+\sqrt{\vec{p}^{\,2}+m^2})}{2\sqrt{\vec{p}^{\,2}+m^2}}}_{\text{does not contribute}}\Bigg)F(x) \\ &\quad= \frac{F(\sqrt{\vec{p}^{\,2}+m^2})}{2\sqrt{\vec{p}^{\,2}+m^2}}\,.\end{aligned} \tag{4.296}$$

Renaming x as p^0 and using the definition $p^2 \equiv p^{0\,2} - \vec{p}^{\,2}$, this can be written as

$$\int_{-\infty}^{\infty} dp^0 \delta(p^2-m^2)\theta(p^0)F(p^0) = \left.\frac{F(p^0)}{2p^0}\right|_{p^0=\sqrt{\vec{p}^{\,2}+m^2}}\,, \tag{4.297}$$

where $\theta(p^0)$ is inserted and the integration range is extended to $(-\infty,\infty)$. Note that p^0 on the left-hand side is an integration variable and not fixed to $\sqrt{\vec{p}^{\,2}+m^2}$. Assuming that F is a function also of $\vec{p}$ and integrating over $\vec{p}$, we obtain (4.292). ∎

Using (4.292), $\Delta_+(x)$ is now written as

$$\Delta_+(x) = \frac{1}{(2\pi)^3}\int d^4p\,\delta(p^2-m^2)\theta(p^0)e^{-ip\cdot x} \tag{4.298}$$

where the integration $d^4p \equiv dp^0 d^3p$ is now over the entire four-dimensional space; in particular, the p^0 range is from $-\infty$ to ∞ thanks to the step function $\theta(p^0)$. Now, is this Lorentz-invariant? Namely, does $\Delta_+(x') = \Delta_+(x)$ hold if $x' = \Lambda x$ where Λ is a proper and orthochronous transformation? Let's see; we take $x' = \Lambda x$ as the argument and relabel $p^\mu \to p^{\mu'}$ which is allowed since they are dummy integration variables:

$$\Delta_+(x') = \frac{1}{(2\pi)^3} \int d^4p' \, \delta(p'^2 - m^2)\theta(p'^0) e^{-ip'\cdot x'} \,. \tag{4.299}$$

Then, we change the integration variables to p which is related to p' by

$$p = \Lambda^{-1} p' \,. \tag{4.300}$$

Since $x' = \Lambda x$ and $p' = \Lambda p$, we have

$$p'^2 = p^2 \,, \quad p' \cdot x' = p \cdot x \,, \tag{4.301}$$

and since Λ is proper and orthochronous,

$$d^4p' = \underbrace{\det \Lambda}_{1} d^4p = d^4p \,, \quad \theta(p'^0) = \theta(p^0) \,. \tag{4.302}$$

The last relation $\theta(p'^0) = \theta(p^0)$ may not be trivial. This relation holds because as long as $p^2 = m^2 > 0$, as guaranteed by $\delta(p^2 - m^2)$, an orthochronous transformation cannot change the sign of the energy p^0. Putting all together, $\Delta_+(x')$ is now

$$\Delta_+(x') = \frac{1}{(2\pi)^3} \int d^4p \, \delta(p^2 - m^2)\theta(p^0) e^{-ip\cdot x} = \Delta_+(x) \,. \tag{4.303}$$

Thus, $i\Delta(x) = \Delta_+(x) - \Delta_+(-x)$ is Lorentz-invariant; namely, it is a function only of x^2, and this completes the proof of the causality. At this point, it is worthwhile to point out the difference between the Lorentz-invariance of a scalar field $\phi(x)$ and that of $\Delta_+(x)$. Both are functions of x, but the scalar field changes its functional form under a Lorentz transformation: $\phi'(x') = \phi(x)$, while $\Delta(x)$ does not:

$$\Delta(x') = \Delta(x) \,. \tag{4.304}$$

This distinction is important; otherwise, any scalar field $\phi(x)$ would be a function only of x^2 !

We have seen that measurements of the Klein-Gordon field at two space-time points are indeed independent if they are separated by a space-like distance. It is interesting to see how this has happened. Since $i\Delta(x-y) = \Delta_+(x-y) - \Delta_+(y-x) = 0$ for space-like $x - y$, we see that $\Delta_+(x - y) = \Delta_+(y - x)$ for a space-like $x - y$. On the other hand, $\Delta_+(x - y)$ had a loose interpretation as the amplitude for a particle

created at y to propagate to x, and according to the formula (4.277) (or Figure 4.3) it is non-zero even when $x-y$ is space-like. Then, the causality is accomplished by the amplitude of propagation from y to x cancelling out that of propagation from x to y.

Exercise 4.8 *Microscopic causality of charged Klein-Gordon field.*
(*a*) *Evaluate the commutator*

$$[\phi(x)\,,\phi^\dagger(y)]$$

where x and y are arbitrary space-time points, and express it in terms of the function Δ_+ (or Δ).
(*b*) *In the case of hermitian field, there was an interpretation for the vanishing of the commutator for x and y separated by a space-like distance: cancellation of the amplitude for a particle to propagate from x to y and that for the propagation from y to x. Find a similar interpretation for this case.*

Lorentz invariance of the quantization procedure

The procedure of canonical quantization is performed in a given frame using equal-time commutation relations. Then, a natural question is whether quantizations performed in different frames result in consistent quantized systems. In order to study this, it is convenient to re-write the sum over momentum as an integral. The reason is that the cube of volume $V=L^3$ is obviously not Lorentz-invariant and thus makes it difficult to connect formulations in different frames.

We start from the momentum expansion of $\phi(x)$ (4.173) and use the identity $\sum_{\vec{p}} = V/(2\pi)^3 \int d^3p$ (4.159):

$$\phi(x) = \sum_{\vec{p}} \Big(a_{\vec{p}}\, e_{\vec{p}}(x) + a^\dagger_{\vec{p}}\, e^*_{\vec{p}}(x) \Big) = \frac{V}{(2\pi)^3} \int d^3p \Big(a_{\vec{p}}\, e_{\vec{p}}(x) + a^\dagger_{\vec{p}}\, e^*_{\vec{p}}(x) \Big)\,, \tag{4.305}$$

and define new annihilation operators and normal-mode functions by

$$\tilde{a}_{\vec{p}} \stackrel{\text{def}}{\equiv} \sqrt{\frac{V}{(2\pi)^3}}\, a_{\vec{p}}\,, \qquad \tilde{e}_{\vec{p}}(x) \stackrel{\text{def}}{\equiv} \sqrt{\frac{V}{(2\pi)^3}}\, e_{\vec{p}}(x) = \frac{e^{-ip\cdot x}}{\sqrt{(2\pi)^3\, 2p^0}} \tag{4.306}$$

to write it as

$$\phi(x) = \int d^3p \Big(\tilde{a}_{\vec{p}}\, \tilde{e}_{\vec{p}}(x) + \tilde{a}^\dagger_{\vec{p}}\, \tilde{e}^*_{\vec{p}}(x) \Big)\,. \tag{4.307}$$

Note that in changing the sum to an integral in (4.305), the operator value of ϕ stays the same. It can be readily verified that the new normal-mode functions satisfy the orthonormal relations given by

$$\begin{aligned} &\int d^3x\, \tilde{e}^*_{\vec{p}}(x) i \overset{\leftrightarrow}{\partial}_0 \tilde{e}_{\vec{p}'}(x) = \delta^3(\vec{p}-\vec{p}')\,, \quad \int d^3x\, \tilde{e}_{\vec{p}}(x) i \overset{\leftrightarrow}{\partial}_0 \tilde{e}^*_{\vec{p}'}(x) = -\delta^3(\vec{p}-\vec{p}')\,, \\ &\int d^3x\, \tilde{e}_{\vec{p}}(x) i \overset{\leftrightarrow}{\partial}_0 \tilde{e}_{\vec{p}'}(x) = 0\,, \qquad\qquad \int d^3x\, \tilde{e}^*_{\vec{p}}(x) i \overset{\leftrightarrow}{\partial}_0 \tilde{e}^*_{\vec{p}'}(x) = 0\,. \end{aligned} \tag{4.308}$$

What is the commutator $[\tilde{a}_{\vec{p}}\,,\tilde{a}^\dagger_{\vec{p}'}]$? Since the discrete-$\vec{p}$ version is $\delta_{\vec{p},\vec{p}'}$, one expects that this is proportional to $\delta^3(\vec{p}-\vec{p}')$. In fact, using the definition (4.306), we find

$$\int d^3p\,[\tilde{a}_{\vec{p}}\,,\tilde{a}^\dagger_{\vec{p}'}] = \underbrace{\int d^3p\,\frac{V}{(2\pi)^3}}_{\sum_{\vec{p}}}\underbrace{[a_{\vec{p}}\,,a^\dagger_{\vec{p}'}]}_{\delta_{\vec{p},\vec{p}'}} = 1\,; \tag{4.309}$$

namely, $[\tilde{a}_{\vec{p}}\,,\tilde{a}^\dagger_{\vec{p}'}]$ acts as $\delta^3(\vec{p}-\vec{p}')$:

$$[\tilde{a}_{\vec{p}}\,,\tilde{a}^\dagger_{\vec{p}'}] = \delta^3(\vec{p}-\vec{p}')\,. \tag{4.310}$$

Since $a_{\vec{p}}$ and $a_{\vec{p}'}$ commute regardless of $\vec{p}$ and $\vec{p}'$ in the discrete-$\vec{p}$ version, they will commute in the continuous-$\vec{p}$ version also:

$$[\tilde{a}_{\vec{p}}\,,\tilde{a}_{\vec{p}'}] = [\tilde{a}^\dagger_{\vec{p}}\,,\tilde{a}^\dagger_{\vec{p}'}] = 0\,. \tag{4.311}$$

We have seen before that $a^\dagger_{\vec{p}}$ creates one particle in the volume V. To see what $\tilde{a}^\dagger_{\vec{p}}$ does, we write the total energy (4.199) in the integral form (without changing its operator value):

$$H = \underbrace{\sum_{\vec{p}}}_{V/(2\pi)^3\int d^3p}\,p^0\,\underbrace{a^\dagger_{\vec{p}}a_{\vec{p}}}_{(2\pi)^3/V\,\tilde{a}^\dagger_{\vec{p}}\tilde{a}_{\vec{p}}} = \int d^3p\,p^0\tilde{a}^\dagger_{\vec{p}}\tilde{a}_{\vec{p}}\,, \tag{4.312}$$

and apply it to the state $\tilde{a}^\dagger_{\vec{p}}|0\rangle$:

$$H\tilde{a}^\dagger_{\vec{p}}\,|0\rangle = \Big(\int d^3p'p^{0\prime}\tilde{a}^\dagger_{\vec{p}'}\underbrace{\tilde{a}_{\vec{p}'}}_{\delta^3(\vec{p}-\vec{p}')+\cancel{\tilde{a}^\dagger_{\vec{p}}\tilde{a}_{\vec{p}'}}}\Big)\tilde{a}^\dagger_{\vec{p}}\,|0\rangle = p^0\,\tilde{a}^\dagger_{\vec{p}}\,|0\rangle\,. \tag{4.313}$$

Thus, the total energy of the state $\tilde{a}^\dagger_{\vec{p}}\,|0\rangle$ is p^0, indicating that $\tilde{a}^\dagger_{\vec{p}}$ creates one particle with momentum $\vec{p}$ in the entire universe. This is not surprising since $\tilde{a}^\dagger_{\vec{p}}$ is proportional to $a^\dagger_{\vec{p}}$, and thus $\tilde{a}^\dagger_{\vec{p}}\,|0\rangle$ has the same eigenvalue as $a^\dagger_{\vec{p}}|0\rangle$; simply, the volume V is now the entire universe. It is usually clear whether we are using discrete $\vec{p}$ or continuous $\vec{p}$; thus, we will hereafter omit the '~' on $a_{\vec{p}}$ and $e_{\vec{p}}(x)$.

Now, let's get back to the question of Lorentz-invariance of the quantization procedure, and define exactly what we mean by the statement. In a frame K, we take the Lagrangian of the Klein-Gordon field ϕ, find the conjugate field π, and introduce the commutation relations among them. When the field is momentum-expanded, the quantization condition becomes equivalent to the commutation relations among the expansion coefficients, namely among $a_{\vec{p}}$'s and $a^\dagger_{\vec{p}}$'s:

$$\begin{array}{c}[a_{\vec{p}}\,,a^\dagger_{\vec{p}'}] = \delta^3(\vec{p}-\vec{p}')\\ \text{all else} = 0\end{array} \quad\leftrightarrow\quad \begin{array}{c}[\phi(t,\vec{x}),\pi(t,\vec{x}')] = i\delta^3(\vec{x}-\vec{x}')\\ \text{all else} = 0\end{array} \tag{4.314}$$

Exercise 4.9 *Continous $\vec{p}$ formalisms.*
Start from the momentum expansion of the charged scalar field

$$\phi(x) = \int d^3p \, (a_{\vec{p}} e_{\vec{p}}(x) + b^\dagger_{\vec{p}} e^*_{\vec{p}}(x)) \tag{4.315}$$

with

$$e_{\vec{p}}(x) \equiv \frac{1}{\sqrt{(2\pi)^3 2p^0}} e^{-ip\cdot x} \tag{4.316}$$

and the commutation relations among creation and annihilation operators

$$\begin{gathered} [a_{\vec{p}}\,, a^\dagger_{\vec{p}'}] = [b_{\vec{p}}\,, b^\dagger_{\vec{p}'}] = \delta^3(\vec{p} - \vec{p}') \\ \textit{all else} = 0 \end{gathered} \tag{4.317}$$

to derive

$$\begin{aligned} [\phi(t,\vec{x}), \pi(t,\vec{x}')] &= i\delta^3(\vec{x} - \vec{x}')\,, \\ [\phi(t,\vec{x}), \phi(t,\vec{x}')] &= [\pi(t,\vec{x}), \pi(t,\vec{x}')] = 0\,, \\ [\phi(t,\vec{x}), \phi^\dagger(t,\vec{x}')] &= [\pi(t,\vec{x}), \pi^\dagger(t,\vec{x}')] = 0\,, \end{aligned} \tag{4.318}$$

where $\pi = \dot{\phi}^\dagger$ and $\pi^\dagger = \dot{\phi}$.

Now, if one performs the same quantization procedure in a different frame K' related to K by a proper and orthochronous transformation Λ, then the question is whether or not the commutation relations in the two frames are consistent. Namely, does the set of quantization conditions in one frame leads to that in another frame and vice versa? We could prove the consistency for the commutation relations for ϕ and π or, equivalently, those for $a_{\vec{p}}$ and $a^\dagger_{\vec{p}}$'s. We choose the latter.

First, we need some principle to relate the operators in the two systems, which is provided by the scalar-field condition

$$\phi'(x') = \phi(x) \qquad (x' = \Lambda x)\,. \tag{4.319}$$

This can be converted to relations for a's and $a^\dagger$'s as follows: Momentum-expanding $\phi'(x')$,

$$\begin{aligned} \phi'(x') &= \int d^3p' \Big(a'_{\vec{p}'} e_{\vec{p}'}(x') + a'^\dagger_{\vec{p}'} e^*_{\vec{p}'}(x') \Big) \\ &= \int d^3p' \frac{1}{\sqrt{(2\pi)^3 2p^{0\prime}}} \Big(a'_{\vec{p}'} e^{-ip'\cdot x'} + a'^\dagger_{\vec{p}'} e^{ip'\cdot x'} \Big) \end{aligned} \tag{4.320}$$

where we labeled the integration variable as $\vec{p}'$ (could be anything since it is a dummy variable), and the prime on $a_{\vec{p}}$ indicates that it is defined in the K' frame. We will

now change the integration variable to p (or more precisely, its space part $\vec{p}$) related to p' by $p' = \Lambda p$. Using the identity (4.292) and (4.302), we have

$$\begin{aligned}\int \frac{d^3p'}{2p^{0\prime}} \cdots &= \int d^4p' \delta(p'^2 - m^2)\theta(p^{0\prime}) \cdots \\ &= \int d^4p\, \delta(p^2 - m^2)\theta(p^0) \cdots = \int \frac{d^3p}{2p^0} \cdots . \end{aligned} \tag{4.321}$$

Thus, replacing d^3p' by $d^3p\,(p^{0\prime}/p^0)$, and using $p' \cdot x' = p \cdot x$, $\phi'(x')$ is now written as

$$\begin{aligned}\phi'(x') &= \int d^3p \frac{1}{\sqrt{(2\pi)^3 2p^0}} \sqrt{\frac{p^{0\prime}}{p^0}} \left(a'_{\vec{p}\prime} e^{-ip\cdot x} + a'^{\dagger}_{\vec{p}\prime} e^{ip\cdot x} \right) \\ &= \int d^3p \sqrt{\frac{p^{0\prime}}{p^0}} \left(a'_{\vec{p}\prime} e_{\vec{p}}(x) + a'^{\dagger}_{\vec{p}\prime} e^*_{\vec{p}}(x) \right) , \end{aligned} \tag{4.322}$$

which should equal to $\phi(x)$; namely,

$$\int d^3p \sqrt{\frac{p^{0\prime}}{p^0}} \left(a'_{\vec{p}\prime} e_{\vec{p}}(x) + a'^{\dagger}_{\vec{p}\prime} e^*_{\vec{p}}(x) \right) = \phi(x) = \int d^3p \left(a_{\vec{p}}\, e_{\vec{p}}(x) + a^{\dagger}_{\vec{p}} e^*_{\vec{p}}(x) \right) . \tag{4.323}$$

Applying $e^*_{\vec{p}}(x) i \overset{\leftrightarrow}{\partial}$ from the left and integrating over $\vec{x}$, the orthonormality relation (4.308) gives

$$\sqrt{\frac{p^{0\prime}}{p^0}} a'_{\vec{p}\prime} = a_{\vec{p}} \quad \rightarrow \quad \sqrt{p^{0\prime}} a'_{\vec{p}\prime} = \sqrt{p^0} a_{\vec{p}} , \tag{4.324}$$

which means that a particle with 4-momentum $p' = \Lambda p$ in the frame K' and a particle with 4-momentum p in the frame K are represented by the same state in the Hilbert space (the over-all constants do not affect the physical meaning of the states).

We now assume that $a_{\vec{p}}$ and $a^{\dagger}_{\vec{q}}$ satisfy $[a_{\vec{p}}, a^{\dagger}_{\vec{q}}] = \delta^3(\vec{p} - \vec{q})$ in the frame K, and derive the corresponding relations in the frame K', $[a'_{\vec{p}\prime}, a'^{\dagger}_{\vec{q}\prime}] = \delta^3(\vec{p}\prime - \vec{q}\prime)$. Using the relation (4.324), the commutation relation between $a'_{\vec{p}\prime}$ and $a'^{\dagger}_{\vec{q}\prime}$ is then

$$[a'_{\vec{p}\prime}, a'^{\dagger}_{\vec{q}\prime}] = \sqrt{\frac{p^0 q^0}{p^{0\prime} q^{0\prime}}} [a_{\vec{p}}\,, a^{\dagger}_{\vec{q}}] = \frac{p^0}{p^{0\prime}} \delta^3(\vec{p} - \vec{q}) , \tag{4.325}$$

where $\vec{p}$ and $\vec{q}$ are given by $p' = \Lambda p$ and $q' = \Lambda q$ (or the space parts thereof), and when the delta function forces $\vec{p}$ and $\vec{q}$ to be equal it also forces $p_0 = q_0$ since $p_0 \equiv \sqrt{\vec{p}^{\,2} + m^2}$ and $q_0 \equiv \sqrt{\vec{q}^{\,2} + m^2}$ by defintiion. Thus, the delta function forces $p = q$ which also leads to $p' = q'$. On the other hand, one can show that

$$p^{0\prime} \delta^3(\vec{p}\prime - \vec{q}\prime) = p^0 \delta^3(\vec{p} - \vec{q}) \tag{4.326}$$

as follows: Using (4.321) and noting that $\vec{p}' = \vec{q}'$ corresponds to $\vec{p} = \vec{q}$, we have

$$\begin{aligned} \int \underbrace{\frac{d^3p}{2p^0}}_{d^3p'/(2p^{0\prime})} \Big[2p^{0\prime}\delta^3(\vec{p}' - \vec{q}')\Big] f(\vec{p}) &= \int d^3p' \delta^3(\vec{p}' - \vec{q}') f(\vec{p}) \\ &= f(\vec{p})\Big|_{\vec{p}'=\vec{q}'} = f(\vec{q})\,, \end{aligned} \tag{4.327}$$

where $f(\vec{p})$ is an arbitrary function of $\vec{p}$. Namely, $2p^{0\prime}\delta^3(\vec{p}' - \vec{q}')$ behaves the same way as $2p^0\delta^3(\vec{p} - \vec{q})$:

$$\int \frac{d^3p}{2p^0}\Big[2p^0\delta^3(\vec{p} - \vec{q})\Big] f(\vec{p}) = \int d^3p\, \delta^3(\vec{p} - \vec{q}) f(\vec{p}) = f(\vec{q})\,, \tag{4.328}$$

thus proving (4.326). Then, from (4.325) and (4.326), we have

$$[a'_{\vec{p}'}, a'^{\dagger}_{\vec{q}'}] = \delta^3(\vec{p}' - \vec{q}')\,, \tag{4.329}$$

which shows that the same quantization condition holds in the K' frame. Thus, we have shown that the quantization procedure is Lorentz-invariant.

4.6 Quantization of the Dirac field

Quantization of the Dirac field proceeds similarly to that of the Klein-Gordon field, except for one major difference: once the classical field is expanded into normal modes, the quantization condition should be imposed such that it is consistent with Pauli's exclusion principle; namely, each state can be occupied by at most one quantum. We have seen that such condition can be implemented if we use anticommutators instead of commutators among the creation and annihilation operators. Thus, we will first find the Lagrangian that gives the Dirac field, derive the conjugate field and the Hamiltonian, and upon momentum-expanding the field, we will introduce anticommutation relations among the expansion coefficients. Then, anticommutators among the field and its conjugate field will follow, time variations of operators will be given by Heisenberg's equations of motion as before, and the quantized system will be established. Our initial motivation for using the anticommutators is thus experimental - namely, nature demands it in the form of Pauli's exclusion principle. We will find, however, that it is also demanded by the theory if we require that both particle and antiparticle have positive energy, and that the Dirac field respects microscopic causality.

Lagrangian formulation of the Dirac field

As we will see shortly, the Lagrangian density for the Dirac equation is given by

$$\boxed{\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi\,.} \tag{4.330}$$

To show that this Lagrangian indeed gives the Dirac equation, we go back to the action principle itself:

$$\delta S = 0\,, \qquad S \equiv \int d^4x \mathcal{L} = \int d^4x\, \psi^\dagger \gamma^0 (i\partial\!\!\!/ - m)\psi\,, \tag{4.331}$$

where the space integral is over the volume V (or the entire universe) and the time integral is from t_1 to t_2 which is the time window of interest. Here, we have 4-component field ψ with each component being complex. As before, ψ_n and ψ_n^* ($n = 1, 2, 3, 4$) can be regarded as independent variables. Thus, we first take all possible small variations of $\psi^\dagger = (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*)$ keeping ψ unchanged. We will vary ψ later. In fact, it does not matter which of the eight independent variables are changed; one could vary only one at a time if one wishes; the result will be the same. Then, the variation in S in this case is very simple:

$$\delta S = \int d^4x\, \delta\psi^\dagger\, \gamma^0 (i\partial\!\!\!/ - m)\psi = 0\,, \tag{4.332}$$

with

$$\delta\psi^\dagger \equiv (\delta\psi_1^*, \delta\psi_2^*, \delta\psi_3^*, \delta\psi_4^*)\,. \tag{4.333}$$

This should hold for all possible variations of ψ; thus, each of the four components of $\gamma^0(i\partial\!\!\!/ - m)\psi$ should be zero. Multiplying γ^0 from the left,

$$\gamma^0(i\partial\!\!\!/ - m)\psi = 0\,, \qquad \rightarrow \quad (i\partial\!\!\!/ - m)\psi = 0\,, \tag{4.334}$$

which is the Dirac equation as promised.

You may find, however, something odd about the form of this Lagrangian: it is patently asymmetric between ψ and $\psi^\dagger$, and it is not real either. In fact, if we take complex conjugate of $\mathcal{L}$, we get

$$\begin{aligned} \mathcal{L}^* &= \left[\bar\psi(i\partial\!\!\!/ - m)\psi\right]^* = \left[\bar\psi\gamma^\mu i\partial_\mu\psi - m\bar\psi\psi\right]^* \\ &= (-i\partial_\mu\bar\psi)\underbrace{\overline{\gamma^\mu}}_{\gamma^\mu}\psi - m\bar\psi\psi \\ &= -\bar\psi(i\overleftarrow{\partial\!\!\!/} + m)\psi\,. \end{aligned} \tag{4.335}$$

which seems to differ from the original $\mathcal{L}$. It does, however, have the form from which we can extract the Dirac equation for $\psi^\dagger$ (or equivalently $\bar\psi$) by varying the four components of ψ:

$$\begin{aligned} \delta\int d^4x\, \mathcal{L}^* &= -\int d^4x\, \bar\psi(i\overleftarrow{\partial\!\!\!/} + m)\delta\psi = 0 \\ &\rightarrow\ \bar\psi(i\overleftarrow{\partial\!\!\!/} + m) = 0\,. \end{aligned} \tag{4.336}$$

which is the Dirac equation for $\bar{\psi}$ (3.71). In fact, $\mathcal{L}$ and $\mathcal{L}^*$ are related by partial integration:

$$\begin{aligned}\delta S &= \delta\int d^4x\,\mathcal{L} = \delta\int d^4x\big[i\underbrace{\bar{\psi}\gamma^\mu\partial_\mu\psi}-m\bar{\psi}\psi\big]\\ &\qquad\qquad \underbrace{\partial_\mu(\bar{\psi}\gamma^\mu\psi)}_{\to 0}-(\partial_\mu\bar{\psi})\gamma^\mu\psi\\ &= -\delta\int d^4x\,\bar{\psi}(i\overleftarrow{\partial\!\!\!/}+m)\psi\\ &= \delta\int d^4x\,\mathcal{L}^*\,. \end{aligned}\tag{4.337}$$

The discarding of the integral $\int d^4x\,\partial_\mu(\bar{\psi}\gamma^\mu\psi)$ requires some clarification. First, we are not using the conservation of current $\partial_\mu j^\mu = 0$ ($j^\mu \equiv \bar{\psi}\gamma^\mu\psi$) since ψ does not necessarily satisfy the Dirac equation. It does not contribute, however, when δS is evaluated:

$$\begin{aligned}\delta\int d^4x\,\partial_\mu j^\mu &= \delta\int dt\int d^3x(\partial_0 j^0+\underbrace{\vec{\nabla}\cdot\vec{j}}_{\to\ 0\text{ upon }\int d^3x})\\ &= \int_{t_1}^{t_2}dt\,\partial_0\Big(\int d^3x\,\delta\,j^0\Big)=\Big[\int d^3x\,\delta\,j^0\Big]_{t_1}^{t_2}=0\,,\end{aligned}\tag{4.338}$$

where the last equality is due to the constraint $\delta\psi(t_1)=\delta\psi(t_2)=0$ (4.76) which leads to $\delta j^0(t_1)=\delta j^0(t_2)=0$. Thus, the Lagrangian densities $\mathcal{L}$ and $\mathcal{L}^*$ are equivalent and results in the same equation of motion. In fact, $\bar{\psi}(i\overleftarrow{\partial\!\!\!/}+m)=0$ is just the spinor adjoint of $(i\partial\!\!\!/-m)\psi=0$. One could force the Lagrangian to be real by

$$\begin{aligned}\mathcal{L}^R\equiv\frac{1}{2}(\mathcal{L}+\mathcal{L}^*\big) &= \frac{1}{2}\bar{\psi}(i\partial\!\!\!/-m-i\overleftarrow{\partial\!\!\!/}-m\big)\psi\\ &= \bar{\psi}\Big(i\frac{\overleftrightarrow{\partial\!\!\!/}}{2}-m\Big)\psi\,.\end{aligned}\tag{4.339}$$

We will use $\mathcal{L}=\bar{\psi}(i\partial\!\!\!/-m)\psi$ most of the time.

The field conjugate to the n-th component of ψ is, by definition,

$$\begin{aligned}\pi_n\equiv\frac{\partial\mathcal{L}}{\partial\dot{\psi}_n}=\frac{\partial}{\partial\dot{\psi}_n}[\underbrace{\bar{\psi}\gamma^\mu i\partial_\mu\psi}-m\bar{\psi}\psi\big]=i\psi_n^*\\ \underbrace{\bar{\psi}\gamma^0 i\partial_0\psi}+\cdots\\ i\psi_n^*\dot{\psi}_n\end{aligned}$$

$$\to\ \boxed{\pi=i\psi^\dagger}\,.\tag{4.340}$$

How about the field conjugate to ψ_n^*? Since there is no $\dot{\psi}_n^*$ appearing in $\mathcal{L}$, it does not have a conjugate field. In fact, the above expression for π tells us that ψ_n^* itself is conjugate to ψ_n up to a constant. The situation is similar if we start from $\mathcal{L}^*$ instead of $\mathcal{L}$; in this case, ψ_n's do not have conjugate field. In either case, we have four fields and their conjugate fields. If this is confusing, one can always take the real and imaginary parts of each field to be the independent fields, in which case we have eight fields to start with. Then, regardless of which Lagrangian one uses [$\mathcal{L}$, $\mathcal{L}^*$, or $\mathcal{L}^R = (\mathcal{L} + \mathcal{L}^*)/2$], one finds that four fields become conjugate to the other four fields or linear combinations thereof, ending up with four dynamical degrees of freedom in each case. In the case of the complex Klein-Gordon field, we saw that the field conjugate to ϕ was the time derivative of ϕ^* up to a constant. Why don't we get time derivatives in this case when we extract conjugate fields? Apparently, that is because the Lagrangian is linear in the time derivative of fields which in turn is because the Dirac equation is linear in time derivative.

Using the Lagrangian $\mathcal{L}$ and the conjugate fields π given above, and regarding $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)$ as independent fields, the Hamiltonian is

$$\begin{aligned}\mathcal{H} &\equiv \sum_{n=1}^{4} \underbrace{\pi_n}_{i\psi_n^*} \dot{\psi}_n - \underbrace{\mathcal{L}}_{\bar{\psi}\gamma^\mu i\partial_\mu\psi - m\bar{\psi}\psi} \\ &= \cancel{i\psi^\dagger\dot{\psi}} - \Big(\underbrace{i\bar{\psi}\gamma^0\partial_0\psi}_{\cancel{i\psi^\dagger\dot{\psi}}} + \underbrace{i\bar{\psi}\gamma^k\partial_k\psi}_{i\psi^\dagger \underbrace{\gamma^0\gamma^k}_{\alpha_k}\partial_k\psi} - m\psi^\dagger \underbrace{\gamma^0}_{\beta}\psi \Big) ,\end{aligned} \tag{4.341}$$

where k is the space index which is summed, or

$$\boxed{\mathcal{H} = \psi^\dagger(-i\vec{\alpha}\cdot\vec{\nabla} + m\beta)\psi} . \tag{4.342}$$

Note that the operator sandwiched by $\psi^\dagger$ and ψ is nothing but the 'Hamiltonian' we encountered when we introduced the Dirac equation: $i\partial_0\psi = (-i\vec{\alpha}\cdot\vec{\nabla} + m\beta)\psi$. When ψ satisfies the Dirac equation, the total Hamiltonian can be written as

$$H \equiv \int d^3x\, \mathcal{H} = \int d^3x\, \psi^\dagger i\partial_0\psi \,. \tag{4.343}$$

The total momentum is obtained by applying the general form (4.122):

$$\vec{P} \equiv -\int d^3x \underbrace{\pi_n}_{i\psi_n^*}\vec{\nabla}\psi_n = \int d^3x\, \psi^\dagger(-i\vec{\nabla})\psi \,. \tag{4.344}$$

As in the case of the Klein-Gordon field (4.207), we note the form for the total energy and momentum where the corresponding differential operator $\mathcal{O} = i\partial_0$ or $-i\vec{\nabla}$ is

sandwiched by the appropriate inner product of field: $\frac{1}{2}\phi(i\overleftrightarrow{\partial_0})\mathcal{O}\phi$ for the real Klein-Gordon field, and $\psi^\dagger\mathcal{O}\psi$ for the Dirac field.

Momentum expansion of the Dirac field

We now proceed to quantize the Dirac field. The first step is to expand the general solution of the Dirac equation into normal modes. Suppose $\psi(t,\vec{x})$ is an arbitrary solution of the Dirac equation; then, at a given time, say $t=0$, each component $\psi_n(0,\vec{x})$ can be uniquely Fourier-expanded:

$$\psi(0,\vec{x}) = \begin{pmatrix}\psi_1(0,\vec{x})\\ \psi_2(0,\vec{x})\\ \psi_3(0,\vec{x})\\ \psi_4(0,\vec{x})\end{pmatrix} = \begin{pmatrix}\sum_{\vec{p}} c_{1\vec{p}}\, e^{i\vec{p}\cdot\vec{x}}\\ \sum_{\vec{p}} c_{2\vec{p}}\, e^{i\vec{p}\cdot\vec{x}}\\ \sum_{\vec{p}} c_{3\vec{p}}\, e^{i\vec{p}\cdot\vec{x}}\\ \sum_{\vec{p}} c_{4\vec{p}}\, e^{i\vec{p}\cdot\vec{x}}\end{pmatrix} = \sum_{\vec{p}} \begin{pmatrix}c_{1\vec{p}}\\ c_{2\vec{p}}\\ c_{3\vec{p}}\\ c_{4\vec{p}}\end{pmatrix} e^{i\vec{p}\cdot\vec{x}}\,, \tag{4.345}$$

where $c_{n\vec{p}}$ $(n=1,2,3,4)$ are uniquely-determined complex coefficients. For each $\vec{p}$, we can use the orthonormal set of spinors $(u_{\vec{p},\pm\vec{s}}\,, v_{-\vec{p},\pm\vec{s}})$ (3.308) to uniquely expand $c_{n\vec{p}}$:

$$\begin{pmatrix}c_{1\vec{p}}\\ c_{2\vec{p}}\\ c_{3\vec{p}}\\ c_{4\vec{p}}\end{pmatrix} = \sum_{\vec{s}} (A_{\vec{p},\vec{s}}\, u_{\vec{p},\vec{s}} + B_{-\vec{p},\vec{s}}\, v_{-\vec{p},\vec{s}})\,, \tag{4.346}$$

where the sum over $\vec{s}$ is understood to be over $\pm\vec{s}$ where $\vec{s}$ is some fixed unit vector, which could be a function of $\vec{p}$ ($\vec{s}=\hat{p}$, for example) or fixed in space ($\vec{s}=\hat{z}$, for example). It is important to note that for any pair $(\vec{p},\vec{s})$, we have an orthonormal set $(u_{\vec{p},\pm\vec{s}}\,, v_{-\vec{p},\pm\vec{s}})$ by which any complex 4-component vector can be uniquely expanded. In particular, the set of four spinors $(u_{\vec{p},\pm\vec{s}}\,, v_{-\vec{p},\pm\vec{s}})$ are different for different $\vec{p}$. We choose to expand the 4-component spinor $(c_{1\vec{p}}, c_{2\vec{p}}, c_{3\vec{p}}, c_{4\vec{p}})$ using the orthonormal set $(u_{\vec{p},\pm\vec{s}}\,, v_{-\vec{p},\pm\vec{s}})$ where $\vec{p}$ is the same one that appears in the indexes of $c_{n\vec{p}}$'s. We now have a unique expansion of a general solution of the Dirac equation at $t=0$:

$$\begin{aligned}\psi(0,\vec{x}) &= \sum_{\vec{p},\vec{s}} (A_{\vec{p},\vec{s}}\, u_{\vec{p},\vec{s}} + B_{-\vec{p},\vec{s}}\, v_{-\vec{p},\vec{s}}) e^{i\vec{p}\cdot\vec{x}}\\ &= \sum_{\vec{p},\vec{s}} (A_{\vec{p},\vec{s}}\, u_{\vec{p},\vec{s}}\, e^{i\vec{p}\cdot\vec{x}} + \underbrace{B_{\vec{p},\vec{s}}\, v_{\vec{p},\vec{s}}\, e^{-i\vec{p}\cdot\vec{x}}}_{\text{relabeled } \vec{p}\leftrightarrow -\vec{p}})\,.\end{aligned} \tag{4.347}$$

What is the time dependence of $\psi(t,\vec{x})$? Actually, we already know the answer: in order to be a solution of the Dirac equation, $u_{\vec{p},\vec{s}}\, e^{i\vec{p}\cdot\vec{x}}$ should be attached to the positive frequency $e^{-ip^0 t}$, and $v_{\vec{p},\vec{s}}\, e^{-i\vec{p}\cdot\vec{x}}$ should be attached to the negative frequency $e^{ip^0 t}$ with $p^0 \equiv \sqrt{\vec{p}^{\,2}+m^2}$:

$$u_{\vec{p},\vec{s}}\, e^{-ip\cdot x} = u_{\vec{p},\vec{s}}\, e^{-ip^0 t + i\vec{p}\cdot\vec{x}}\,, \qquad v_{\vec{p},\vec{s}}\, e^{ip\cdot x} = v_{\vec{p},\vec{s}}\, e^{ip^0 t - i\vec{p}\cdot\vec{x}}\,. \tag{4.348}$$

Thus, the general solution is now written as

$$\psi(t,\vec{x}) = \sum_{\vec{p},\vec{s}} (A_{\vec{p},\vec{s}}\, u_{\vec{p},\vec{s}}\, e^{-ip\cdot x} + B_{\vec{p},\vec{s}}\, v_{\vec{p},\vec{s}}\, e^{ip\cdot x})\,. \tag{4.349}$$

This expansion is general and unique; namely, if you give me an arbitrary solution of the Dirac equation, I can uniquely expand it at a given time t in the Fourier series in the 3-dimensional space, and then the time dependence is uniquely determined for the function to be a solution of the Dirac equation. Note that, in addition to the appropriate time dependence, $u_{\vec{p},\vec{s}}$ had to be attached to $e^{i\vec{p}\cdot\vec{s}}$ and $v_{\vec{p},\vec{s}}$ to $e^{-i\vec{p}\cdot\vec{s}}$. The required opposite sign of $\vec{p}$ in the subscript of $v_{\vec{p},\vec{s}}$ and that in the corresponding exponent would not have resulted if we had used the other orthonormal set $(u_{\vec{p},\pm\vec{s}}\,, v_{\vec{p},\pm\vec{s}})$ which is defined by the inner product $\bar{a}b$ (try it).

In the case of the Klein-Gordon field, the normal mode $\phi(x) = e_{\vec{p}}(x)$ was normalized such that the classical probability density $\phi^* i \overset{\leftrightarrow}{\partial}_0 \phi$ gives unity when integrated over V as seen in (4.181). In the case of the Dirac field, the classical probability density is given by $j^0 = \psi^\dagger\psi$; thus, we take the normal-mode functions to be

$$\boxed{f_{\vec{p},\vec{s}}(x) \overset{\text{def}}{\equiv} \frac{u_{\vec{p},\vec{s}}}{\sqrt{2p^0V}} e^{-ip\cdot x}\,, \quad g_{\vec{p},\vec{s}}(x) \overset{\text{def}}{\equiv} \frac{v_{\vec{p},\vec{s}}}{\sqrt{2p^0V}} e^{ip\cdot x}\,,} \tag{4.350}$$

which is normalized properly:

$$\int d^3x f^\dagger_{\vec{p},\vec{s}}(x) f_{\vec{p},\vec{s}}(x) = \int_V d^3x \frac{1}{2p^0V} \underbrace{u^\dagger_{\vec{p},\vec{s}} u_{\vec{p},\vec{s}}}_{2p^0 \text{ by (3.308)}} = 1\,, \text{ etc.} \tag{4.351}$$

Then, the expansion is now

$$\boxed{\psi(x) = \sum_{\vec{p},\vec{s}} \Big(a_{\vec{p},\vec{s}}\, f_{\vec{p},\vec{s}}(x) + b^\dagger_{\vec{p},\vec{s}}\, g_{\vec{p},\vec{s}}(x) \Big)\,,} \tag{4.352}$$

where

$$a_{\vec{p},\vec{s}} \equiv \sqrt{2p^0V}\, A_{\vec{p},\vec{s}}\,, \qquad b^\dagger_{\vec{p},\vec{s}} \equiv \sqrt{2p^0V}\, B_{\vec{p},\vec{s}}\,. \tag{4.353}$$

Note that we used $b^\dagger_{\vec{p},\vec{s}}$ instead of $b_{\vec{p},\vec{s}}$ in the second equation above. This anticipates that it will be a creation operator, rather than an annihilation operator, of antiparticle. We will discuss this choice later in the context of the energy sign of antiparticle. The spin sum is over $\pm\vec{s}$, where $\vec{s}$ is a unit vector which could in general be a function of $\vec{p}$. It is easily verified that the normal-mode functions satisfy the orthonormality relations given by

$$\boxed{\begin{aligned} \int d^3x f^\dagger_{\vec{p},\vec{s}}(x) f_{\vec{p}',\vec{s}'}(x) &= \int d^3x\, g^\dagger_{\vec{p},\vec{s}}(x) g_{\vec{p}',\vec{s}'}(x) = \delta_{\vec{p},\vec{p}'}\delta_{\vec{s},\vec{s}'} \\ \int d^3x f^\dagger_{\vec{p},\vec{s}}(x) g_{\vec{p}',\vec{s}'}(x) &= \int d^3x\, g^\dagger_{\vec{p},\vec{s}}(x) f_{\vec{p}',\vec{s}'}(x) = 0 \end{aligned}}\,, \tag{4.354}$$

where $\vec{s}$ and $\vec{s}'$ are plus or minus some direction which is in general a function of $\vec{p}$.

Exercise 4.10 *Verify (4.354).*

Then, we can use the orthonormality relations to write a's and $b^\dagger$'s as

$$\begin{aligned} a_{\vec{p},\vec{s}} &= \int d^3x\, f^\dagger_{\vec{p},\vec{s}}(x)\psi(x) \\ b^\dagger_{\vec{p},\vec{s}} &= \int d^3x\, g^\dagger_{\vec{p},\vec{s}}(x)\psi(x)\,. \end{aligned} \tag{4.355}$$

Up to this point, $\psi(x)$ is just a complex wave function. We will now regard a's and $b^\dagger$'s as operators in the Hilbert space and impose anticommutation relations given by

$$\boxed{\begin{array}{c}\{a_{\vec{p},\vec{s}}\,, a^\dagger_{\vec{p}',\vec{s}'}\} = \{b_{\vec{p},\vec{s}}\,, b^\dagger_{\vec{p}',\vec{s}'}\} = \delta_{\vec{p},\vec{p}'}\delta_{\vec{s},\vec{s}'} \\ \text{all others} = 0\end{array}}\,. \tag{4.356}$$

This will assure that the oscillator associated with each normal mode will be occupied by at most one quantum; namely, the number operators $a^\dagger_{\vec{p},\vec{s}}\,a_{\vec{p},\vec{s}}$ and $b^\dagger_{\vec{p},\vec{s}}\,b_{\vec{p},\vec{s}}$ have eigenvalues 0 and 1 only. Note that the operator field $\psi(x)$ satisfies the Dirac equation since $f_{\vec{p},\vec{s}}(x)$ and $g_{\vec{p},\vec{s}}(x)$ in the expansion (4.352) do:

$$(i\partial\!\!\!/ - m)\psi(x) = 0 \qquad (\psi : \text{operator})\,. \tag{4.357}$$

Using the momentum expansion, we can derive aniticommutation relations among fields. Let's define

$$\psi = \psi_a + \psi_{b^\dagger}\,, \quad \text{with} \quad \psi_a \stackrel{\text{def}}{\equiv} \sum_{\vec{p},\vec{s}} a_{\vec{p},\vec{s}}\, f_{\vec{p},\vec{s}}\,, \quad \psi_{b^\dagger} \stackrel{\text{def}}{\equiv} \sum_{\vec{p},\vec{s}} b^\dagger_{\vec{p},\vec{s}}\, g_{\vec{p},\vec{s}}\,. \tag{4.358}$$

Since only the combinations $\{a, a^\dagger\}$ or $\{b, b^\dagger\}$ survive,

$$\{\psi_n(t,\vec{x})\,, \psi_m(t,\vec{x}')\} = \{\psi^\dagger_n(t,\vec{x})\,, \psi^\dagger_m(t,\vec{x}')\} = 0\,. \tag{4.359}$$

The anticommutator $\{\psi_n(t,\vec{x})\,, \psi^\dagger_m(t,\vec{x}')\}$ becomes

$$\begin{aligned} \{\psi_n(t,\vec{x})\,, \psi^\dagger_m(t,\vec{x}')\} &= \{\psi_{a_n}(t,\vec{x}) + \psi_{b^\dagger n}(t,\vec{x})\,, \psi^\dagger_{a m}(t,\vec{x}') + \psi^\dagger_{b^\dagger m}(t,\vec{x}')\} \\ &= \{\psi_{a_n}(t,\vec{x})\,, \psi^\dagger_{a m}(t,\vec{x}')\} + \{\psi_{b^\dagger n}(t,\vec{x})\,, \psi^\dagger_{b^\dagger m}(t,\vec{x}')\} \end{aligned} \tag{4.360}$$

Using the anticommutation relation (4.356) and the explicit expression for $f_{\vec{p},\vec{s}}$ (4.350), we have

$$\begin{aligned} \{\psi_{a_n}(t,\vec{x})\,, \psi^\dagger_{a m}(t,\vec{x}')\} &= \sum_{\vec{p},\vec{s},\vec{p}',\vec{s}'} \underbrace{\{a_{\vec{p},\vec{s}}\,, a^\dagger_{\vec{p}',\vec{s}'}\}}_{\delta_{\vec{p},\vec{p}'}\delta_{\vec{s},\vec{s}'}} f_{\vec{p},\vec{s}_n}(t,\vec{x}) f^\dagger_{\vec{p}',\vec{s}'_m}(t,\vec{x}') \\ &= \sum_{\vec{p},\vec{s}} f_{\vec{p},\vec{s}_n}(t,\vec{x}) f^\dagger_{\vec{p},\vec{s}_m}(t,\vec{x}') \\ &= \sum_{\vec{p}} \frac{e^{i\vec{p}\cdot(\vec{x}-\vec{x}')}}{2p^0 V} \sum_{\vec{s}} (u_{\vec{p},\vec{s}} u^\dagger_{\vec{p},\vec{s}})_{nm}\,. \end{aligned} \tag{4.361}$$

Similarly,

$$\begin{aligned}\{\psi_{b^\dagger n}(t,\vec{x})\,,\psi^\dagger_{b^\dagger m}(t,\vec{x}\,')\} &= \sum_{\vec{p}}\frac{e^{-i\vec{p}\cdot(\vec{x}-\vec{x}\,')}}{2p^0V}\sum_{\vec{s}}(v_{\vec{p},\vec{s}}v^\dagger_{\vec{p},\vec{s}})_{nm}\\ (\text{relabel }\vec{p}\to-\vec{p}) &= \sum_{\vec{p}}\frac{e^{i\vec{p}\cdot(\vec{x}-\vec{x}\,')}}{2p^0V}\sum_{\vec{s}}(v_{-\vec{p},\vec{s}}v^\dagger_{-\vec{p},\vec{s}})_{nm}\,.\end{aligned}\tag{4.362}$$

Then, the anticommutator $\{\psi_n(t,\vec{x})\,,\psi^\dagger_m(t,\vec{x}\,')\}$ is, adding (4.361) and (4.362),

$$\begin{aligned}\{\psi_n(t,\vec{x})\,,\psi^\dagger_m(t,\vec{x}\,')\} &= \underbrace{\sum_{\vec{p}}\frac{1}{V}e^{i\vec{p}\cdot(\vec{x}-\vec{x}\,')}}_{\delta^3(\vec{x}-\vec{x}\,')\text{ by (4.157)}}\ \underbrace{\frac{1}{2p^0}\sum_{\vec{s}}(u_{\vec{p},\vec{s}}u^\dagger_{\vec{p},\vec{s}}+v_{-\vec{p},\vec{s}}v^\dagger_{-\vec{p},\vec{s}})_{nm}}_{\delta_{nm}\text{ by (3.323)}}\\ &= \delta_{nm}\delta^3(\vec{x}-\vec{x}\,')\,.\end{aligned}\tag{4.363}$$

Thus, using $\pi = i\psi^\dagger$ and together with (4.359),

$$\begin{aligned}\{\psi_n(t,\vec{x})\,,\pi_m(t,\vec{x}\,')\} &= i\delta_{nm}\delta^3(\vec{x}-\vec{x}\,')\\ \{\psi_n(t,\vec{x})\,,\psi_m(t,\vec{x}\,')\} &= \{\pi_n(t,\vec{x})\,,\pi_m(t,\vec{x}\,')\} = 0\end{aligned}\ .\tag{4.364}$$

Namely, the field ψ and its conjugate π satisfy similar quantization conditions as those of the Klein-Gordon fields, but with commutators replaced by anticommutators.

We have seen that one complex field represents a charged spin-0 particle and a set of four complex fields represents a spin-1/2 particle. For a charged spin-0 particle with a given moemtnum, there are two degrees of freedom: a particle and its antiparticle. On the other hand, for a spin-1/2 particle with a given momentum, there are four degrees of freedom: electron, positron and spin up, down. The question is then why a spinor field requires factor of two more complex fields per degree of freedom compared to a scalar field. One way to understand this is to count the number of independent *canonical pairs* of fields. For a charged spin-0 field, there are two such pairs (ϕ,π) and $(\phi^\dagger,\pi^\dagger)$. For a spin-1/2 field, the conjugate field is given by $\pi = i\psi^\dagger$, and thus there are only four independent canonical pairs. Another way is to simply count the number of independent complex coefficients required for a given $\vec{p}$ in the Fourier expansion of field. For a complex scalar field, there are two time dependences that can be assigned for a given $\vec{p}$; namely, $e^{-ip\cdot x}$ and $e^{ip\cdot x}$. Then, after quantization, the corresponding coefficients $a_{\vec{p}}$ and $b^\dagger_{\vec{p}}$ are interpreted as the annihilation operator of particle and the creation opearator of antiparticle, respectively. For a spinor field, when the spatial dependence is Fourier transformed and expanded into u and v spinors there are four coefficients for a given $\vec{p}$. However, the time dependences corresponding to $u_{\vec{p}\vec{s}}$ or $v_{\vec{p}\vec{s}}$ are already uniquely defined, and thus one has only four independent coefficients, leading to four degrees of freedom for a given $\vec{p}$. The difference between the spin-0

and spin-1/2 cases arises essentially from the fact that the Klein-Gordon equation is second-order in time derivative while the Dirac equation is first-order.

Total energy and momentum

We will now express the total energy $H = \int d^3x\, \psi^\dagger i\partial_0\psi$ (4.343) in terms of annihilation and creation operators. Using the expansion (4.352) and the orthonormality relations (4.354), we obtain

$$\begin{aligned} H &= \int d^3x\, \psi^\dagger i\partial_0\psi \\ &= \int d^3x \sum_{\vec{p},\vec{s}} (a^\dagger_{\vec{p},\vec{s}} f^\dagger_{\vec{p},\vec{s}} + b_{\vec{p},\vec{s}} g^\dagger_{\vec{p},\vec{s}}) \sum_{\vec{p}',\vec{s}'} p^{0\prime} (a_{\vec{p}',\vec{s}'} f_{\vec{p}',\vec{s}'} - b^\dagger_{\vec{p}',\vec{s}'} g_{\vec{p}',\vec{s}'}) \\ &= \sum_{\vec{p},\vec{s},\vec{p}',\vec{s}'} p^{0\prime} (a^\dagger_{\vec{p},\vec{s}} a_{\vec{p}',\vec{s}'} \delta_{\vec{p},\vec{p}'}\delta_{\vec{s},\vec{s}'} - b_{\vec{p},\vec{s}} b^\dagger_{\vec{p}',\vec{s}'} \delta_{\vec{p},\vec{p}'}\delta_{\vec{s},\vec{s}'}) \\ &= \sum_{\vec{p},\vec{s}} p^0 (a^\dagger_{\vec{p},\vec{s}} a_{\vec{p},\vec{s}} - b_{\vec{p},\vec{s}} b^\dagger_{\vec{p},\vec{s}}) \end{aligned} \tag{4.365}$$

So far, we have used only the orthonormality relations of normal-mode functions, and no operator relations have been used. In fact, the above is valid for the non-quantized fields also if hermitian conjugation is understood to be complex conjugation. Now, we can change the second term to a number operator by $\{b_{\vec{p},\vec{s}}\,, b^\dagger_{\vec{p},\vec{s}}\} = 1$ or $b_{\vec{p},\vec{s}} b^\dagger_{\vec{p},\vec{s}} = 1 - b^\dagger_{\vec{p},\vec{s}} b_{\vec{p},\vec{s}}$:

$$H = \sum_{\vec{p},\vec{s}} p^0 (a^\dagger_{\vec{p},\vec{s}} a_{\vec{p},\vec{s}} + b^\dagger_{\vec{p},\vec{s}} b_{\vec{p},\vec{s}} - 1)\,, \tag{4.366}$$

which shows that both a-particles and b-particles contribute positively to the total energy. Since, in the expansion of ψ, a-particles are associated with electron solutions $u_{\vec{p},\vec{s}}\, e^{-ip\cdot x}$ and b-particle with positron solutions $v_{\vec{p},\vec{s}}\, e^{ip\cdot x}$, one expects that a-particles are electrons and b-particles positrons (or fermions and anti-fermions). Such an interpretation will be justified later when we find 'charge' carried by those particles. Thus, we see that both electrons and positrons carry positive energy. Suppose we had quantized by commutators instead of anticommutators; namely, suppose the braces in (4.356) were square brackets. Then, (4.365) would still be valid, and using the commutator $[b_{\vec{p},\vec{s}}\,, b^\dagger_{\vec{p},\vec{s}}] = 1$, the total energy would then be

$$H = \sum_{\vec{p},\vec{s}} p^0 (a^\dagger_{\vec{p},\vec{s}} a_{\vec{p},\vec{s}} - b^\dagger_{\vec{p},\vec{s}} b_{\vec{p},\vec{s}} - 1) \quad \text{(commutators used)}\,, \tag{4.367}$$

which indicates that b-particles carry negative energy. Since the commutation relation indicates that any number of quanta can occupy a given normal mode, we see that there is no lowest energy state which would have defined the vacuum. Thus, it was critical that we quantize the Dirac field by anticommutation relations in order to avoid the negative energy problem.

It is instructive to examine the origin of the negative sign on the second term $b_{\vec{p},\vec{s}}b^\dagger_{\vec{p},\vec{s}}$ in (4.365). This came about due to the first-order time derivative $\dot{\psi}$ that appear in $\mathcal{H}$ which picked up the factor $(-ip^0)$ from $f_{\vec{p},\vec{s}}$ and ip^0 from $g_{\vec{p},\vec{s}}$. The corresponding sign in the case of the Klein-Gordon field was positive as seen in (4.198), and this is because there are two time derivatives per term for the Klein-Gordon Hamiltonian $\mathcal{H} = (\dot{\phi}^2 - \phi\ddot{\phi})/2$.

One subtle point is left: when we expanded ϕ in (4.352), we could have labeled the coefficient of the second term as $b_{\vec{p},\vec{s}}$ instead of $b^\dagger_{\vec{p},\vec{s}}$ with all else identical including the anticommutation relations which are symmetric between $b_{\vec{p},\vec{s}}$ and $b^\dagger_{\vec{p},\vec{s}}$. Then, the total energy (4.365) before any quantization condition is used would be

$$H = \sum_{\vec{p},\vec{s}} p^0 (a^\dagger_{\vec{p},\vec{s}} a_{\vec{p},\vec{s}} - b^\dagger_{\vec{p},\vec{s}} b_{\vec{p},\vec{s}}) \,, \tag{4.368}$$

which is already expressed in terms of number operators. Then, does this mean that the energy of the b-particle can be negative? Not so. We recall that the formulation of fermionic oscillator was symmetric under the exchange of $a^\dagger \leftrightarrow a$ and $|0\rangle \leftrightarrow |1\rangle$. Which to use was to be determined by defining the lower-energy state to be the empty state $|0\rangle$. The above Hamiltonian shows that the state $|1\rangle$ for a given $\vec{p}$ of b-particle contributes $-p^0 < 0$ to the total energy while the $|0\rangle$ contributes zero. This indicates that we have mislabeled $|0\rangle$ and $|1\rangle$ for the b-particles. Thus, we have to redefine as $|0\rangle \leftrightarrow |1\rangle$ and also $b^\dagger_{\vec{p},\vec{s}} \leftrightarrow b_{\vec{p},\vec{s}}$, which recovers the original derivation (4.366).

Just as in the case of the Klein-Gordon field, the total Hamiltonian (4.366) contains an apparent infinity $\sum_{\vec{p},\vec{s}}(-p^0)$. Again, we regard it as a harmless constant offset and choose to discard it by normal ordering. This time, however, we have to change the sign of the term when the creation and annihilation operators are swapped:

$$: a^\dagger_{\vec{p},\vec{s}} a_{\vec{p},\vec{s}} - b_{\vec{p},\vec{s}} b^\dagger_{\vec{p},\vec{s}} : = a^\dagger_{\vec{p},\vec{s}} a_{\vec{p},\vec{s}} + b^\dagger_{\vec{p},\vec{s}} b_{\vec{p},\vec{s}} \,. \tag{4.369}$$
$$\uparrow$$
$$\text{sign flip}$$

Thus, the normal ordering procedure is now extended to include fermion operators: it simply reorders the annihilation and creation operators such that all creation operators are to the left of all annihilation operators, and add a minus sign if odd number of swaps of fermion operators are needed for the reordering. If both boson and fermion operators are present, it is assumed that fermion operators *commute* with boson operators. For example, if all are fermion operators,

$$: a_1 a_2^\dagger : = -a_2^\dagger a_1 \,, \quad : b_1 a_1 a_2^\dagger : = a_2^\dagger \underbrace{b_1 a_1}_{-a_1 b_1} = -a_2^\dagger a_1 b_1 \,, \text{ etc.} \tag{4.370}$$

Note that, when fermions are involved, one does have to keep track of the ordering among creation operators or that among annihilation operators.

Using this extended definition of normal ordering, the total Hamiltonian can be written as

$$H \equiv :\int d^3x\, \psi^\dagger i\partial_0 \psi : = \sum_{\vec{p},\vec{s}} p^0 (a^\dagger_{\vec{p},\vec{s}} a_{\vec{p},\vec{s}} + b^\dagger_{\vec{p},\vec{s}} b_{\vec{p},\vec{s}})\,. \tag{4.371}$$

Similarly, the total momentum (4.344) can be expressed in terms of a's and b's as

$$\vec{P} \equiv :\int d^3x\, \psi^\dagger (-i\vec{\nabla})\psi : = \sum_{\vec{p},\vec{s}} \vec{p} (a^\dagger_{\vec{p},\vec{s}} a_{\vec{p},\vec{s}} + b^\dagger_{\vec{p},\vec{s}} b_{\vec{p},\vec{s}})\,. \tag{4.372}$$

Using the above expressions of $P^\mu = (H, \vec{P})$, it is straightforward to show

$$[P^\mu, \psi(x)] = -i\partial^\mu \psi(x)\,, \quad [P^\mu, \pi(x)] = -i\partial^\mu \pi(x)\,, \tag{4.373}$$

which can be extended to any polynomial function $F(\psi, \pi)$:

$$[P^\mu, F(\psi, \pi)] = -i\partial^\mu F(\psi, \pi)\,. \tag{4.374}$$

Thus, the total energy-momentum operators act as space-time translation operators just as in the case of the Klein-Gordon field.

Exercise 4.11 *Heisenberg's equation of motion for the Dirac field.*
Use the momentum expansion of Dirac field and the total Hamiltonian (expressed in terms of the number operators) to show that the Dirac field obeys Heisenberg's equation of motion (note the commutator, not anticommutator):

$$-i\dot{\psi}(x) = [H, \psi(x)]\,. \tag{4.375}$$

(*hint: You may find the following identity handy:* $[AB, C] = A\{B, C\} - \{A, C\}B$.)

Exercise 4.12 *Use the momentum expansion of the Dirac field to verify (4.372).*

The Noether current for the phase transformation
The Lagrangian $\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi$ is invariant under the phase rotation

$$\psi' = e^{i\theta}\psi \quad \rightarrow \quad \bar{\psi}' = e^{-i\theta}\bar{\psi}\,, \tag{4.376}$$

where θ is a real parameter and all four components of ψ are rotated by the same angle simultaneously. If each component of ψ is phase-rotated individually, then the Lagrangian is not invariant because of the off-diagonal terms such as $\psi_1^* \partial_\mu \psi_2$ (due to the off diagonal elements of the γ matrices). The derivation of the Noether current corresponding to this phase rotation is identical to the case of charged Klein-Gordon

field (4.259) except that this time there are four fields ψ_n ($n = 1, 2, 3, 4$) whose contribution to the variation of $\mathcal{L}$ should be added up:

$$\partial_\mu j^\mu = 0\,, \quad j^\mu \equiv i\Big(\underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_n^*)}}_{0} \psi_n^* - \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_n)}}_{i(\bar\psi\gamma^\mu)_n} \psi_n \Big)\,, \tag{4.377}$$

namely,

$$\boxed{j^\mu = \bar\psi \gamma^\mu \psi}\,. \tag{4.378}$$

This is exactly the same current as the 'probability current' (3.67) we have derived directly from the Dirac equation in the non-quantized theory; we noted that the time component $j^0 = \psi^\dagger\psi$ was always positive and consistent with the interpretation that it is the probability density. We will now see that the same quantity in the framework of the quantum field theory is interpreted as the charge current, and that the conserved quantity - the total charge - can take both positive and negative values.

The conserved quantify is the space integral of the time component of a conserved current:

$$Q \equiv \int d^3x\, j^0 = \int d^3x\, \psi^\dagger \psi\,. \tag{4.379}$$

We use the momentum expansion of ψ and the orthonormality relations (4.354) to obtain

$$\begin{aligned} Q &= \int d^3x \sum_{\vec p,\vec s,\vec p',\vec s'} (a^\dagger_{\vec p,\vec s} f^\dagger_{\vec p,\vec s} + b_{\vec p,\vec s} g^\dagger_{\vec p,\vec s})(a_{\vec p',\vec s'} f_{\vec p',\vec s'} + b^\dagger_{\vec p',\vec s'} g_{\vec p',\vec s'}) \\ &= \sum_{\vec p,\vec s,\vec p',\vec s'} (a^\dagger_{\vec p,\vec s} a_{\vec p',\vec s'}\, \delta_{\vec p,\vec p'}\delta_{\vec s,\vec s'} + b_{\vec p,\vec s} b^\dagger_{\vec p',\vec s'}\, \delta_{\vec p,\vec p'}\delta_{\vec s,\vec s'}) \\ &= \sum_{\vec p,\vec s} (a^\dagger_{\vec p,\vec s} a_{\vec p,\vec s} + \underbrace{b_{\vec p,\vec s} b^\dagger_{\vec p,\vec s}}_{1 - b^\dagger_{\vec p,\vec s} b_{\vec p,\vec s} \quad \leftarrow \quad \{b_{\vec p,\vec s}\,, b^\dagger_{\vec p,\vec s}\} = 1}) \\ &= \sum_{\vec p,\vec s} (a^\dagger_{\vec p,\vec s} a_{\vec p,\vec s} - b^\dagger_{\vec p,\vec s} b_{\vec p,\vec s} + 1)\,. \end{aligned} \tag{4.380}$$

We see that a-particles contribute to the quantity Q by $+1$ each regardless of momentum and spin, and b-particles contribute by -1 each. Thus, one can interpret that a-particles are electrons carrying $+1$ 'electron number' each and b-particles are positrons carrying -1 electron number each. The extra term $\sum_{\vec p,\vec s} 1$ is again infinite, which reminds us of the *electron sea* in the hole theory. Again, we discard the constant offset by normal ordering:

$$: a^\dagger_{\vec p,\vec s} a_{\vec p,\vec s} + b_{\vec p,\vec s} b^\dagger_{\vec p,\vec s} : = a^\dagger_{\vec p,\vec s} a_{\vec p,\vec s} - b^\dagger_{\vec p,\vec s} b_{\vec p,\vec s}\,; \tag{4.381}$$

namely,

$$Q \equiv :\int d^3x\, \psi^\dagger \psi : = \sum_{\vec{p},\vec{s}} (a^\dagger_{\vec{p},\vec{s}} a_{\vec{p},\vec{s}} - b^\dagger_{\vec{p},\vec{s}} b_{\vec{p},\vec{s}})\,. \tag{4.382}$$

Note that $\int d^3x\, \psi^\dagger \psi$ is a linear combination of terms which are product of creation and annihilation operators, and that the normal ordering applies directly to each term.

Noether current of rotational invariance (general)

In (3.246), the hole theory guided us to assign spin $+1/2$ to the spinor $v_{\vec{0},+\vec{s}}$ even though its egenvalue of the spin operator $\Sigma\cdot\vec{s}/2$ was $-1/2$. It is now time to show that the creation operator $b^\dagger_{\vec{0},\vec{s}}$ which is associated with $v_{\vec{0},+\vec{s}}$ indeed creates a state with spin $+1/2$ in the $\vec{s}$ direction. To do so, we have to find the total angular momentum operator which should contain the spin term. As we will see below, the orbital angular momentum $\vec{x}\times\vec{p}$ shows up as a part of the conserved quantity corresponding to the invariance of Lagrangian under rotation. The rest of the conserved quantity is then identified as the spin of the particle. We will first derive a general expression for the total angular momentum.

Our starting point is that the Lagrangian density is a Lorentz scalar:

$$\mathcal{L}'(x') = \mathcal{L}(x) \qquad (x' = \Lambda x)\,, \tag{4.383}$$

where Λ is a proper and orthochronous Lorentz transformation. Let us be specific about the meaning of this relation. In general, a Lagrangian density is a function of a set of fields $\tilde{\phi} = (\phi_1, \ldots, \phi_n)$ and its derivative $\partial_\mu\tilde{\phi}$:

$$\mathcal{L}(x) \stackrel{\text{def}}{\equiv} \mathcal{L}(\tilde{\phi}(x), \partial_\mu\tilde{\phi}(x)) \tag{4.384}$$

Under a Lorentz transformation, the field $\tilde{\phi}$ transforms by a certain $n\times n$ matrix S:

$$\phi'_a(x') = S_{ab}\phi_b(x) \qquad (a,b = 1,\ldots,n)\,. \tag{4.385}$$

The Lagrangian in the transformed frame $\mathcal{L}'(x')$ is then defined by *the same functional form* as $\mathcal{L}(\tilde{\phi}, \partial_\mu\tilde{\phi})$ where $\tilde{\phi}(x)$ is replaced by $\tilde{\phi}'(x')$ and x by x':

$$\mathcal{L}'(x') \stackrel{\text{def}}{\equiv} \mathcal{L}(\tilde{\phi}'(x'), \partial'_\mu\tilde{\phi}'(x'))\,. \tag{4.386}$$

We have already seen in (4.130) that, with these definitions, the spin-0 Lagrangian indeed satisfies the relation $\mathcal{L}'(x') = \mathcal{L}(x)$. It is also straightforward to show that the Dirac field Lagrangian $\bar{\psi}(i\partial\!\!\!/ - m)\psi$ is a Lorentz scalar.

Exercise 4.13 *Show that the Lagrangian density of free Dirac field $\bar{\psi}(i\partial\!\!\!/ - m)\psi$ is indeed a Lorentz scalar as defined above.*

As can be seen in the definitions (4.384) and (4.386), $\mathcal{L}(x)$ as a function of $(\tilde{\phi}(x), \partial_\mu \tilde{\phi}(x))$ and $\mathcal{L}'(x')$ as a function of $(\tilde{\phi}'(x'), \partial'_\mu \tilde{\phi}'(x'))$ have the same functional form; thus, we can express $\mathcal{L}'(x') - \mathcal{L}(x) = 0$ in terms of the changes in the arguments

$$\delta\phi_a \stackrel{\text{def}}{\equiv} \phi'_a(x') - \phi_a(x) \quad \text{and} \quad \delta(\partial_\mu \phi_a) \stackrel{\text{def}}{\equiv} \partial'_\mu \phi'_a(x') - \partial_\mu \phi_a(x)\,. \tag{4.387}$$

Namely,

$$\begin{aligned} 0 &= \mathcal{L}(\tilde{\phi}'(x'), \partial'_\mu \tilde{\phi}'(x')) - \mathcal{L}(\tilde{\phi}(x), \partial_\mu \tilde{\phi}(x)) \\ &= \underbrace{\frac{\partial \mathcal{L}}{\partial \phi_a}}\, \delta\phi_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta(\partial_\mu \phi_a)\,. \end{aligned} \tag{4.388}$$

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \text{ (by equation of motion)}$$

There are six independent generators $M^{\alpha\beta}$ for proper and orthochronous Lorentz transformations Λ, and correspondingly there should be six independent generators in the space of n fields $\tilde{\phi}$. Namely, corresponding to the infinitesimal Lorentz transformation (1.89)

$$\Lambda^\mu{}_\nu = g^\mu{}_\nu + \omega^\mu{}_\nu = g^\mu{}_\nu + \frac{1}{2}\omega_{\alpha\beta}(M^{\alpha\beta})^\mu{}_\nu \tag{4.389}$$

the matrix S can be written as

$$S_{ab} = \delta_{ab} + \frac{1}{2}\omega_{\alpha\beta}(T^{\alpha\beta})_{ab}\,, \tag{4.390}$$

where $T^{\alpha\beta}$ are the generators in the space of $\tilde{\phi}$ and are antisymmetric under the exchange $\alpha \leftrightarrow \beta$:

$$T^{\alpha\beta} = -T^{\beta\alpha} \tag{4.391}$$

making the number of independent generators to be six. Then $\delta\phi_a$ and $\delta(\partial_\mu \phi_a)$ are

$$\begin{aligned} \delta\phi_a &= \phi'_a(x') - \phi_a(x) \\ &= S_{ab}\phi_b(x) - \phi_a(x) \\ &= \frac{1}{2}\omega_{\alpha\beta}(T^{\alpha\beta})_{ab}\phi_b(x) \end{aligned} \tag{4.392}$$

$$\begin{aligned} \delta(\partial_\mu \phi_a) &= \partial'_\mu \phi'_a(x') - \partial_\mu \phi_a(x) \\ &= (\Lambda_\mu{}^\nu \partial_\nu) S_{ab}\phi_b(x) - \partial_\mu \phi_a(x) \\ &= (g_\mu{}^\nu + \omega_\mu{}^\nu)\partial_\nu\Big(\delta_{ab} + \frac{1}{2}\omega_{\alpha\beta}(T^{\alpha\beta})_{ab}\phi_b(x)\Big) - \partial_\mu \phi_a(x) \\ &= (\partial_\mu + \omega_\mu{}^\nu \partial_\nu)\Big(\phi_a(x) + \frac{1}{2}\omega_{\alpha\beta}(T^{\alpha\beta})_{ab}\phi_b(x)\Big) - \partial_\mu \phi_a(x) \\ &= \omega_\mu{}^\nu \partial_\nu \phi_a(x) + \frac{1}{2}\omega_{\alpha\beta}(T^{\alpha\beta})_{ab}\partial_\mu \phi_b(x)\,, \end{aligned} \tag{4.393}$$

where in the last step we discarded the term that is second order in ω. Using these in (4.388),

$$\begin{aligned} 0 &= \left(\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\right)\left(\frac{1}{2}\omega_{\alpha\beta}(T^{\alpha\beta})_{ab}\phi_b\right) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\left(\omega_\mu{}^\nu \partial_\nu \phi_a + \frac{1}{2}\omega_{\alpha\beta}(T^{\alpha\beta})_{ab}\partial_\mu \phi_b\right) \\ &= \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\frac{1}{2}\omega_{\alpha\beta}(T^{\alpha\beta})_{ab}\phi_b\right) + \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\omega_{\mu\nu}\partial^\nu \phi_a}_{\displaystyle \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi_a)}\omega_{\alpha\beta}\partial^\beta \phi_a} \\ &= \omega_{\alpha\beta}\left[\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\frac{(T^{\alpha\beta})_{ab}}{2}\phi_b\right) + \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi_a)}\partial^\beta \phi_a\right]. \end{aligned} \tag{4.394}$$

This holds for any (small) $\omega_{\alpha\beta}$. Then, can we set the quantity inside the square bracket to be zero? No, since $\omega^{\alpha\beta}$'s are antisymmetric with respect to (α, β) and are not independent. We can, however, pick a specific pair (α, β) and set

$$\omega_{\mu\nu} = 0 \quad \text{for all } \mu \text{ and } \nu, \text{ except} \quad \omega_{\alpha\beta} = -\omega_{\beta\alpha}\,. \tag{4.395}$$

Then using $T^{\alpha\beta} = -T^{\beta\alpha}$, (4.394) becomes (no sum over α, β)

$$\begin{aligned} 0 &= \omega_{\alpha\beta}\left[\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\frac{(T^{\alpha\beta})_{ab}}{2}\phi_b\right) + \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi_a)}\partial^\beta \phi_a \quad - \quad (\alpha \leftrightarrow \beta)\right] \\ &= \omega_{\alpha\beta}\Bigg[\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\frac{(T^{\alpha\beta})_{ab} - (T^{\beta\alpha})_{ab}}{2}\phi_b\right) + \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi_a)}\partial^\beta \phi_a - \frac{\partial \mathcal{L}}{\partial(\partial_\beta \phi_a)}\partial^\alpha \phi_a \\ &\qquad\qquad \underbrace{-\,g^{\alpha\beta}\mathcal{L} \qquad\qquad +\,g^{\alpha\beta}\mathcal{L}}_{0}\Bigg] \\ &= \omega_{\alpha\beta}\left[\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}(T^{\alpha\beta})_{ab}\phi_b\right) + J^{\alpha\beta} - J^{\beta\alpha}\right] \end{aligned} \tag{4.396}$$

where in the second line we have added and subtracted $\mathcal{L}\, g^{\alpha\beta}$, and $J^{\alpha\beta}$ is the energy momentum tensor defined in (4.121):

$$J^{\alpha\beta} \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi_a)}\partial^\beta \phi_a - \mathcal{L}\, g^{\alpha\beta}\,.$$

Using the conservation of $J^{\alpha\beta}$, $J^{\alpha\beta} - J^{\beta\alpha}$ can be written as

$$\begin{aligned} \partial_\mu(x^\alpha J^{\mu\beta} - x^\beta J^{\mu\alpha}) &= \delta_{\mu\alpha}J^{\mu\alpha} + x^\alpha \underbrace{\partial_\mu J^{\mu\alpha}}_{0} + \delta_{\mu\beta}J^{\mu\beta} + x^\beta \underbrace{\partial_\mu J^{\mu\beta}}_{0} \\ &= J^{\alpha\beta} - J^{\beta\alpha}\,. \end{aligned} \tag{4.397}$$

Then (4.396) becomes

$$0 = \omega_{\alpha\beta}\partial_\mu M^{\mu\alpha\beta} \tag{4.398}$$

with

$$\boxed{M^{\mu\alpha\beta} \equiv \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}(T^{\alpha\beta})_{ab}\phi_b + (x^\alpha J^{\mu\beta} - x^\beta J^{\mu\alpha})}\,. \tag{4.399}$$

Since there is no sum over α and β as prescribed in (4.395), we immeidately obtain the conservation relation

$$\boxed{\partial_\mu M^{\mu\alpha\beta} = 0}\,, \tag{4.400}$$

which is valid for any (α, β) since the choice of (α, β) was arbitrary in (4.395). Note that $M^{\mu\alpha\beta}$ is also antisymmetric under the exchange $\alpha \leftrightarrow \beta$.

The conserved quantity corresponding to the rotation around the k-th axis is then $(i, j, k$: cyclic)

$$\begin{aligned} J^k &= \int d^3x\, M^{0ij} \\ &= \int d^3x \left[\frac{\partial\mathcal{L}}{\partial\dot{\phi}_a}(T^{ij})_{ab}\phi_b + (x^i J^{0j} - x^j J^{oi})\right] \\ &= \int d^3x \left[\pi_a(T^{ij})_{ab}\phi_b + (\vec{x}\times\vec{J}^0)_k\right] \end{aligned} \tag{4.401}$$

where $\vec{J}^0 \equiv (J^{01}, J^{02}, J^{03})$ is the momentum density and used the definition of conjugate field $\pi_a \equiv \delta\mathcal{L}/\delta\dot{\phi}_a$. The second term has the form of orbital angular momentum around the origin, and the first term is then interpreted as the spin angular momentum carried by the particle. The quantity J^k is the total angular momentum which is a constant of motion only after spin of the particle is added to the orbital angular momentum.

You may be wondering what are the 'conserved quantities' of (4.400) for $\alpha = 0$ or $\beta = 0$ which should correspond to the generators of boost. The time derivative of such quantities would indeed be zero; because of the explicit time dependence that appear in the definition (4.399), however, such quantities are not genuine physical conserved quantities.

Spin of electron

We can now apply (4.401) to the Dirac field Lagrangian to obtain the corresponding total angular momentum operator. Using the definition (4.121), the momentum density J^{0i} is (using $\pi = i\psi^\dagger$)

$$J^{0i} = \frac{\partial\mathcal{L}}{\partial\dot{\psi}_a}\partial^i\psi_a - g^{0i}\mathcal{L} = \pi_a\partial^i\psi_a = \psi^\dagger(-i\nabla_i)\psi\,. \tag{4.402}$$

The generators in the spinor space are $T^{ij} = B^{ij} = -i\Sigma_k/2$ (3.182). The total angular momentum (4.401) is then

$$J^k = \int d^3x\, \psi^\dagger \left(\frac{\Sigma_k}{2} + (\vec{x} \times \vec{p})_k \right) \psi \qquad (\vec{p} \equiv -i\vec{\nabla})\,, \tag{4.403}$$

and the component along some fixed direction $\vec{s}$ is given by

$$\vec{s} \cdot \vec{J} = \int d^3x\, \psi^\dagger \left(\frac{\vec{s} \cdot \vec{\Sigma}}{2} + \vec{s} \cdot (\vec{x} \times \vec{p}) \right) \psi\,. \tag{4.404}$$

We will now evaluate the expectation value of this operator for a state that represents a positron at rest: $b^\dagger_{\vec{0},\vec{s}}|0\rangle$ where $\vec{s}$ is taken to be the same $\vec{s}$ that is used in $\vec{s} \cdot \vec{J}$. Momentum-expanding the fields and recalling the implicit normal ordering, we get

$$\begin{aligned}
&\langle 0|b_{\vec{0},\vec{s}}(\vec{s} \cdot \vec{J}) b^\dagger_{\vec{0},\vec{s}}|0\rangle \\
&\quad = \langle 0|b_{\vec{0},\vec{s}} \sum_{\vec{p},\vec{s}_1,\vec{p}',\vec{s}_2} \int d^3x\, :(\cancel{a^\dagger_{\vec{p},\vec{s}_1}} \cancel{f^\dagger_{\vec{p},\vec{s}_1}} + b_{\vec{p},\vec{s}_1} g^\dagger_{\vec{p},\vec{s}_1}) \\
&\qquad \times \left(\frac{\vec{s} \cdot \vec{\Sigma}}{2} + \vec{s} \cdot (\vec{x} \times \vec{p}) \right) (\cancel{a_{\vec{p}',\vec{s}_2}} \cancel{f_{\vec{p}',\vec{s}_2}} + b^\dagger_{\vec{p}',\vec{s}_2} g_{\vec{p}',\vec{s}_2}) : b^\dagger_{\vec{0},\vec{s}}|0\rangle \\
&\quad = \sum_{\vec{p},\vec{s}_1,\vec{p}',\vec{s}_2} \langle 0|b_{\vec{0},\vec{s}} \underbrace{: b_{\vec{p},\vec{s}_1} b^\dagger_{\vec{p}',\vec{s}_2} :}_{\underbrace{-b^\dagger_{\vec{p}',\vec{s}_2} b_{\vec{p},\vec{s}_1}}_{-\delta_{\vec{0},\vec{p}'}\delta_{\vec{s},\vec{s}_2}\delta_{\vec{p},\vec{0}}\delta_{\vec{s}_1,\vec{s}}}} b^\dagger_{\vec{0},\vec{s}}|0\rangle \int d^3x\, g^\dagger_{\vec{p},\vec{s}_1} \left(\frac{\vec{s} \cdot \vec{\Sigma}}{2} + \vec{s} \cdot (\vec{x} \times \vec{p}) \right) g_{\vec{p}',\vec{s}_2} \\
&\quad = -\int d^3x\, g^\dagger_{\vec{0},\vec{s}} \left(\frac{\vec{s} \cdot \vec{\Sigma}}{2} + \vec{s} \cdot (\vec{x} \times \vec{p}) \right) g_{\vec{0},\vec{s}}\,.
\end{aligned} \tag{4.405}$$

Since applying $\vec{p} = -i\vec{\nabla}$ on $g_{\vec{0},\vec{s}}$ annihilates it and

$$(\vec{s} \cdot \vec{\Sigma}) g_{\vec{0},\vec{s}} = -g_{\vec{0},\vec{s}} \tag{4.406}$$

by construction [see (3.263)], we have

$$\langle 0|b_{\vec{0},\vec{s}}(\vec{s} \cdot \vec{J}) b^\dagger_{\vec{0},\vec{s}}|0\rangle = \frac{1}{2} \int d^3x\, g^\dagger_{\vec{0},\vec{s}} g_{\vec{0},\vec{s}} = \frac{1}{2}\,. \tag{4.407}$$

Thus, the angular momentum of the positron state $b^\dagger_{\vec{0},\vec{s}}|0\rangle$ is indeed $+1/2$ in the $\vec{s}$ direction. Note the crucial minus sign when the normal ordering was applied to $b_{\vec{p},\vec{s}_1} b^\dagger_{\vec{p}',\vec{s}_2}$ which cancelled the minus sign in (4.406). When the state is changed to an electron at rest, then all needed is to change $b^{(\dagger)}_{\vec{0},\vec{s}}$ to $a^{(\dagger)}_{\vec{0},\vec{s}}$ in the procedure above. One

sees that the relevant product $a^\dagger_{\vec{p},\vec{s}_1} a_{\vec{p}',\vec{s}_2}$ in the angular momentum operator is already normal ordered and there is no minus sign in (4.406) when g is replaced by f, and one obtains

$$\langle 0|a_{\vec{0},\vec{s}}(\vec{s}\cdot\vec{J})a^\dagger_{\vec{0},\vec{s}}|0\rangle = \frac{1}{2}\,. \tag{4.408}$$

Thus, the spin indexes of $a^\dagger_{\vec{p},\vec{s}}$ and $b^\dagger_{\vec{p},\vec{s}}$ both correctly represent the corresponding physical spin.

Microscopic causality of the Dirac field

We have seen that the Klein-Gordon field satisfied microscopic causality; namely, field operators $\phi(x)$ and $\phi(y)$ commuted if x and y are separated by a space-like distance:

$$[\phi(x),\phi(y)] = i\Delta(x-y) = 0\,, \quad \text{if } (x-y)^2 < 0\,, \tag{4.409}$$

where $i\Delta(x)$ is given by (4.290). In the case of the Dirac field also, in order for two measurements to be independent, the corresponding operators A and B should *commute* and not *anticommute*. In fact, independence of two meansurements means that for any eigenvalue a of the operator A and any eigenvalue b of the operator B, there exists a simultaneous eigenstate $|a,b\rangle$ that allows measurements of the two observables with infinite accuracy. Then, the operators A and B should commute:

$$\begin{aligned}(AB-BA)|a,b\rangle &= (ab-ba)|a,b\rangle = 0 \quad (\text{for all } a,b)\\ &\to \quad [A,B] = 0\,.\end{aligned} \tag{4.410}$$

On the other hand, the operator relations for the Dirac fields are given as anticommutators, and commutators such as $[\psi_n(x),\psi^\dagger_m(y)]$ do not vanish for $(x-y)^2<0$; in fact, it is a messy expression of a's and b's and not even a c-number. Physically relevant quantities, however, always appear as bilinear covariants which have the form

$$\bar\psi\,\Gamma\psi = \Gamma_{nm}\bar\psi_n\psi_m\,, \tag{4.411}$$

where Γ is a 4×4 matrix. Thus, if

$$[\bar\psi_n(x)\psi_m(x),\bar\psi_k(y)\psi_l(y)] = 0 \quad (x-y)^2<0 \quad (\text{for any } n,m,k,l), \tag{4.412}$$

then we conclude that microscopic causality is satisfied for the Dirac field. In proving this, we first note that *if* field components at x, $\bar\psi_n(x)$ and $\psi_m(x)$, *anticommute* with those at y, $\bar\psi_k(y)$ and $\psi_l(y)$, then $\bar\psi_n(x)\psi_m(x)$ commutes with $\bar\psi_k(y)\psi_l(y)$:

$$\begin{aligned}\overleftarrow{\bar\psi_n(x)\psi_m(x)}^{(-1)^2}\boxed{\bar\psi_k(y)}\psi_l(y) &= \bar\psi_k(y)\,\overleftarrow{\bar\psi_n(x)\psi_m(x)}^{(-1)^2}\boxed{\psi_l(y)}\\ &= \bar\psi_k(y)\psi_l(y)\;\bar\psi_n(x)\psi_m(x)\,.\end{aligned} \tag{4.413}$$

Thus, we need to show that, for any $n, m = 1, 2, 3, 4$,

$$\begin{cases} \{\psi_n(x), \psi_m(y)\} = 0 \\ \{\bar{\psi}_n(x), \bar{\psi}_m(y)\} = 0 \\ \{\psi_n(x), \bar{\psi}_m(y)\} = 0 \end{cases} \quad \text{for } (x-y)^2 < 0\,. \tag{4.414}$$

First two are trivial since the only non-zero anticommutators are of the type $\{a, a^\dagger\}$ or $\{b, b^\dagger\}$:

$$\begin{array}{lcl} \psi \text{ contains } a \text{ and } b^\dagger \text{ only} & \rightarrow & \{\psi_n(x), \psi_m(y)\} = 0\,, \\ \bar{\psi} \text{ contains } a^\dagger \text{ and } b \text{ only} & \rightarrow & \{\bar{\psi}_n(x), \bar{\psi}_m(y)\} = 0\,. \end{array} \tag{4.415}$$

The evaluation of the anticommutator $\{\psi_n(x), \bar{\psi}_m(y)\}$ requires some care. For convenience, let's extend the definition of matrix of the form ab^T, where a and b are column vectors, to the anticommutators (and similarly to the commutators): $\{a, b^T\}$ is a matrix whose components are defined by

$$\{a, b^T\}_{nm} \stackrel{\text{def}}{\equiv} \{a_n\,, b_m\}\,, \tag{4.416}$$

which is clearly linear with respect to a and b.t Then, using the division of ψ into creation and annihilation parts (4.358),

$$\begin{aligned} \{\psi(x), \bar{\psi}(y)\} &= \{\psi_a(x) + \psi_{b^\dagger}(x)\,, \bar{\psi}_a(y) + \bar{\psi}_{b^\dagger}(y)\} \\ &= \{\psi_a(x)\,, \bar{\psi}_a(y)\} + \{\psi_{b^\dagger}(x)\,, \bar{\psi}_{b^\dagger}(y)\}\,. \end{aligned} \tag{4.417}$$

Following the procedure similar to (4.361),

$$\begin{aligned} \{\psi_a(x)\,, \bar{\psi}_a(y)\} &= \sum_{\vec{p},\vec{s},\vec{p}',\vec{s}'} \underbrace{\{a_{\vec{p},\vec{s}}\,, a^\dagger_{\vec{p}',\vec{s}'}\}}_{\delta_{\vec{p},\vec{p}'}\delta_{\vec{s},\vec{s}'}} f_{\vec{p},\vec{s}}(x)\overline{f}_{\vec{p}',\vec{s}'}(y) = \sum_{\vec{p},\vec{s}} f_{\vec{p},\vec{s}}(x)\overline{f}_{\vec{p},\vec{s}}(y) \\ &= \underbrace{\sum_{\vec{p}} \frac{1}{2p^0 V}}_{\frac{1}{(2\pi)^3}\int \frac{d^3p}{2p^0}} \underbrace{\sum_{\vec{s}} u_{\vec{p},\vec{s}}\,\bar{u}_{\vec{p},\vec{s}}}_{(\underbrace{\not{p}}_{i\not{\partial}_x} + m) \text{ by (3.317)}} e^{-ip\cdot(x-y)} \\ &= (i\not{\partial}_x + m)\frac{1}{(2\pi)^3}\int \frac{d^3p}{2p^0} e^{-ip\cdot(x-y)} \\ &= (i\not{\partial}_x + m)\Delta_+(x-y)\,, \end{aligned} \tag{4.418}$$

where $\partial^\mu_x \equiv \partial/\partial x_\mu$ (namely, operates on x and not on y). Similarly,

$$\{\psi_{b^\dagger}(x)\,, \bar{\psi}_{b^\dagger}(y)\} = \sum_{\vec{p},\vec{s},\vec{p}',\vec{s}'} \underbrace{\{b^\dagger_{\vec{p},\vec{s}}\,, b_{\vec{p}',\vec{s}'}\}}_{\delta_{\vec{p},\vec{p}'}\delta_{\vec{s},\vec{s}'}} g_{\vec{p},\vec{s}}(x)\overline{g}_{\vec{p}',\vec{s}'}(y) = \sum_{\vec{p},\vec{s}} g_{\vec{p},\vec{s}}(x)\overline{g}_{\vec{p},\vec{s}}(y)$$

$$
\begin{aligned}
&= \underbrace{\sum_{\vec{p}} \frac{1}{2p^0 V}}_{\frac{1}{(2\pi)^3}\int \frac{d^3p}{2p^0}} \underbrace{\sum_{\vec{s}} v_{\vec{p},\vec{s}}\, \bar{v}_{\vec{p},\vec{s}}\, e^{ip\cdot(x-y)}}_{(\underbrace{\not{p}}_{-i\not{\partial}_x} - m) \text{ by (3.317)}} \\
&= -(i\not{\partial}_x + m)\frac{1}{(2\pi)^3}\int \frac{d^3p}{2p^0} e^{ip\cdot(x-y)} \\
&= -(i\not{\partial}_x + m)\Delta_+(y-x)\,. \qquad (4.419)
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
\{\psi(x), \bar{\psi}(y)\} &= (i\not{\partial}_x + m)\left[\Delta_+(x-y) - \Delta_+(y-x)\right] \\
&= (i\not{\partial}_x + m) i\Delta(x-y) \\
&\stackrel{\text{def}}{\equiv} -iS(x-y) \qquad (4.420)
\end{aligned}
$$

Now, the function $i\Delta(x-y)$ is already known to vanish for $(x-y)^2 < 0$; thus, $\{\psi(x), \bar{\psi}(y)\} = 0$ for $(x-y)^2 < 0$, and all bilinear covariants commute at space-like distances. Thus, the quantized Dirac field satisfies microscopic causality.

What would have happened to the microscopic causality if we had quantized the Dirac field with commutators instead of anticommutators? Then, we would have

$$
[a_{\vec{p},\vec{s}}\,, a^\dagger_{\vec{p}',\vec{s}'}] = [b_{\vec{p},\vec{s}}\,, b^\dagger_{\vec{p}',\vec{s}'}] = \delta_{\vec{p},\vec{p}'}\delta_{\vec{s},\vec{s}'}\,, \qquad (4.421)
$$

and all other *commutators* would be zero. The momentum expansion in terms of normal modes would be the same as before, and we would evaluate $[\psi_n(x), \bar{\psi}_m(y)]$ instead of the corresponding anticommutator. The critical difference occurs when we use $[b^\dagger_{\vec{p},\vec{s}}\,, b_{\vec{p}',\vec{s}'}] = -[b_{\vec{p}',\vec{s}'}\,, b^\dagger_{\vec{p},\vec{s}}] = -\delta_{\vec{p},\vec{p}'}\delta_{\vec{s},\vec{s}'}$ which will change the sign in the first line of (4.419), and as a result we would obtain

$$
[\psi(x), \bar{\psi}(y)] = (i\not{\partial}_x + m)\left[\Delta_+(x-y) + \Delta_+(y-x)\right], \qquad (4.422)
$$

which does not vanish for $(x-y)^2 < 0$ since $\Delta_+(x-y) = \Delta^*_+(y-x)$ and $\Delta_+(x-y)$ is real and positive in the space-like region. Thus, in addition to the positive definiteness of the energy, the microscopic causality also requires that the Dirac field be quantized by anticommutators. How did it work out for the case of the Klein-Gordon field? There, we did not have the minus sign in the commutator corresponding to (4.419) and the necessary relative minus sign between $\Delta_+(x-y)$ and $\Delta_+(y-x)$ came from the property of commutator $[\phi^\dagger_a(x), \phi_a(y)] = -[\phi_a(y), \phi^\dagger_a(x)]$ used in (4.287). In 1940, Pauli extended the above argument of microscopic causality to particles with general spins, and established the fundamental connection between spin and statistics:

$$
\begin{aligned}
\text{Integer spin} &\leftrightarrow \text{Bose-Einstein statistics,} \\
\text{Half-integer spin} &\leftrightarrow \text{Fermi-Dirac statistics,}
\end{aligned} \qquad (4.423)
$$

where 'Bose-Einstein statistics' means that the field must be quantized by commutators and thus a given state can be occupied by any number of particles, and 'Fermi-Dirac statistics' means that the field must be quantized by anticommutators and thus a given state can be occupied by at most one particle.

Lorentz invariance of the Dirac-field quantization

Let's now turn to the question of whether systems quantized in different Lorentz frames are equivalent or not. As in the case of the Klein-Gordon field, we could prove the equivalence of the anticommutation relations of fields in different frames, or those of creation and annihilation operators. Let's take the fields this time. First, we will see below that the general-time anticommutation relation (4.415) and (4.420) reduces to the equal-time anticommutation relation (4.364) when we set $x^0 = y^0$. In fact, setting $x^0 = y^0 = t$ in (4.415), we obtain

$$\{\psi_n(t,\vec{x}),\psi_m(t,\vec{y})\} = 0\,, \quad \{\pi_n(t,\vec{x}),\pi_m(t,\vec{y})\} = 0\,. \tag{4.424}$$

Setting $x^0 = y^0$ in $\{\psi(t,\vec{x}),\bar{\psi}(t,\vec{y})\}$ is equivalent to repeating the derivation (4.420) with $x^0 = y^0$ from the beginning. We start from just before $\not{p}$ is replaced by the differential operator in (4.418) and (4.419):

$$\begin{aligned}
\{\psi(x)\,,\bar{\psi}(y)\}_{x^0=y^0} &= \frac{1}{(2\pi)^3}\int\frac{d^3p}{2p^0}\Big[\underbrace{(\not{p}}_{p^0\gamma^0-\vec{p}\cdot\vec{\gamma}}+m)e^{i\vec{p}\cdot(\vec{x}-\vec{y})}+\underbrace{(\not{p}}_{p^0\gamma^0-\vec{p}\cdot\vec{\gamma}\ \text{relabel}\ \vec{p}\to-\vec{p}}-m)e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}\Big] \\
&= \frac{1}{(2\pi)^3}\int\frac{d^3p}{2p^0}\Big[\underbrace{(p^0\gamma^0-\vec{p}\cdot\vec{\gamma}+m)+(p^0\gamma^0+\vec{p}\cdot\vec{\gamma}-m)}_{2p^0\gamma^0}\Big]e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \\
&= \gamma^0\frac{1}{(2\pi)^3}\int d^3p\,e^{i\vec{p}\cdot(\vec{x}-\vec{y})} = \gamma^0\delta^3(\vec{x}-\vec{y})\,,
\end{aligned} \tag{4.425}$$

where $\vec{\gamma}\overset{\text{def}}{\equiv}(\gamma^1,\gamma^2,\gamma^3)$. Now, the definition (4.416) leads to

$$\Big(\{a,b^T\}M\Big)_{nm} = \underbrace{\{a,b^T\}_{nk}}_{\{a_n,b_k\}}M_{km} = \{a_n,\ \underbrace{b_kM_{km}}_{(b^TM)_m}\} = \{a,b^TM\}_{nm}\,; \tag{4.426}$$

where a_n and b_m are operators while M_{nm} is a c-number; namely,

$$\{a,b^T\}M = \{a,b^TM\}\,, \quad \text{and similarly,} \quad M\{a,b^T\} = \{Ma,b^T\}\,. \tag{4.427}$$

Thus, we have

$$\{\psi(t,\vec{x}),\underbrace{\psi^\dagger(t,\vec{y})}_{\bar{\psi}\gamma^0}\} = \underbrace{\{\psi(t,\vec{x}),\bar{\psi}(t,\vec{y})\}}_{\gamma^0\delta^3(\vec{x}-\vec{y})}\gamma^0 = I\delta^3(\vec{x}-\vec{y})\,. \tag{4.428}$$

Since $\pi = i\psi^\dagger$, this is equivalent to $\{\psi(t,\vec{x}),\pi(t,\vec{y})\} = iI\delta^3(\vec{x}-\vec{y})$ which is the equal-time quantization condition.

Thus, the Lorentz invariance of the quantization procedure is proven if we can show that

$$\begin{cases} \{\psi_n(x),\psi_m(y)\} = 0 \\ \{\bar\psi_n(x),\bar\psi_m(y)\} = 0 \\ \{\psi(x),\bar\psi(y)\} = -iS(x-y) \end{cases} \tag{4.429}$$

in one frame leads to the same relations in another frame where the field in the new frame are related to the original field by the spinor representation of Lorentz transformation:

$$\psi'(x') = S(\Lambda)\psi(x)\,, \qquad \psi'(y') = S(\Lambda)\psi(y)\,, \tag{4.430}$$

where $x' = \Lambda x$ and $y' = \Lambda y$ (Λ: proper and orthochronous). The first two relations are trivial:

$$\{\psi'_n(x'),\psi'_m(y')\} = \{S_{nk}\psi_k(x), S_{ml}\psi_l(y)\} = S_{nk}S_{ml}\{\psi_k(x),\psi_l(y)\} = 0\,, \tag{4.431}$$

and similarly $\{\bar\psi'_n(x'),\bar\psi'_m(y')\} = 0$. Using (4.427),

$$\begin{aligned} \{\psi'(x'),\bar\psi'(y')\} &= \{S\psi(x),\bar\psi(y)\bar S\} = S\,\underbrace{\{\psi(x),\bar\psi(y)\}}_{(i\partial\!\!\!/_x + m)i\Delta(x-y)}\bar S \\ &= (i\partial_{x\mu}\underbrace{S\gamma^\mu\bar S}_{\Lambda_\nu{}^\mu\gamma^\nu} + m)\underbrace{i\Delta(x-y)}_{i\Delta(x'-y')} \\ &= (i\partial_{x'\nu}\gamma^\nu + m)i\Delta(x'-y') \\ &= -iS(x'-y')\,, \end{aligned} \tag{4.432}$$

where we have used the Lorentz invariance of the function $i\Delta$: $i\Delta(x') = i\Delta(x)$, the property $S\gamma^\mu\bar S = \Lambda_\nu{}^\mu\gamma^\nu$ (3.274), and $\partial_{x'\nu} = \Lambda_\nu{}^\mu\partial_{x\mu}$. Thus, the same set of anticommutation relations are satisfied in the new frame, and the Lorentz invariance of the quantization procedure is proven.

Incidentally, $\Delta_+(x)$, and thus $i\Delta(x) = \Delta_+(x) - \Delta_+(-x)$, is a solution of the Klein-Gordon equation:

$$(\partial^2 + m^2)\Delta_+(x) = \frac{1}{(2\pi)^3}\int\frac{d^3p}{2p^0}\underbrace{(\partial^2+m^2)}_{(-p^2+m^2)=0}e^{-ip\cdot x} = 0\,, \tag{4.433}$$

and $-iS(x)$ is a solution of the Dirac equation:

$$\begin{aligned} (i\partial\!\!\!/ - m)(-iS(x)) &= \underbrace{(i\partial\!\!\!/ - m)(i\partial\!\!\!/ + m)}_{(-\underbrace{\partial\!\!\!/\partial\!\!\!/}_{\partial^2} - m^2)}i\Delta(x) \\ &= -(\partial^2+m^2)i\Delta(x) = 0\,. \end{aligned} \tag{4.434}$$

Problems

4.1 *Momentum expansion.*
In the momentum expansion of a hermitian Klein-Gordon field

$$\phi(x) = \sum_{\vec{p}} \Big(a_{\vec{p}}\, e_{\vec{p}}(x) + a^\dagger_{\vec{p}}\, e^*_{\vec{p}}(x) \Big),$$

suppose one takes the normalization of the normal mode functions as

$$e_{\vec{p}}(x) \stackrel{\text{def}}{\equiv} \frac{e^{-ip\cdot x}}{\sqrt{2c_p V}}$$

where c_p is some real function of $|\vec{p}|$. This defines the normalization of the operators $a_{\vec{p}}$ and $a^\dagger_{\vec{p}}$. Assume that the standard equal-time commutation relations among ϕ and π hold:

$$[\phi(t,\vec{x}), \pi(t,\vec{x}')] = i\delta(\vec{x}-\vec{x}')\,, \quad [\phi(t,\vec{x}), \phi(t,\vec{x}')] = [\pi(t,\vec{x}), \pi(t,\vec{x}')] = 0\,.$$

(a) Obtain the orthonormality relations of the normal mode functions; namely, calculate $\int d^3x\, e^*_{\vec{p}}(x) i \overset{\leftrightarrow}{\partial_0} e_{\vec{p}'}(x)$ and $\int d^3x\, e_{\vec{p}}(x) i \overset{\leftrightarrow}{\partial_0} e_{\vec{p}'}(x)$.
(b) Express $a_{\vec{p}}$ and $a^\dagger_{\vec{p}}$ in terms of ϕ and π. Note that the relation $\pi = \dot{\phi}$ is a result of the Heisenberg's equation of motion for ϕ which results from the commutation relations among ϕ and π and not affected by the normalization of the normal modes.
(c) Require that the equal-time commutation relations among ϕ and π lead to $[a_{\vec{p}}, a^\dagger_{\vec{p}'}] = \delta_{\vec{p},\vec{p}'}$, and find the correct normalization factor c_p.

4.2 *Lorentz invariance of quantization procedure.*
In the text, we have proven the Lorentz invariance of the quantization of a spin-0 field by showing that the commutation relations among creation and annihilation operators in one frame lead to those in the other frame. It is of course possible to show that the equal-time commutation relations among field and its conjugate filed lead to those in the other frame. Take a hermitian spin-0 field $\phi(x)$. The field in another frame, $\phi'(x')$, is given by the scalar field condition:

$$\phi'(x') = \phi(x) \qquad (x' = \Lambda x)\,,$$

where Λ is a proper and orthochronous Lorentz transformation. The conjugate field in each frame is defined as the time derivative of the field in each frame:

$$\pi(x) \equiv \frac{\partial}{\partial x^0}\phi(x)\,, \quad \pi'(x') \equiv \frac{\partial}{\partial x^{0\prime}}\phi'(x')\,.$$

(*a*) *Start from the commutation relation*

$$[\phi(x), \phi(y)] = i\Delta(x-y) = \Delta_+(x-y) - \Delta_+(y-x)$$

with

$$\Delta_+(z) = \frac{1}{(2\pi)^3}\int \frac{d^3p}{2p^0} e^{-ip\cdot z}\,,$$

and show that this leads to the commutation relations

$$[\phi(x), \phi(y)]_{x^0=y^0} = 0\,, \quad [\pi(x), \pi(y)]_{x^0=y^0} = 0\,, \quad [\phi(x), \pi(y)]_{x^0=y^0} = i\delta^3(\vec{x}-\vec{y})\,.$$

Do not use annihilation and creation operators.
(*b*) *Complete the proof that the above equal-time commutation relations among the field and its conjugate field lead to those in another frame with the same form; namely,*

$$[\phi'(x'), \phi'(y')]_{x'^0=y'^0} = 0\,, \quad [\pi'(x'), \pi'(y')]_{x'^0=y'^0} = 0\,, \quad [\phi'(x'), \pi'(y')]_{x'^0=y'^0} = i\delta^3(\vec{x}'-\vec{y}')\,.$$

4.3 *Lagrangian density for the Dirac field.*
The Lagrangian density for the Dirac field ψ is given by

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi\,.$$

In the text, we have derived the Dirac equation directly from the action principle. This time, apply the Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial \phi_k} = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_k)}$$

to each component of ψ; namely take ψ_a $(a = 1, \ldots, 4)$ as ϕ_k, to derive the Dirac's equation the Dirac equation for $\bar{\psi}$:

$$\bar{\psi}(i\overleftarrow{\partial\!\!\!/} + m) = 0\,,$$

and apply it for $\psi_a^\dagger$ $(a = 1, \ldots, 4)$ to obtain

$$(i\partial\!\!\!/ - m)\psi = 0\,.$$

Use the same Lagrangian density given above for both cases, and also do not resort to the Dirac or other explicit representation.

4.4 *Continous-$\vec{p}$ formalisms for Dirac field.*
For continuous momentum, the normal mode functions of the Dirac field are given by

$$f_{\vec{p},\vec{s}}(x) \equiv \frac{u_{\vec{p},\vec{s}} e^{-ip\cdot x}}{\sqrt{(2\pi)^3 2p^0}}\,, \quad g_{\vec{p},\vec{s}}(x) \equiv \frac{v_{\vec{p},\vec{s}} e^{ip\cdot x}}{\sqrt{(2\pi)^3 2p^0}}\,,$$

and the momentum expansion is

$$\psi(x) = \sum_{\vec{s}} \int d^3p (a_{\vec{p},\vec{s}} f_{\vec{p},\vec{s}}(x) + b^\dagger_{\vec{p},\vec{s}} g_{\vec{p},\vec{s}}(x)) \,.$$

First, show that the following normalizations hold:

$$\begin{aligned} \int d^3x f^\dagger_{\vec{p},\vec{s}}(x) f_{\vec{p}',\vec{s}'}(x) &= \int d^3x g^\dagger_{\vec{p},\vec{s}}(x) g_{\vec{p}',\vec{s}'}(x) = \delta^3(\vec{p}-\vec{p}')\delta_{\vec{s},\vec{s}'} \\ \int d^3x f^\dagger_{\vec{p},\vec{s}}(x) g_{\vec{p}',\vec{s}'}(x) &= \int d^3x g^\dagger_{\vec{p},\vec{s}}(x) f_{\vec{p}',\vec{s}'}(x) = 0 \end{aligned} \,.$$

Then, repeat the derivation of

$$\{\psi(x), \bar{\psi}(y)\} = -iS(x-y) \,, \quad \textit{with} \quad -iS(x) \stackrel{\text{def}}{\equiv} (i\partial\!\!\!/ + m) i\Delta(x) \,.$$

Use the anticommutation relations

$$\begin{gathered} \{a_{\vec{p},\vec{s}} \,, a^\dagger_{\vec{p}',\vec{s}'}\} = \{b_{\vec{p},\vec{s}} \,, b^\dagger_{\vec{p}',\vec{s}'}\} = \delta^3(\vec{p}-\vec{p}')\delta_{\vec{s},\vec{s}'} \\ \textit{all others} = 0 \ . \end{gathered}$$

Chapter 5

Interacting Fields

So far, we have dealt only with free fields. Even though we have seen many remarkable features such as the spin of fermions, microscopic causality and the Lorentz invariance of quantized fields, and so on, all measurable effects occur through interactions of fields primarily as decays and scatterings which we will discuss in this chapter.

Before going into details, let's take a rough look at what we will be dealing with. As an example, take the creation of Higgs particle by annihilation of a fermion pair:

$$f + \bar{f} \to H \,, \tag{5.1}$$

where the fermion could be any lepton or quark, and the Higgs is a neutral spin-0 particle that is an important ingredient in the standard model of elementary particles. We can roughly picture this interaction in terms of non-quantized fields as follows: at the beginning we have two overlapping plane waves, one for f and the other for $\bar{f}$. Now, suppose that the overlap of the waves acts as source of the Higgs field. Namely, at every point of the overlap, Higgs field is created and propagates outward and they linearly add up (Huygens' principle). As the end result of the sum of all the waves propagating from every point of overlap, we will have a macroscopic wave of Higgs field coming out of the region of the overlap.

Of course, no macroscopic Higgs wave should be generated unless the invariant mass of the incoming fermion pair happens to be the mass of the Higgs (i.e. 4-momentum conservation):

$$(p_f + p_{\bar{f}})^2 = M^2 \,, \tag{5.2}$$

where p_f and $p_{\bar{f}}$ are 4-momenta of the fermion and antifermion, respectively, and M is the mass of the Higgs particle. Such a constraint, as we will see, is automatically built into the wave picture: in short, the microscopic spherical waves from the sources do not add up constructively to form a macroscopic wave unless the above condition is met. Another constraint is that the spins of the fermion pair should properly add up to form a spin-0 particle. This specifies how the four components of the fermion

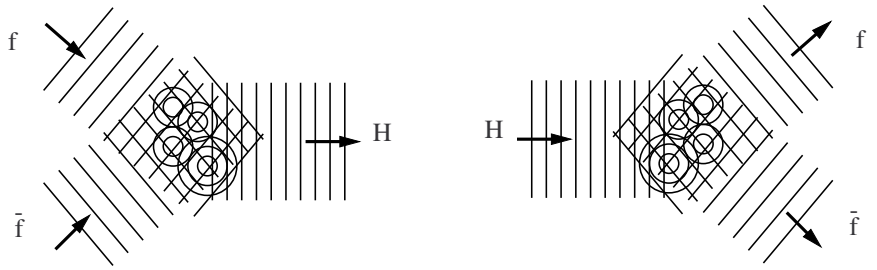


Figure 5.1: Rough graphical representations of the Higgs creation $f\,\bar{f} \to H$ and the Higgs decay $H \to f\,\bar{f}$.

fields should be combined to define the 'overlap', and largely determines the form of the interaction term in the Lagrangian. As we will see later, the existence of such a term in the Lagrangian in turn indicates that the Higgs field itself acts as a source of a fermion pair. It then represents a decay of the Higgs particle to a fermion pair:

$$H \to f\,\bar{f}\,, \tag{5.3}$$

which we will take in the following as an example to introduce the framework and techniques for evaluating interaction rates.

The above picture is not complete; we need to treat it in the framework of quantized field theory. What we are interested in is the probability amplitude for a given initial state be found in a certain final state, and it can in principle be obtained once we know the Hamiltonian of the system. In the Schrödinger picture, we would prepare the initial state and let it evolve according to the Schrödinger equation of motion and then take the inner product of the evolved state and the final state of interest. In the Heisenberg picture, we would find the state that represents the given initial-state at $t = -\infty$ and the state that represents the desired final-state at $t = +\infty$ and take their inner product. Here, we will adopt an intermediate picture, called the interaction picture, where the rapid oscillations of free fields that do not change the physical content (such as particle types and 4-momenta) are contained in the field operators and the states evolve relatively slowly reflecting the change in the physical content. Then the transition amplitude is given by taking the inner product of the evolved state and the given final state as in the Schrödinger picture. Let us now start from examining what form of interaction is possible for the Higgs-fermion coupling.

5.1 Lagrangian for the decay $H \to f\,\bar{f}$

The 'overlap' of two fermion fields ψ_1 and ψ_2 can be written in general as

$$\bar{\psi}_1 \Gamma \psi_2\,, \tag{5.4}$$

where Γ is an arbitrary 4×4 matrix. We have seen that such quantity can be expressed as a linear combination of *bilinear covariants* which transform under Lorentz transformation in well-defined ways:

$$\bar{\psi}_1\psi_2,\ \bar{\psi}_1\gamma^\mu\psi_2,\ \bar{\psi}_1\sigma^{\mu\nu}\psi_2,\ \bar{\psi}_1\gamma_5\gamma^\mu\psi_2,\ \bar{\psi}_1\gamma_5\psi_2\,. \tag{5.5}$$

As we have seen in (4.92), in order for the equation of motion to be Lorentz-invariant, the Lagrangian should be a Lorentz scalar. The simplest choices are

$$\phi\,\bar{\psi}_1\psi_2, \quad \text{or} \quad \phi\,\bar{\psi}_1\gamma_5\psi_2\,, \tag{5.6}$$

where ϕ is the spin-0 Higgs field. Note that the bilinear covariant has the same transformation property under proper and orthochronous transformation as the Higgs field; namely, it is a scalar or a pseudoscalar. This is because when the system is viewed in a different Lorentz frame, the relevant overlap should transform in the same way as the field created by the source in order for the same wave-source picture to be valid in the new frame. As we will see in a later chapter, which of the two forms to take depends on the relative orbital angular momentum L of the fermion pair: L=odd for the scaler and L=even for the pseudoscalar. Here, we assume the proper overlap is a scalar. Since the fermion and its antiparticle are represented by the same field (as e^- and e^+ are representated by a single Dirac field ψ), the interaction term is then

$$\mathcal{L}_{\text{int}} = \lambda\,\phi\,\bar{\psi}\psi\,. \tag{5.7}$$

where λ is a real constant that specifies the strength of the source for a given overlap, and is called the coupling constant. Note that the interaction term is hermitian; if it were not hermitian, then its hermitian conjugate would have to be added to make the whole hermitian.

What is the dimension of the coupling constant λ? To find it, we need to know the dimensions of the fields ϕ and ψ. Since $c = 1$, mass (m) and energy-momentum (p^μ) have the same dimension:

$$\dim(m) = \dim(P^\mu) \stackrel{\text{def}}{\equiv} E \quad (\mu = 0, 1, 2, 3)\,. \tag{5.8}$$

Since $p\cdot x$ appears in exponents of exponentials ($e^{-ip\cdot x}$), $p\cdot x$ should be dimensionless; thus, x^μ should have dimension E^{-1}:

$$\dim(x^\mu) = E^{-1} \quad (\mu = 0, 1, 2, 3)\,. \tag{5.9}$$

The dimension of total Lagrangian $L = \int d^3x\mathcal{L}$ is E since $L = T - V$; then the Lagrangian density $\mathcal{L}$ should have dimension E^4:

$$\dim(d^3x\mathcal{L}) = E \quad \to \quad \dim(\mathcal{L}) = E^4\,. \tag{5.10}$$

The term $m^2\phi^2$ in the Klein-Gordon Lagrangian $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2)$ indicates that

$$\dim(m^2\phi^2) = E^4 \quad \rightarrow \quad \dim(\phi) = E \quad (\phi: \text{ scalar field}), \tag{5.11}$$

and the term $m\bar{\psi}\psi$ in $\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi$ gives the dimension of fermion field:

$$\dim(m\bar{\psi}\psi) = E^4 \quad \rightarrow \quad \dim(\psi) = E^{\frac{3}{2}} \quad (\psi: \text{ fermion field}). \tag{5.12}$$

Then, in order for the interaction terms to have dimension E^4, λ has to be dimensionless:

$$\dim(\lambda\,\phi\,\bar{\psi}\psi) = E^4 \quad \rightarrow \quad \dim(\lambda) = E^0. \tag{5.13}$$

The Lagrangian density of the system is sum of free-field terms and the interaction term:

$$\boxed{\begin{gathered}\mathcal{L} = \mathcal{L}_H + \mathcal{L}_f + \mathcal{L}_{\text{int}}, \\ \mathcal{L}_H = \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - M^2\phi^2), \quad \mathcal{L}_f = \bar{\psi}(i\partial\!\!\!/ - m)\psi, \\ \mathcal{L}_{\text{int}} = \lambda\,\phi\,\bar{\psi}\psi.\end{gathered}} \tag{5.14}$$

The equation of motion for the scalar field can be obtained from the Lagrangian density using the Euler-Lagrange equation:

$$\underbrace{\frac{\partial\mathcal{L}}{\partial\phi}}_{-M^2\phi+\lambda\bar{\psi}\psi} = \underbrace{\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}}_{\partial^2\phi} \quad \rightarrow \quad (\partial^2 + M^2)\phi = \lambda\,\bar{\psi}\psi, \tag{5.15}$$

which is probably a more familiar form that shows that the quantity $\bar{\psi}\psi$ is acting as a source of the scalar field.

The conjugate fields are the same as before:

$$\pi \equiv \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi}, \quad \pi_n^f \equiv \frac{\partial\mathcal{L}}{\partial\dot{\psi}_n} = i\psi_n^\dagger, \tag{5.16}$$

where the superscript f of π^f indicates that it is the field conjugate to the fermion field. The Hamiltonian density of the system is then

$$\begin{aligned}\mathcal{H} &\equiv \sum_k \pi_k\dot{\phi}_k - \mathcal{L} \\ &= (\pi\dot{\phi} - \mathcal{L}_H) + (\sum_n \pi_n^f\dot{\psi}_n - \mathcal{L}_f) - \mathcal{L}_{\text{int}} \\ &= \mathcal{H}_H + \mathcal{H}_f + \mathcal{H}_{\text{int}},\end{aligned} \tag{5.17}$$

where k runs over all fields. The free-field Hamiltonians $\mathcal{H}_H$ and $\mathcal{H}_f$ are the same as before, and

$$\boxed{\mathcal{H}_{\text{int}} = -\mathcal{L}_{\text{int}}}, \tag{5.18}$$

which is true in general as long as there is no time derivative of fields in $\mathcal{L}_{\rm int}$, in which case the definition of the conjugate fields would be altered and thus $\mathcal{H}$ would have extra terms.

Quantization proceeds as usual; namely, fields are now considered to be operators in the Heisenberg picture, and commutation and anticommutation relations are imposed as before among the fields. The fermion field and the scalar field are assumed to commute (not anticommute). Then, all time dependences are given by Heisenberg's equations of motion using the total Hamiltonian obtained above, and in principle we should be able to calculate the probability for a given initial configuration to end up as a given final configuration.

5.2 Interaction Picture and the Dyson Series

We have been using the *Heisenberg picture* in which all time dependences are in the operators and the states do not change with time. An alternative picture is the *Schrödinger picture* in which all time dependences are in the states and the operators do not vary with time. To deal with interactions, however, it is convenient to take an intermediate picture in which the operators (i.e., fields) vary according to the free-field Hamiltonian, and the states change according to the interaction part of the Hamiltonian - called the *interaction picture*. As we will see, these pictures are related by unitary transformations of states and operators:

$$|a'\rangle = V|a\rangle\,, \quad O' = VOV^\dagger\,. \tag{5.19}$$

where V is a time-dependent unitary operator. Since all measurements are given by matrix elements of some operators sandwiched between states, $\langle a|O|b\rangle$, the three approaches are effectively identical since matrix elements are invariant under unitary transformations:

$$\langle a'|O'|b'\rangle = \langle a|V^\dagger VOV^\dagger V|b\rangle = \langle a|O|b\rangle\,. \tag{5.20}$$

We start by reviewing the relations among the three pictures.

Interaction picture

The following discussion is valid for any closed quantum mechanical system; we keep in mind, however, that in our case the operators are fields ($\phi\,\bar\psi\psi$, ϕ^2, etc.) and the states belong to the Hilbert space ($a^\dagger_{\vec p}|0\rangle$, $b^\dagger_{\vec p,\vec s}|0\rangle$, etc.).

In the Heisenberg picture, the time dependences of any state $|a\rangle$ and any operator O_H are given by

$$\boxed{|\dot a\rangle_H = 0\,, \quad \dot O_H = i[H_H\,, O_H]} \quad \text{(Heisenberg picture)}\,, \tag{5.21}$$

where the subscript H indicates the Heisenberg picture. In general, operators in Heisenberg picture depend on time, but the Hamiltonian itself does not since

$$\dot{H}_H = i[H_H\,, H_H] = 0\,. \tag{5.22}$$

Transition to the Schrödinger picture is achieved by the transformation

$$|a\rangle_S \stackrel{\rm def}{\equiv} V|a\rangle_H\,, \quad O_S \stackrel{\rm def}{\equiv} VO_HV^\dagger\,, \quad \boxed{V \stackrel{\rm def}{\equiv} e^{-iH_Ht}}\,, \tag{5.23}$$

where V is unitary since H_H is hermitian, and the subscript S indicates the Schrödinger picture. Since H_H commutes with $V = e^{-iH_Ht} = \sum_n(-iH_Ht)^n/n\,!$, we have

$$H_S \equiv \underbrace{VH_H}_{\rm commute}\,V^\dagger = H_H\underbrace{VV^\dagger}_{1} = H_H\,. \tag{5.24}$$

Since H_H does not depend on time, we can use $(\partial/\partial x)e^{xA} = Ae^{xA}$ (1.107) to obtain

$$\dot{V} = (-iH_H)V = V(-iH_H)\,. \tag{5.25}$$

Taking the time derivative of $|a\rangle_S = V|a\rangle_H$ and noting that $\dot{|a\rangle}_H = 0$ and $H_H = H_S$, we get

$$\dot{|a\rangle}_S = \dot{V}|a\rangle_H = (-iH_H\underbrace{V)|a\rangle_H}_{|a\rangle_S} = -iH_H|a\rangle_S = -iH_S|a\rangle_S\,. \tag{5.26}$$

Similarly, taking the time derivative of $O_S = VO_HV^\dagger$,

$$\begin{aligned}\dot{O}_S &= \dot{V}O_HV^\dagger + VO_H\dot{V}^\dagger + V\dot{O}_HV^\dagger \\ &= \underbrace{V(-iH_H)O_HV^\dagger + VO_H(iH_HV^\dagger)}_{V\,i[O_H,H_H]V^\dagger} + V\underbrace{\dot{O}_H}_{i[H_H,O_H]}V^\dagger = 0\,.\end{aligned} \tag{5.27}$$

Thus, in the Schrödinger picture, the time dependences are given by

$$\boxed{\dot{|a\rangle}_S = -iH_S|a\rangle_S\,, \quad \dot{O}_S = 0} \quad \text{(Schrödinger picture)}\,, \tag{5.28}$$

namely, the operators are now constants and states change with time.

Now, we move to the interaction picture by starting from the Schrödinger picture and dividing the Hamiltonian into two parts

$$H_S = H_S^0 + h_S\,. \tag{5.29}$$

where the two pieces are hermitian and do not depend on time. In general they do not commute. Later, H_S^0 is taken to be the free field part and h_S the interaction part,

but at present the division is arbitrary. Then, we apply a transformation by a unitary operator $V^0 = e^{iH_S^0 t}$:

$$|a\rangle_I \stackrel{\text{def}}{\equiv} V^0|a\rangle_S\,, \quad O_I \stackrel{\text{def}}{\equiv} V^0 O_S V^{0\dagger}\,, \quad \boxed{V^0 \stackrel{\text{def}}{\equiv} e^{iH_S^0 t}}\,, \tag{5.30}$$

where the subscript I indicates the interaction picture. Since $\dot{H}_S^0 = 0$, we can use (1.107) to get

$$\dot{V}^0 = iH_S^0 V^0 = V^0 iH_S^0\,. \tag{5.31}$$

Then, the time derivative of $|a\rangle_I = V^0|a\rangle_S$ becomes

$$\begin{aligned} |\dot{a}\rangle_I &= \underbrace{\dot{V}^0}_{V^0 iH_S^0}|a\rangle_S + V^0\underbrace{|\dot{a}\rangle_S}_{-iH_S|a\rangle_S \text{ by (5.28)}} \\ &= iV^0\overbrace{(H_S^0 - H_S)}^{-h_S}\underbrace{|a\rangle_S}_{V^{0\dagger}V^0} = -i\underbrace{V^0 h_S V^{0\dagger}}_{h_I}\underbrace{V^0|a\rangle_S}_{|a\rangle_I} \end{aligned} \tag{5.32}$$

and the time derivative of $O_I = V^0 O_S V^{0\dagger}$ is (using $\dot{O}_S = 0$)

$$\begin{aligned} \dot{O}_I &= \dot{V}^0 O_S V^{0\dagger} + V^0 O_S \dot{V}^{0\dagger} \\ &= (iH_S^0\underbrace{V^0)O_S V^{0\dagger}}_{O_I} + \underbrace{V^0 O_S\big(V^{0\dagger}}_{O_I}(-iH_S^0)\big) \\ &= i[H_S^0\,, O_I] = i[H_I^0\,, O_I] \end{aligned} \tag{5.33}$$

In the last step, we have used

$$H_I^0 \equiv \underbrace{V^0 H_S^0}_{\text{commute}} V^{0\dagger} = H_S^0\,. \tag{5.34}$$

Thus, the time dependences in the interaction picture are

$$\boxed{|\dot{a}\rangle_I = -ih_I|a\rangle_I\,, \quad \dot{O}_I = i[H_I^0\,, O_I]} \quad \text{(interaction picture)}\,, \tag{5.35}$$

At this point, we take the free-field part of the Hamiltonian as H^0, and the interaction part as h. If there is no explicit time derivatives in these terms, they have the same form in all three pictures. For example,

$$\begin{aligned} h_H &= \int d^3x\, \phi_H \bar{\psi}_H \psi_H\,, \\ h_S &= V\int d^3x\, \phi_H(V^\dagger V)\bar{\psi}_H(V^\dagger V)\psi_H V^\dagger = \int d^3x\, \phi_S \bar{\psi}_S \psi_S\,, \\ h_I &= V^0\int d^3x\, \phi_S(V^{0\dagger}V^0)\bar{\psi}_S(V^{0\dagger}V^0)\psi_S V^{0\dagger} = \int d^3x\, \phi_I \bar{\psi}_I \psi_I\,. \end{aligned} \tag{5.36}$$

If there are time derivatives in h, the time dependences of the unitary operators V and V^0 can in principle generate extra terms and break the above form invariance. The time derivatives of fields, however, are usually replaced by the conjugate fields in the Hamiltonian (for example, $\dot{\phi}$ is replaced by π) and thus h does not contain time derivatives of fields even when $\mathcal{L}_{\rm int}$ does.

In the interaction picture, fields have the same time dependence as that of free fields, and thus they can be expanded using the same normal-mode functions as before with creation and annihilation operators that do not depend on time. To see this more clearly, let's work it out for the H-$f\bar{f}$ Lagrangian (5.14). We have imposed the commutation relations in the Heisenberg picture; we note, however, that the commutation relations are invariant under change of pictures:

$$\begin{aligned}[A_H, B_H] = C_H \;\to\; V \times \quad & \Big(\overbrace{A_H B_H}^{V^\dagger V} - \overbrace{B_H A_H}^{V^\dagger V} = C_H \Big) \quad \times V^\dagger \\ \to\; [A_S, B_S] = C_S\,, \quad & \stackrel{\text{similarly}}{\longrightarrow} [A_I, B_I] = C_I\,. \end{aligned} \tag{5.37}$$

Similarly, anticommutation relations are also invariant under change of pictures. Thus, for example, the commutation relations for the scalar field in the interaction picture are

$$[\phi_I(t,\vec{x})\,, \pi_I(t,\vec{x}')] = i\delta^3(\vec{x}-\vec{x}')\,, \quad \text{all others} = 0\,. \tag{5.38}$$

The free-field Hamiltonian in the interaction picture has the same form as in the Heisenberg picture as seen in (5.36):

$$H_I^0 \equiv \int d^3x \mathcal{H}_I^0\,, \quad \mathcal{H}_I^0 = \frac{1}{2}(\pi_I^2 + (\vec{\nabla}\phi_I)^2 + m^2\phi_I^2)\,. \tag{5.39}$$

Then, following exactly the same steps as in the case of free fields, the commutation relations and the equation of motion $\dot{O}_I = i[H_I^0\,, O_I]$ leads to

$$\pi_I = \dot{\phi}_I\,, \quad (\partial^2 + m^2)\phi_I = 0\,. \tag{5.40}$$

Thus, it satisfies the free field Klein-Gordon equation, and again following exactly the same procedure as before, the field ϕ_I can be momentum-expanded as

$$\phi_I(x) = \sum_{\vec{p}} (a_{\vec{p}I} e_{\vec{p}}(x) + a_{\vec{p}I}^\dagger e_{\vec{p}}^*(x))\,, \tag{5.41}$$

where $a_{\vec{p}}$'s are constants of motion, $e_{\vec{p}}(x)$ is the same normal-mode function as before, and the commutation relations among fields lead to those among $a_{\vec{p}}$'s:

$$[a_{\vec{p}I}\,, a_{\vec{p}'I}^\dagger] = \delta_{\vec{p},\vec{p}'}\,, \quad [a_{\vec{p}I}\,, a_{\vec{p}'I}] = [a_{\vec{p}I}^\dagger\,, a_{\vec{p}'I}^\dagger] = 0\,. \tag{5.42}$$

We then define the lowest energy state as the vacuum $|0\rangle$ and identify $a_{\vec{p}}^\dagger|0\rangle$ as the state where there is one scalar particle with momentum $\vec{p}$ in the entire universe. The

situation is the same for fermion fields. Note that the corresponding commutation relations among a's in other pictures also hold since commutators are 'invariant' as we have seen.

The advantage of the interaction picture is that the operator fields carry the rapid phase oscillations of the type $e^{-ip\cdot x}$ which do not change the physical quantities such as particle type, energies and momenta, spins, etc., and the states evolve much slower representing changes in physical contents. For the rest of the chapter, we will stay in the interaction picture and drop the subscript I. The differential equation for states

$$|\dot{a}\rangle = -ih(t)|a\rangle\,, \tag{5.43}$$

can then be solved for a small h. Suppose we have an initial state $|i\rangle$ at $t = 0$. Then, the change in a small time dt is $-ih(t)dt\,|i\rangle$, and as long as the sum total of change is much smaller than $|i\rangle$, it will linearly add up over some finite time T (even though h itself in general oscillates rapidly covering many periods in the duration T):

$$|a(T)\rangle \approx |i\rangle - i\Big(\int_0^T h(t)dt\Big)|i\rangle\,, \tag{5.44}$$

Assuming that initial and final states are orthogonal and normalized as

$$\langle i|i\rangle = 1\,, \quad \langle f|f\rangle = 1\,, \tag{5.45}$$

the amplitude to find a given state $|f\rangle$ at time T is given by

$$\langle f|a(T)\rangle = -i\langle f|\int_0^T h(t)dt\,|i\rangle = -i\int_0^T dt\int d^3x\langle f|\mathcal{H}_{\rm int}(x)|i\rangle\,, \tag{5.46}$$

where we have used $h(t) \equiv \int d^3x\mathcal{H}_{\rm int}(x)$. This is the first order transition matrix element. In order to find the answer to all orders, we have to proceed more systematically.

U and S operators

Each of the basis states of the Hilbert space at time t_0, $a^\dagger_{\vec{p}}|0\rangle$, $a^\dagger_{\vec{p},\vec{s}}b^\dagger_{\vec{p}',\vec{s}'}|0\rangle$ etc., will evolve according to $|\dot{a}\rangle = -ih|a\rangle$, and at a later time t they will become some linear combinations of the original basis states with some complex coefficients. These sets of coefficients form a gigantic matrix and define an operator which transforms any state at time t_0 to a state at a later time t:

$$|a(t)\rangle = U(t,t_0)|a(t_0)\rangle\,, \tag{5.47}$$

which is sometimes called the evolution operator, or the U operator (or the U matrix). Clearly, $U(t_0,t_0)$ is the identity

$$U(t_0,t_0) = I\,. \tag{5.48}$$

Substituting $|a(t)\rangle = U(t,t_0)|a(t_0)\rangle$ in $|\dot{a}\rangle = -ih|a\rangle$,

$$\dot{U}(t,t_0)|a(t_0)\rangle = -ihU(t,t_0)|a(t_0)\rangle\,. \tag{5.49}$$

Since this holds for any state $|a(t_0)\rangle$, we have an operator equation for $U(t,t_0)$:

$$\boxed{\dot{U}(t,t_0) = -ihU(t,t_0)}\,. \tag{5.50}$$

At t_0, $U(t_0,t_0) = I$ is obviously unitary: $U^\dagger(t_0,t_0)U(t_0,t_0) = 1$. Taking the hermitian conjugate of above, and using $h^\dagger = h$

$$\dot{U}^\dagger = iU^\dagger h^\dagger = iU^\dagger h\,. \tag{5.51}$$

Then, $U^\dagger U$ is seen to be constant of motion:

$$\frac{d}{dt}(U^\dagger U) = \underbrace{\dot{U}^\dagger}_{iU^\dagger h}\,U + U^\dagger\,\underbrace{\dot{U}}_{-ihU} = 0\,. \tag{5.52}$$

Thus, $U^\dagger U = 1$ at any time; namely, $U(t,t_0)$ is always unitary.

The S operator (or S matrix) is defined by taking the limit $t_0 \to -\infty$ and $t \to \infty$:

$$\boxed{S \stackrel{\text{def}}{\equiv} \lim_{\substack{t_0\to-\infty\\ t\to\infty}} U(t,t_0)}\,; \tag{5.53}$$

namely, it evolves the 'initial states' to the 'final states'. By 'infinite time', we actually mean some time duration T long enough to cover many oscillations of fields. In fact, we will later find that transition probabilities calculated perturbatively are proportional to T, and if T is truly taken to be infinity, the transition probabilities will diverge or linear approximation will break down.

Now, since U is unitary, so is S:

$$S^\dagger S = I\,. \tag{5.54}$$

As an initial state $|i\rangle$ will evolve to $S|i\rangle$ at $t = \infty$, the transition amplitude $i \to f$ is given by $\langle f|S|i\rangle$ and the probability to find the final state in $|f\rangle$ is then $|\langle f|S|i\rangle|^2$. The unitarity of S then reads

$$\langle i| \times \overbrace{S^\dagger S = I}^{\sum_f |f\rangle\langle f|} \times |i\rangle \quad \to \quad \sum_f |\langle f|S|i\rangle|^2 = 1\,, \tag{5.55}$$

where the sum is over all possible final states that are assumed to form a complete orthonormal set. Thus, the unitarity of S means that probability is conserved.

The next step is to solve the differential equation $\dot{U}(t,t_0) = -ihU(t,t_0)$ for a given interaction h. Our goal is to express U, and thus S, as a perturbation series in powers of h. In doing so, we have to be careful about the operator nature of h. Recall the definition of differentiation and integration of a matrix function:

$$\frac{dA(t)}{dt} \equiv \frac{A(t+dt) - A(t)}{dt}\,, \tag{5.56}$$

and

$$\int_{t_0}^{t_1} dt A(t) \equiv \sum_{n=1}^{N} A(t_0 + ndt)dt \quad (t_1 - t_0 \equiv Ndt)\,. \tag{5.57}$$

Strictly following these definitions, we see that

$$\begin{aligned} \frac{d}{dt}\int_{t_0}^{t} dt' A(t') \;&\equiv\; \frac{1}{dt}\Big(\int_{t_0}^{t+dt} dt' A(t') - \int_{t_0}^{t} dt' A(t')\Big) \\ [\text{by (5.57)}\,] \;&=\; \frac{1}{dt}\,A(t)dt = A(t)\,; \\ &\to \boxed{\frac{d}{dt}\int_{t_0}^{t} dt' A(t') = A(t)}\,. \end{aligned} \tag{5.58}$$

Then, *if* in general

$$\frac{d}{dt}e^{A(t)} = \dot{A}(t)e^{A(t)} \qquad (?)\,, \tag{5.59}$$

the solution of $\dot{U} = -ihU$ would be given by

$$U(t,t_0) = e^{-i\int_{t_0}^{t} dt' h(t')}\,, \tag{5.60}$$

as can be readily verified using (5.58). The relation (5.59), however, does not hold unless $[A, \dot{A}] = 0$:

$$\begin{aligned} \Big(\frac{d}{dt}e^{A(t)}\Big)dt \;&=\; \underbrace{e^{A(t+dt)}}_{e^{(A(t)+\dot{A}(t)dt)}\ \text{by (5.56)}} - e^{A(t)} \\ (\text{if } [A,\dot{A}] = 0\ ) \;&=\; e^{A(t)}e^{\dot{A}(t)dt} - e^{A(t)} \quad [\text{ used (1.108) }] \\ &=\; e^{A(t)}\Big(\underbrace{e^{\dot{A}(t)dt} - 1}_{\dot{A}(t)dt}\Big) \\ &=\; \dot{A}(t)e^{A(t)}dt\,. \end{aligned} \tag{5.61}$$

Earlier, we have used $(\partial/\partial x)e^{xA} = Ae^{xA}$ for a constant operator A which is consistent with the above observation since

$$[xA, \frac{d}{dx}(xA)] = [xA, A] = 0\,. \tag{5.62}$$

In general, however, the condition is not satisfied as can be seen in $[\phi(t,\vec{x}),\dot{\phi}(t,\vec{x}')] \neq 0$. We will now show by direct substitution that the solution is given by

$$\begin{aligned} U(t,t_0) = 1 \;&+\; (-i)\int_{t_0}^{t} dt_1 h(t_1) + \ldots \\ &+\; (-i)^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n \, h(t_1)h(t_2)\cdots h(t_n) \;+\ldots \end{aligned} \tag{5.63}$$

Taking the time derivative of the above and using (5.58),

$$\dot{U}(t,t_0) = (-i)h(t) + \ldots + (-i)^n \int_{t_0}^{t} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n \, h(t)h(t_2)\cdots h(t_n) \;+\ldots \tag{5.64}$$

Taking $h(t)$ in the integrand out in front and relabeling $(t_i \to t_{i-1})$,

$$\begin{aligned} \dot{U}(t,t_0) \;&=\; -ih(t)\Big(1 + \ldots + (-i)^{n-1} \int_{t_0}^{t} dt_1 \cdots \int_{t_0}^{t_{n-2}} dt_{n-1}\, h(t_1)\cdots h(t_{n-1}) + \ldots\Big) \\ &=\; -ih\, U(t,t_0)\,, \end{aligned} \tag{5.65}$$

which shows that the series (5.63) is indeed a solution of $\dot{U} = -ihU$.

It will be convenient later if we express the solution using the same integration range (t_0,t) for all integrals. To do so, we note

$$I_n \stackrel{\text{def}}{\equiv} \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n \, h(t_1)\cdots h(t_n) = \int\limits_{V_n(t_1>\cdots>t_n)} dt_1 \cdots dt_n \, h(t_1)\cdots h(t_n)\,, \tag{5.66}$$

where V_n is the n-dimensional cube defined by $t_0 < t_i < t$ $(i = 1,\ldots,n)$, and the integration range $V_n(t_1 > \cdots > t_n)$ is its sub-volume limited to $(t_1 > \cdots > t_n)$. Relabeling $(t_1,\ldots,t_n) \to (t_{i_1},\ldots,t_{i_n})$, where $(i_1,\ldots,i_n)$ is any permutation of $(1,\ldots,n)$,

$$\begin{aligned} I_n \;&=\; \int\limits_{V_n(t_{i_1}>\ldots>t_{i_n})} dt_{i_1}\cdots dt_{i_n}\, h(t_{i_1})\cdots h(t_{i_n}) \\ &=\; \int\limits_{V_n(t_{i_1}>\ldots>t_{i_n})} dt_1 \cdots dt_n \, T\Big(h(t_1)\cdots h(t_n)\Big)\,, \end{aligned} \tag{5.67}$$

where we have defined *time-ordered product* by reordering of operators in the descending order of times:

$$T\Big(A_1(t_1)\cdots A_n(t_n)\Big) \stackrel{\text{def}}{\equiv} A_{i_1}(t_{i_1})\cdots A_{i_n}(t_{i_n}) \quad (t_{i_1} \geq \cdots \geq t_{i_n})\,, \tag{5.68}$$

where $A_i(t)$'s could in general be different functions of time. For completeness, operators with the same time are defined to keep the original order. This procedure

is understood to be a simple re-ordering of operators that are functions of time as they are written, and one should not redefine the functions before the time-ordering is done. For example,

$$A(1.5)B(2.0) = A'(2.5)B(2.0) \quad \text{if } A(t) \stackrel{\text{def}}{\equiv} A'(t+1)\,, \tag{5.69}$$

but

$$\begin{array}{rcll} T(A(1.5)B(2.0)) & = & B(2.0)A(1.5) & \text{flips the ordering,} \\ T(A'(2.5)B(2.0)) & = & A'(2.5)B(2.0) & \text{does not}\,, \end{array} \tag{5.70}$$

and the results are in general different.

In the case at hand, we have $A_i(t) = h(t)$ for all i. Noting that sum of the regions $V_n(t_{i_1} > \ldots > t_{i_n})$ with all possible permutations ($n\,!$ of them) is just V_n, and that for each permutation, I_n is expressed as (5.67) with the *same integrand* $T(A_1(t_1)\cdots A_n(t_n))$, we have

$$\begin{aligned} n\,! I_n &= \sum_{\substack{\text{all} \\ \text{perm.}}} \int\limits_{V_n(t_{i_1}>\ldots>t_{i_n})} dt_1 \cdots dt_n\, T\Big(h(t_1)\cdots h(t_n)\Big) \\ &= \int_{V_n} dt_1 \cdots dt_n\, T\Big(h(t_1)\cdots h(t_n)\Big)\,, \\ \to\ I_n &= \frac{1}{n\,!}\int_{t_0}^{t} dt_1 \cdots \int_{t_0}^{t} dt_n\, T\Big(h(t_1)\cdots h(t_n)\Big)\,. \end{aligned} \tag{5.71}$$

Thus, the solution (5.63) is now written as

$$U(t,t_0) = 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n\,!} \int_{t_0}^{t} dt_1 \cdots \int_{t_0}^{t} dt_n\, T\Big(h(t_1)\cdots h(t_n)\Big)\,. \tag{5.72}$$

Let's compare this with the 'wrong' solution (5.60) which can be expanded as

$$\begin{aligned} e^{-i\int_{t_0}^{t} h(t')dt'} &= 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n\,!} \left(\int_{t_0}^{t} h(t')dt' \right)^n \\ &= 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n\,!} \int_{t_0}^{t} dt_1 \cdots \int_{t_0}^{t} dt_n\, h(t_1)\cdots h(t_n)\,. \end{aligned} \tag{5.73}$$

We see that the only difference is the time-ordering on each term.

The S matrix is then obtained by taking the limit $t \to \infty$ and $t_0 \to -\infty$.

$$\boxed{S = 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n\,!} \int_{-\infty}^{\infty} dt_1 \cdots \int_{-\infty}^{\infty} dt_n\, T\Big(h(t_1)\cdots h(t_n)\Big)}\,, \tag{5.74}$$

which is known as the Dyson series. The advantage of such an expansion is that often the interaction term is proportional to a small coupling constant and the higher-order terms become progressively insignificant.

Incidentally, the time-ordered product can be defined for cases where fermion fields are included:

$$\boxed{T(A_1(t_1)\cdots A_n(t_n)) \stackrel{\text{def}}{\equiv} s_P A_{i_1}(t_{i_1})\cdots A_{i_n}(t_{i_n}) \quad (t_{i_1}\geq\cdots\geq t_{i_n})}\,, \tag{5.75}$$

where $A_i(t)$ are time-dependent operators and s_P is $+1(-1)$ if the number of swaps of fermion operators needed for the re-ordering is even (odd). With this definition, the difference between before and after the time-ordering becomes a c-number as in the case of normal ordering. In deriving the Dyson series, we needed $s_P = +1$ in (5.67) which is actually the case even if fermion fields are involved since fermion fields always appear in pairs in the interaction term h.

5.3 Evaluation of the decay rate $H \to f\bar{f}$

Taking only the first-order term in the Dyson series,

$$S = -i\int_{-\infty}^{\infty} dt\, h(t) = -i\int_{-\infty}^{\infty} dt \int d^3x\, \mathcal{H}_{\rm int}\,, \tag{5.76}$$

which is the same as the earlier result (5.46) if T is taken to be large ('infinity'). Assuming that there is no time derivatives in the interaction term, we have $\mathcal{H}_{\rm int} = -\mathcal{L}_{\rm int}$ (5.18), and

$$\boxed{S = i\int d^4x\, \mathcal{L}_{\rm int}} \quad \text{(first order)}\,. \tag{5.77}$$

In our case, $\mathcal{L}_{\rm int} = \lambda\phi\,\bar{\psi}\psi$ (5.14), where normal ordering is implicit, and thus,

$$S_{fi} = \langle f|S|i\rangle = i\lambda\int d^4x\, \langle f|\phi\,\bar{\psi}\psi|i\rangle\,. \tag{5.78}$$

Let's define the spins and 4-momenta of the particles involved as

$$\begin{array}{lll} H: & P = (P^0, \vec{P})\,, & P^0 = \sqrt{\vec{P}^{\,2} + M^2}\,, \\ f: \ \vec{s}_1\,, & p_1 = (p_1^0\,, \vec{p}_1)\,, & p_1^0 = \sqrt{\vec{p}_1^{\,2} + m^2}\,, \\ \bar{f}: \ \vec{s}_2\,, & p_2 = (p_2^0\,, \vec{p}_2)\,, & p_2^0 = \sqrt{\vec{p}_2^{\,2} + m^2}\,. \end{array} \tag{5.79}$$

Then, the initial and final states are

$$\begin{aligned} |i\rangle &= a^\dagger_{\vec{P}}|0\rangle\,, \\ |f\rangle &= a^\dagger_{\vec{p}_1,\vec{s}_1} b^\dagger_{\vec{p}_2,\vec{s}_2}|0\rangle\,, \end{aligned} \tag{5.80}$$

where if annihilation or creation operators have only a momentum subscript, they are understood to be for H, and if they have momentum and spin subscripts, then they are for f or $\bar{f}$.

We will now use momentum expansions of fields to write the matrix element S_{fi} in terms of creation and annihilation operators. Since S_{fi} is just a number, all creation and annihilation operators should disappear in the end and we will be left with numbers such as $e_{\vec{P}}$, $\bar{f}_{\vec{p},\vec{s}}$, and $g_{\vec{p},\vec{s}}$. With momentum expansions, we have

$$\begin{aligned}\langle f|\phi\,\bar{\psi}\psi|i\rangle &= \overbrace{\langle 0|b_{\vec{p}_2,\vec{s}_2}a_{\vec{p}_1,\vec{s}_1}}^{\langle f|} \\ &\times \overbrace{\sum_{\vec{q}}(a_{\vec{q}}e_{\vec{q}}+a^\dagger_{\vec{q}}e^*_{\vec{q}})}^{\phi}\overbrace{\sum_{\vec{p},\vec{s}}(a^\dagger_{\vec{p},\vec{s}}\bar{f}_{\vec{p},\vec{s}}+b_{\vec{p},\vec{s}}\bar{g}_{\vec{p},\vec{s}})}^{\bar{\psi}}\overbrace{\sum_{\vec{p}',\vec{s}'}(a_{\vec{p}',\vec{s}'}f_{\vec{p}',\vec{s}'}+b^\dagger_{\vec{p}',\vec{s}'}g_{\vec{p}',\vec{s}'})}^{\psi}\overbrace{a^\dagger_{\vec{P}}|0\rangle}^{|i\rangle}\,. \quad (5.81)\end{aligned}$$

There are many terms in $\phi\,\bar{\psi}\psi$, but there is only one term that survives - the term with matching annihilation and creation operators that counter those in $|i\rangle$ and $\langle f|$. First, all terms that contain $a^\dagger_{\vec{q}}$ vanish since they commute with fermion operators on their left and face the vacuum $\langle 0|$. Also, all terms with $a_{\vec{q}}$ ($\vec{q}\neq\vec{P}$) vanish since they commute with all operators on their right to face $|0\rangle$, leaving $a_{\vec{P}}e_{\vec{P}}$ as the only term in ϕ that survives. Similarly, the only term that survives in ψ is $b^\dagger_{\vec{p}_2,\vec{s}_2}g_{\vec{p}_2,\vec{s}_2}$ and the only term that survives in $\bar{\psi}$ is $a^\dagger_{\vec{p}_1,\vec{s}_1}\bar{f}_{\vec{p}_1,\vec{s}_1}$. One subtlety may be the vanishing of terms with $b_{\vec{p},\vec{s}}$ in $\bar{\psi}$ which need to go beyond $b^\dagger_{\vec{p}',\vec{s}'}$ of ψ to face $|0\rangle$. Actually, $b_{\vec{p},\vec{s}}$'s are already to the right of $b^\dagger_{\vec{p}',\vec{s}'}$'s because of the implicit normal ordering. Even if normal ordering is not assumed, when $b_{\vec{p},\vec{s}}$ is moved past $b^\dagger_{\vec{p}',\vec{s}'}$ it leaves behind $\delta_{\vec{p},\vec{p}'}\delta_{\vec{s},\vec{s}'}$ and such term vanishes anyway since the annihilation operator $a_{\vec{p}_1,\vec{s}_1}$ in $\langle f|$ will face $|0\rangle$ to the right.

We thus have

$$\begin{aligned}\langle f|\phi\,\bar{\psi}\psi|i\rangle &= \langle 0|b_{\vec{p}_2,\vec{s}_2}a_{\vec{p}_1,\vec{s}_1}(a_{\vec{P}}e_{\vec{P}})(a^\dagger_{\vec{p}_1,\vec{s}_1}\bar{f}_{\vec{p}_1,\vec{s}_1})(b^\dagger_{\vec{p}_2,\vec{s}_2}g_{\vec{p}_2,\vec{s}_2})a^\dagger_{\vec{P}}|0\rangle \\ &= \underbrace{\langle 0|b_{\vec{p}_2,\vec{s}_2}a_{\vec{p}_1,\vec{s}_1}a_{\vec{P}}\,a^\dagger_{\vec{p}_1,\vec{s}_1}b^\dagger_{\vec{p}_2,\vec{s}_2}a^\dagger_{\vec{P}}|0\rangle}_{1:\ \text{norm of } a^\dagger_{\vec{p}_1,\vec{s}_1}b^\dagger_{\vec{p}_2,\vec{s}_2}a^\dagger_{\vec{P}}|0\rangle}\,e_{\vec{P}}(\bar{f}_{\vec{p}_1,\vec{s}_1}g_{\vec{p}_2,\vec{s}_2}) \\ &= e_{\vec{P}}(\bar{f}_{\vec{p}_1,\vec{s}_1}g_{\vec{p}_2,\vec{s}_2})\,, \quad (5.82)\end{aligned}$$

which is now explicitly just a number (a function of x). In the above, the fact that the vacuum expectation value of the operators came out to be 1 and not -1 is a coincidence. If we had defined the final state as $b^\dagger a^\dagger|0\rangle$ rather than $a^\dagger b^\dagger|0\rangle$, for example, it would have been -1. Such sign, however, is an overall sign of the amplitude, and does not affect the rate of the interaction. When there are more than

one terms that contribute to the interaction, one needs to pay attention to the relative signs of the amplitudes.

The transition amplitude S_{fi} is then

$$\begin{aligned} S_{fi} &= i\lambda \int d^4x\, e_{\vec{P}}(x)\, \bar{f}_{\vec{p}_1,\vec{s}_1}(x) g_{\vec{p}_2,\vec{s}_2}(x) \\ &= i\lambda \int d^4x \frac{e^{-iP\cdot x}}{\sqrt{2P^0V}} \frac{\bar{u}_{\vec{p}_1,\vec{s}_1} e^{ip_1\cdot x}}{\sqrt{2p_1^0V}} \frac{v_{\vec{p}_2,\vec{s}_2} e^{ip_2\cdot x}}{\sqrt{2p_2^0V}} \\ &= \frac{i\lambda\, \bar{u}_{\vec{p}_1,\vec{s}_1} v_{\vec{p}_2,\vec{s}_2}}{\sqrt{(2P^0V)(2p_1^0V)(2p_2^0V)}} \underbrace{\int d^4x\, e^{i(p_1+p_2-P)\cdot x}}_{(2\pi)^4\delta^4(p_1+p_2-P)} . \end{aligned} \tag{5.83}$$

The four-dimensional delta function arose from the phase terms of the normal-mode functions, and indicates that the transition amplitude is zero unless the energy-momentum conservation $P = p_1 + p_2$ is satisfied.

Let's define the Lorentz-invariant matrix element $\mathcal{M}$ by

$$S_{fi} \equiv \frac{(2\pi)^4\delta^4\Big(p_1+p_2-P\Big)}{\sqrt{(2P^0V)(2p_1^0V)(2p_2^0V)}}\,\mathcal{M}\,, \tag{5.84}$$

where $\mathcal{M}$ is

$$\mathcal{M} = i\lambda\, \bar{u}_{\vec{p}_1,\vec{s}_1} v_{\vec{p}_2,\vec{s}_2}\,, \tag{5.85}$$

which is a scalar bilinear covariant and thus Lorentz-invariant. The probability to find the final state f is then the square of the amplitude:

$$|S_{fi}|^2 = \frac{\Big[(2\pi)^4\delta^4(p_1+p_2-P)\Big]^2}{(2P^0V)(2p_1^0V)(2p_2^0V)} |\mathcal{M}|^2\,. \tag{5.86}$$

What is this square of the delta function? This oddity came about because we pretended that we are integrating over infinite space and time even though we are actually dealing with finite space V and finite time T. Thus, let's recover the origin of the delta functions (for one of the two δ's) and write

$$\begin{aligned} \Big[(2\pi)^4\delta^4(p_1+p_2-P)\Big]^2 &= (2\pi)^4\delta^4(p_1+p_2-P) \underbrace{\int_T dt \int_V d^3x\, e^{i(p_1+p_2-P)\cdot x}}_{TV\,:\,(P=p_1+p_2)} \\ &= TV(2\pi)^4\delta^4(p_1+p_2-P)\,, \end{aligned} \tag{5.87}$$

where the delta function allowed us to set $P = p_1 + p_2$ in the integrand. Thus, the transition probability is

$$|S_{fi}|^2 = \frac{(2\pi)^4\delta^4(p_1+p_2-P)TV}{(2P^0V)(2p_1^0V)(2p_2^0V)} |\mathcal{M}|^2\,. \tag{5.88}$$

This is the probability to find the final state with particular discrete momenta $\vec{p}_1$ and $\vec{p}_2$.

What we usually need is the probability to find the final-state momenta in some ranges d^3p_1 and d^3p_2 which are small but still contain many discrete values of $\vec{p}_1$ and $\vec{p}_2$, which should be equal to the corresponding decay rate $d\Gamma$ times T. The number of grid points in the volume d^3p is d^3p/dv where $dv = (2\pi/L)^3 = (2\pi)^3/V$ is the volume per grid point. Since the volume d^3p is small, the transition probabilities in that volume are essentially the same. Thus, the summation can be replaced by a multiplication by the number of grid points:

$$\sum_{\vec{p}_i \in d^3p_i} \to \frac{V}{(2\pi)^3} d^3p_i \qquad (i = 1, 2)\,. \tag{5.89}$$

The transition probability $d\Gamma\, T$ is then obtained by summing over all final states with $\vec{p}_1 \in d^3p_1$ and $\vec{p}_2 \in d^3p_2$:

$$\begin{aligned} d\Gamma\, T &= \sum_{\vec{p}_1 \in d^3p_1} \sum_{\vec{p}_2 \in d^3p_2} |S_{fi}|^2 \\ &= \frac{V}{(2\pi)^3} d^3p_1 \, \frac{V}{(2\pi)^3} d^3p_2 \, \frac{(2\pi)^4 \delta^4(p_1 + p_2 - P) VT}{(2P^0V)(2p_1^0V)(2p_2^0V)} \, |\mathcal{M}|^2\,; \end{aligned} \tag{5.90}$$

namely, the decay rate is given by

$$d\Gamma = \frac{(2\pi)^4}{2P^0} \delta^4(p_1 + p_2 - P) \frac{d^3p_1}{(2\pi)^3 2p_1^0} \, \frac{d^3p_2}{(2\pi)^3 2p_2^0} \, |\mathcal{M}|^2\,, \tag{5.91}$$

or

$$\boxed{d\Gamma = \frac{(2\pi)^4}{2P^0} d\Phi_2 |\mathcal{M}|^2}\,, \tag{5.92}$$

with the 2-body Lorentz-invariant phase space $d\Phi_2$ defined as

$$\boxed{d\Phi_2 \stackrel{\text{def}}{\equiv} \delta^4(p_1 + p_2 - P) \frac{d^3p_1}{(2\pi)^3 2p_1^0} \frac{d^3p_2}{(2\pi)^3 2p_2^0}}\,. \tag{5.93}$$

In deriving this formula, we did not require that the mass of the two final-state particles are the same; thus, it is valid for the case the masses are different, in which case we have $p_i^0 \equiv \sqrt{\vec{p}_i^{\,2} + m_i^2}$ $(i = 1, 2)$.

The total decay rate is obtained by integrating over all final states ('phase space'):

$$\Gamma \equiv \frac{(2\pi)^4}{2P^0} \int d\Phi_2 |\mathcal{M}|^2. \tag{5.94}$$

That $d\Phi_2$ is Lorentz-invariant can be seen by the identity (4.292). What do we mean by the Lorentz-invariance of differential expressions? It means that if a Lorentz-invariant function is integrated with the differential piece $d\Phi_2$, then the resulting

quantity will have the same numerical value in any frame. Since $\mathcal{M}$ is Lorentz-invariant, when integrated over all phase-space, the result $\int d\Phi_n |\mathcal{M}|^2$ will be Lorentz-invariant. Thus, the above expression for Γ indicates that the decay rate is inversely proportional to the energy, which is just the time dilation effect due to boost.

We now perform the integration over the phase space in the rest frame of the parent particle. Since H is spinless, there is no preferred spacial direction, and thus the decay should be uniform over 4π steradians in the rest frame of H. Later, we will indeed see that $|\mathcal{M}|^2$ does not depend on direction of $\vec{p}_1$ or $\vec{p}_2$ in the C.M. frame. Using the identity (4.292), we can change d^3p_1 to d^4p_1 and eliminate the 4-dimentional delta function by integrating over p_1. Keeping the two final state masses to be different and denoting the integrand as X,

$$\begin{aligned}
\int d\Phi_2 X &= \int \delta^4(p_1+p_2-P)\,\underbrace{\frac{d^3p_1}{(2\pi)^3 2p_1^0}}\,\frac{d^3p_2}{(2\pi)^3 2p_2^0}\,X \\
&\qquad\qquad \delta(p_1^2-m_1^2)\theta(p_1^0)\frac{d^4p_1}{(2\pi)^3} \;\text{ by (4.292)} \\
&= \int \delta(p_1^2-m_1^2)\theta(p_1^0)\frac{d^3p_2}{(2\pi)^6 2p_2^0}\,X\Big|_{p_1=P-p_2} \\
&= \int \delta((P-p_2)^2-m_1^2)\theta(P^0-p_2^0)\frac{d^3p_2}{(2\pi)^6 2p_2^0}\,X \\
&= \int \delta((M^2+m_2^2-2Mp_2^0)-m_1^2)\theta(M-p_2^0)\frac{d^3p_2}{(2\pi)^6 2p_2^0}\,X\,, \qquad (5.95)
\end{aligned}$$

where in the last step we have used $P=(M,\vec{0})$. Since the integrand does not depend on the direction of $\vec{p}_2$, we can write

$$d^3p_2 = 4\pi \breve{p}_2^2\, d\breve{p}_2 = 4\pi \breve{p}_2\, p_2^0 dp_2^0 \qquad (\breve{p}_2 \equiv |\vec{p}_2|)\,, \qquad (5.96)$$

where we have denoted $|\vec{p}|$ as $\breve{p}$ in order to simplify the notation during the computation, and used

$$p_2^{0\,2} = \breve{p}_2^2 + m_2^2 \quad\to\quad p_2^0 dp_2^0 = \breve{p}_2 d\breve{p}_2\,. \qquad (5.97)$$

We can then complete the integral using the property of the delta function (4.114):

$$\begin{aligned}
\int d\Phi_2 X &= \int \underbrace{\delta(M^2+m_2^2-2Mp_2^0-m_1^2)}\;\theta(M-p_2^0)\,\underbrace{\frac{4\pi\breve{p}_2\,p_2^0 dp_2^0}{(2\pi)^6 2p_2^0}}\,X \\
&\qquad \frac{1}{2M}\delta\Big(p_2^0-\frac{M^2+m_2^2-m_1^2}{2M}\Big) \text{ by (4.114)} \quad \frac{\breve{p}_2 dp_2^0}{(2\pi)^5} \\
&= \frac{\breve{p}_2}{(2\pi)^5 2M}X \qquad (5.98)
\end{aligned}$$

where $\breve{p}_2 = \sqrt{p_2^{0\,2} - m_2^2}$ with $p_2^0 = (M^2 + m_2^2 - m_1^2)/2M$, and $\theta(M - p_2^0) = +1$ since p_2^0 as given is always smaller than M (the energy of the daughter particle is always smaller than that of the parent). Thus, the total decay rate in the C.M. system for a uniform decay is given by (with $X = (2\pi)^4 |\mathcal{M}|^2/2P^0$)

$$\boxed{\Gamma = \frac{|\vec{p}|}{8\pi M^2} \, |\mathcal{M}|^2}, \tag{5.99}$$

where we have defined $|\vec{p}_1| = |\vec{p}_2| \equiv |\vec{p}|$.

We now move on to the evaluation of $|\mathcal{M}|^2$. Since we know the explicit forms of u, v spinors, in principle we can calculate $\mathcal{M} = i\lambda \, \bar{u}_{\vec{p}_1,\vec{s}_1} v_{\vec{p}_2,\vec{s}_2}$ for given momenta and spins. There is, however, much quicker and also representation-independent way. Suppose we are not measuring the spins of the final state. Then, decay rates to all possible spin states (there are 4 of them for given $\vec{p}_1$ and $\vec{p}_2$) should be incoherently summed:

$$\Gamma = \frac{|\vec{p}|}{8\pi M^2} \sum_{\vec{s}_1,\vec{s}_2} |\mathcal{M}|^2 \, . \tag{5.100}$$

As we will see, when the matrix element squared is summed over the spins, we obtain a trace of certain 4×4 matrix in the spinor space which can be evaluated using representation-independent techniques:

$$\begin{aligned}
\sum_{\vec{s}_1,\vec{s}_2} |\mathcal{M}|^2 &= \lambda^2 \sum_{\vec{s}_1,\vec{s}_2} \underbrace{(\bar{u}_{\vec{p}_1,\vec{s}_1} v_{\vec{p}_2,\vec{s}_2})^*}_{\bar{v}_{\vec{p}_2,\vec{s}_2} u_{\vec{p}_1,\vec{s}_1}} (\bar{u}_{\vec{p}_1,\vec{s}_1} v_{\vec{p}_2,\vec{s}_2}) \\
&= \lambda^2 \sum_{\vec{s}_2} \bar{v}_{\vec{p}_2,\vec{s}_2} \underbrace{\Big(\sum_{\vec{s}_1} u_{\vec{p}_1,\vec{s}_1} \bar{u}_{\vec{p}_1,\vec{s}_1}\Big)}_{(\not{p}_1 + m)} v_{\vec{p}_2,\vec{s}_2} \\
\text{(write out components)} \quad &= \lambda^2 \sum_{\vec{s}_2} (\bar{v}_{\vec{p}_2,\vec{s}_2})_n (\not{p}_1 + m)_{nm} (v_{\vec{p}_2,\vec{s}_2})_m \\
&= \lambda^2 \underbrace{\sum_{\vec{s}_2} (v_{\vec{p}_2,\vec{s}_2})_m (\bar{v}_{\vec{p}_2,\vec{s}_2})_n}_{(\not{p}_2 - m)_{mn}} (\not{p}_1 + m)_{nm} \\
&= \lambda^2 \Big[(\not{p}_2 - m)(\not{p}_1 + m) \Big]_{mm} \\
&= \lambda^2 \mathrm{Tr} \Big[(\not{p}_2 - m)(\not{p}_1 + m) \Big] \, .
\end{aligned} \tag{5.101}$$

In general, the spin sum over the form $\bar{u}_{\vec{p},\vec{s}} M u_{\vec{p},\vec{s}}$ or $\bar{v}_{\vec{p},\vec{s}} M v_{\vec{p},\vec{s}}$, where M is any 4×4 matrix, results in a trace: following the last four lines above with $\not{p}_1 + m \to M$, we obtain

$$\begin{aligned}
\sum_{\vec{s}} \bar{u}_{\vec{p},\vec{s}} M u_{\vec{p},\vec{s}} &= \mathrm{Tr} \left[(\not{p} + m) M \right] , \\
\sum_{\vec{s}} \bar{v}_{\vec{p},\vec{s}} M v_{\vec{p},\vec{s}} &= \mathrm{Tr} \left[(\not{p} - m) M \right] .
\end{aligned} \tag{5.102}$$

The expression (5.101) can be readily evaluated using the trace theorems (Exercise 5.1). Since traces of an odd number of gamma matrices are zero,

$$\begin{aligned}\sum_{\vec{s}_1,\vec{s}_2}|\mathcal{M}|^2 &= \lambda^2\big[\underbrace{\mathrm{Tr}\not{p}_2\not{p}_1}_{4p_2\cdot p_1} - m^2\underbrace{\mathrm{Tr}I}_{4}\big] \\ &= 4\lambda^2(p_2\cdot p_1 - m^2)\,. \end{aligned} \tag{5.103}$$

In the Higgs C.M. frame, we have $p_1 \equiv (E,\vec{p})$ and $p_2 = (E,-\vec{p})$; thus,

$$p_2\cdot p_1 - m^2 = E^2 + \vec{p}^{\,2} - m^2 = 2\vec{p}^{\,2}\,. \tag{5.104}$$

Then, using this in (5.100), we finally obtain

$$\Gamma = \frac{\lambda^2|\vec{p}\,|^3}{\pi M^2}\,. \tag{5.105}$$

We can now evaluate the actual decay rate of the Higgs decay to a bottom quark and its antiquark. The mass of bottom quark m_b is approximately 4.5 GeV while the Higgs mass is measured to be 126 GeV. Then the momentum of the bottom quark $|\vec{p}|$ is ~ 62.8 GeV/c. In the standard model, the coupling constant λ is given by

$$\lambda = \frac{gm_b}{2m_W}\,, \tag{5.106}$$

where $g \sim 0.65$ is a universal coupling constant of the standard model and $m_W \sim 80$ GeV is the W boson mass. The numerical value of λ is thus ~ 0.018. There is one more complication; namely, quarks come in three colors, and each one is created with the same decay rate (5.105) thus increasing the rate by factor three. Putting all together, the decay rate $H \to b\bar{b}$ would be

$$\Gamma(H\to b\bar{b}) = \frac{3\lambda^2|\vec{p}\,|^3}{\pi M^2} \sim 5.0\,\mathrm{MeV}\,. \tag{5.107}$$

Let's reflect upon the procedures we just followed. We started from the first-order expression of the S operator and sandwiched it between the initial and final states which led to a space-time integral of $\langle f|\mathcal{L}_{\mathrm{int}}|i\rangle$. When momentum expansions were used for the fields appearing in $\mathcal{L}_{\mathrm{int}}$, only the term that contained the creation and annihilation operators that exactly matched those of the initial and final states survived. In short, the interaction term had to annihilate the particles in the initial state and create the particles in the final state. More specifically, for the creation operator of the initial state $a^\dagger_{\vec{P}}$, the term had to contain the annihilation operator $a_{\vec{P}}$ and the associated normal-mode function $e_{\vec{P}}(x)$ survived into S_{fi}. Similarly, the fermion (antifermion) in the final state came with the annihilation operator $a_{\vec{p}_1,\vec{s}_1}$ ($b_{\vec{p}_2,\vec{s}_2}$) and the surviving term in the interaction had to contain the matching creation

operator $a^\dagger_{\vec{p}_1,\vec{s}_1}$ ($b^\dagger_{\vec{p}_2,\vec{s}_2}$) which contributed the normal-mode function $\bar{f}_{\vec{p}_1,\vec{s}_1}$ ($g_{\vec{p}_2,\vec{s}_2}$) to S_{fi}.

These contributions of the initial and final state particles to S_{fi} are valid for a general transition from an arbitrary number of initial state particles to an arbitrary number of final state particles. We did assume that the creation operators $a^\dagger_{\vec{p},\vec{s}}$ of fermion and the annihilation operators $b_{\vec{p},\vec{s}}$ of antifermion appear in the form $\bar{\psi}$ (and not as $\psi^\dagger$ which can be written as $\bar{\psi}\gamma^0$ in any case) as is the case for any bilinear covariant. These can be readily extended to scalars in the final state and fermions and antifermions in the initial state. Thus, the contributions to S_{fi} from the initial and final states are:

$$\begin{array}{ll}
e_{\vec{p}}(x) \text{ for a scalar in } |i\rangle & (a_{\vec{p}}\, e_{\vec{p}}(x) \text{ to match } a^\dagger_{\vec{p}}|0\rangle\,.) \\
e^*_{\vec{p}}(x) \text{ for a scalar in } |f\rangle & (a^\dagger_{\vec{p}}\, e^*_{\vec{p}}(x) \text{ to match } \langle 0|a_{\vec{p}}\,.) \\
f_{\vec{p},\vec{s}}(x) \text{ for a fermion in } |i\rangle & (a_{\vec{p},\vec{s}}\, f_{\vec{p},\vec{s}}(x) \text{ to match } a^\dagger_{\vec{p},\vec{s}}|0\rangle\,.) \\
\bar{f}_{\vec{p},\vec{s}}(x) \text{ for a fermion in } |f\rangle & (a^\dagger_{\vec{p},\vec{s}}\, \bar{f}_{\vec{p},\vec{s}}(x) \text{ to match } \langle 0|a_{\vec{p},\vec{s}}\,.) \\
\bar{g}_{\vec{p},\vec{s}}(x) \text{ for an antifermion in } |i\rangle & (b_{\vec{p},\vec{s}}\, \bar{g}_{\vec{p},\vec{s}}(x) \text{ to match } b^\dagger_{\vec{p},\vec{s}}|0\rangle\,.) \\
g_{\vec{p},\vec{s}}(x) \text{ for an antifermion in } |f\rangle & (b^\dagger_{\vec{p},\vec{s}}\, g_{\vec{p},\vec{s}}(x) \text{ to match } \langle 0|b_{\vec{p},\vec{s}}\,.)
\end{array} \tag{5.108}$$

In this list, we note that annilation operators, namely the particles in the initial state, are always associted with a positive energy oscillation $e^{-ip\cdot x}$, and creation operators, namely the particles in the final state, are always associated with a negative energy oscillation $e^{ip\cdot x}$, regardless of the type of particle. Thus, if the initial state momenta are $p_i (i = 1, \ldots, m)$ and the final state momenta are $p_f (f = 1, \ldots, n)$, upon integration over space-time, we will have a delta function corresponding to energy-momentum conservation $(2\pi)^4\delta^4(\sum_i p_i - \sum_f p_f)$. In a general transition $p_i (i = 1, \ldots, m)$ to $p_f (f = 1, \ldots, n)$, the delta function for the energy momentum conservation and the normalization factors $1/\sqrt{2p^0 V}$ can be separated out in S_{fi} and the rest defined as the Lorentz-invariant matrix element $\mathcal{M}$:

$$S_{fi} \equiv \frac{(2\pi)^4\delta^4\Big(\sum_i p_i - \sum_f p_f\Big)}{\sqrt{\prod_i (2p_i^0 V)\prod_f (2p_f^0 V)}}\,\mathcal{M}\,. \tag{5.109}$$

As we will see later, this form holds also when spin-1 particles are involved.

The contributions of the inital and final state particles to the Lorentz-invariant matrix element are then

	initial state	final state
scalar	1	1
fermion	$u_{\vec{p},\vec{s}}$	$\bar{u}_{\vec{p},\vec{s}}$
antifermion	$\bar{v}_{\vec{p},\vec{s}}$	$v_{\vec{p},\vec{s}}$

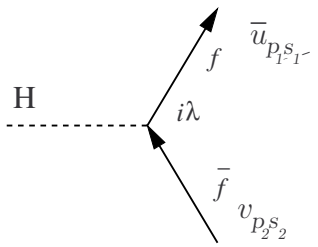
(5.110)

For a n-body decay $P \to p_1, \ldots, p_n$ with masses $M \to m_1, \ldots, m_n$, the same procedure as in the case of 2-body decay can be followed to obtain

$$\begin{aligned} d\Gamma &= \frac{(2\pi)^4}{2P^0} d\Phi_n |\mathcal{M}|^2 \\ d\Phi_n &\stackrel{\text{def}}{\equiv} \delta^4(\sum_f p_f - P) \prod_f \frac{d^3 p_f}{(2\pi)^3 2p_f^0} \end{aligned} \tag{5.111}$$

where $d\Phi_n$ is the Lorentz-invariant n-body phase space and p_i^0 is a function of $\vec{p}_i$ as usual: $p_i^0 \equiv \sqrt{\vec{p}_i^2 + m_i^2}$ $(i = 1, \ldots, n)$.

In the case of the decay $H \to f\bar{f}$, the u, v spinors are combined following the form of the interaction Lagrangian $\mathcal{L} = \lambda \phi \bar{\psi} \psi$ with the constant vertex factor given by i (which came from the Dyson series) times whatever the coupling constant is in $\mathcal{L}$. This can be graphically written as below, and one can immediately write down the Lorentz-invariant matrix element:



$$\mathcal{M} = i\lambda\, \bar{u}_{\vec{p}_1, \vec{s}_1} v_{\vec{p}_2, \vec{s}_2}\,. \tag{5.112}$$

The rules (5.110) together with the vertex factor are part of the calculational rules called the Feynman rules, and the diagram is called the Feynman diagram. In the diagram above, time flows from left to right, but this rule is not strictly followed.

Often the arrow of antifermion is reversed as in the figure above to emphasize that the spinor combination $\bar{u}_{\vec{p}_1, \vec{s}_1} v_{\vec{s}_2, \vec{s}_2}$ forms a bilinear covariant (or a general 'current'). In the framework of the non-quantized hole theory, one may interpret that the reversal of the arrow arises because an antifermion corresponds to a negative-energy fermion with momentum and spin that are opposite to those of the physical anti-fermion. In our framework of quantum fields, however, there is no need to resort to such negative energy states. The fermion and antifermion in the final state naturally formed a bilinear covariant because, in the momentum expansion of fields, creation operators of a fermion were associated with $\bar{u}$ spinors and the creation operators of antifermion with v spinors.

Exercise 5.1 *Traces of gamma matrices.*
Let's prove some of the most often used relations in actual calculation of rates. These traces show up when spin average is taken for fermions. Do NOT rely on any specific representation of γ matrices.

(*a*) *Prove that*

$$\mathrm{Tr}1 = 4\,, \quad \mathrm{Tr}(\not{a}\not{b}) = 4a\cdot b\,. \tag{5.113}$$

(*b*) *Prove the following trace relation that reduces number of* γ *matrixes in traces by two:*

$$\begin{aligned}\mathrm{Tr}(\not{a}_1\not{a}_2\cdots\not{a}_{2n}) &= \sum_{i=2}^{2n}(-1)^i a_1\cdot a_i \mathrm{Tr}(\not{a}_2\cdots\not{a}_{i-1}\not{a}_{i+1}\cdots\not{a}_{2n}) \\ &= a_1\cdot a_2\mathrm{Tr}(\not{a}_3\not{a}_4\cdots\not{a}_{2n}) - a_1\cdot a_3\mathrm{Tr}(\not{a}_2\not{a}_4\cdots\not{a}_{2n}) \\ &\quad +\cdots + a_1\cdot a_{2n}\mathrm{Tr}(\not{a}_2\not{a}_3\cdots\not{a}_{2n-1})\,. \end{aligned}\tag{5.114}$$

(*hint: Take the* γ *matrix on* a_1 *and shift it all the way over to the right using the anticommutation relation.*)
(*c*) *Using the result of* (*b*) *to show that*

$$\mathrm{Tr}(\not{a}\not{b}\not{c}\not{d}) = 4\Big[(a\cdot b)(c\cdot d) - (a\cdot c)(b\cdot d) + (a\cdot d)(b\cdot c)\Big]\,, \tag{5.115}$$

or equivalently

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta) = 4(g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\nu\alpha})\,. \tag{5.116}$$

(*d*) *Prove that the trace of a product of odd number of* γ *matrices* $(\gamma^\mu; \mu = 0,1,2,3)$ *is zero including the case of single* γ *matrix:*

$$\mathrm{Tr}(\gamma^{\mu_1}\ldots\gamma^{\mu_{2n+1}}) = 0 \quad (n = 0,1,2\ldots)\,. \tag{5.117}$$

This means that

$$\mathrm{Tr}(\not{a}_1\cdots\not{a}_{2n+1}) = 0 \quad (n = 0,1,2\ldots)\,. \tag{5.118}$$

(*hint: Multiply* ${\gamma_5}^2 = 1$ *at the end of the trace, and move one* γ_5 *to the left hand most in two ways: one using* $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ *and the other by shifting it using anticommutation rule of* γ *matrices.*)
(*e*) *Furthermore show that*

$$\mathrm{Tr}\gamma_5 = 0\,, \quad \mathrm{Tr}(\not{a}\not{b}\gamma_5) = 0\,. \tag{5.119}$$

(*f*) *Prove that*

$$\mathrm{Tr}(\gamma_5\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta) = 4i\,\epsilon^{\alpha\beta\gamma\delta}\,. \tag{5.120}$$

(*hint: note that* $\mathrm{Tr}(\gamma_5^2) = \mathrm{Tr}1 = 4$. *Replace one of the* γ_5*'s by* $i\gamma^0\gamma^1\gamma^2\gamma^3$.)
(*g*) *And finally,*

$$\mathrm{Tr}(\gamma^{\mu_1}\gamma^{\mu_2}\ldots\gamma^{\mu_{2n}}) = \mathrm{Tr}(\gamma^{\mu_{2n}}\ldots\gamma^{\mu_2}\gamma^{\mu_1}) \quad (\textit{reverse order})\,. \tag{5.121}$$

Exercise 5.2 *More on γ matrixes.*
Prove the following relations:

$$\begin{aligned} \gamma^\mu\gamma_\mu &= 4 \\ \gamma^\mu\gamma^\alpha\gamma_\mu &= -2\gamma^\alpha \\ \gamma^\mu\gamma^\alpha\gamma^\beta\gamma_\mu &= 4g^{\alpha\beta} \\ \gamma^\mu\gamma^\alpha\gamma^\beta\gamma^\delta\gamma_\mu &= -2\gamma^\delta\gamma^\beta\gamma^\alpha\,. \end{aligned} \tag{5.122}$$

[*comment: These relations lead to*

$$\begin{aligned} \gamma^\mu \not{a}\gamma_\mu &= -2\not{a} \\ \gamma^\mu \not{a}\not{b}\gamma_\mu &= 4a\cdot b \\ \gamma^\mu \not{a}\not{b}\not{c}\gamma_\mu &= -2\not{c}\not{b}\not{a}\,. \end{aligned} \tag{5.123}$$

Note that the index μ is summed in all of the above.]

5.4 Muon decay

The muon decay $\mu^- \to \nu_\mu e^- \bar{\nu}_e$, which proceeds with essentially 100% branching fraction, is an example of weak interaction which is mediated by W and Z vector bosons. All particles appearing in the initial and final states are fermions: μ^- is a charged fermion of mass $m = 0.106$ GeV, and ν_μ and $\bar{\nu}_e$ are the muon nutrino and electron antineutrino both assumed to be massless. The mass of the electron (~ 0.0005 GeV) is much smaller than that of the muon, and thus we will ignore the electron mass in the following calculation. For now we will assume that the interaction Lagrangian is given by

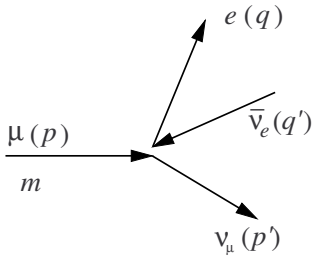
$$\mathcal{L}_{\rm int} = \frac{G_F}{\sqrt{2}}\Big(\bar{\psi}_{\nu_\mu}\gamma_\alpha(1-\gamma_5)\psi_\mu\Big)\Big(\bar{\psi}_e\gamma^\alpha(1-\gamma_5)\psi_{\nu_e}\Big) + h.c. \tag{5.124}$$

where ψ_μ is the muon field, ψ_e is the electron field, ψ_{ν_μ} and ψ_{ν_e} are the neutrino fields. The Lorentz index α is contracted. This form was first suggested by Fermi, and the constant G_F is called the Fermi coupling constant. Since $\dim(\mathcal{L}) = E^4$ and $\dim(\psi) = E^{3/2}$ (see p205), the dimension of the coupling constant should be $\dim(G_F) = E^{-2}$. This is actually a low-energy approximation of a more fundamental interaction involving W vector boson that we will study later. In fact, we will find that the Fermi coupling constant is given by

$$G_F = \frac{g^2}{4\sqrt{2}\,m_W^2}\,, \tag{5.125}$$

where g is the dimensionless universal coupling constant and m_W is the W boson mass, the same quantities we have seen in (5.106). Here, we will take the measured muon lifetime as an input and extract the value of G_F.

We could simply apply the Feynman rules (5.110), but let us review the derivation from the first principles. The first-order S_{fi} is given by



$$S_{fi} = i\frac{G_F}{\sqrt{2}} \int d^4x \, \langle f| \, \bar{\psi}_{\nu_\mu} \gamma_\alpha (1-\gamma_5) \psi_\mu \times \bar{\psi}_e \gamma^\alpha (1-\gamma_5) \psi_{\nu_e} \, |i\rangle \, . \tag{5.126}$$

We did not use the hermitian-conjugate part of (5.124) since, as we will see, the term we took contains all the operators needed to annihilate the iniital state and create the final state; the hermitian-conjugate part would be responsible for the decay of μ^+. Upon momentum-expanding the fields, only the term that annihilates the initial-state muon and creates the final-state particles will survive. The integration over x will give the delta-function for the energy-momentum conservation, and we obtain

$$S_{fi} = \frac{(2\pi)^4 \delta^4(p' + q + q' - p)}{\sqrt{(2p^0 V)(2p'^0 V)(2q^0 V)(2q'^0 V)}} \mathcal{M} \tag{5.127}$$

with

$$\mathcal{M} = i\frac{G_F}{\sqrt{2}} \, (\overline{u}_{\nu_\mu} \gamma_\alpha (1-\gamma_5) u_\mu)(\overline{u}_e \gamma^\alpha (1-\gamma_5) v_{\nu_e}), \tag{5.128}$$

which could also be directly obtained from the Feynman rules where it is understood that the u, v spinors are combined with the γ matrices to form currents corresponding to those in $\mathcal{L}_{\rm int}$.

Summing over all final states with $\vec{p}' \in d^3p'$, $\vec{q} \in d^3q$, and $\vec{q}' \in d^3q'$, the corresponding decay rate is [see (5.111)]

$$d\Gamma = \frac{(2\pi)^4}{2p^0} d\Phi_3 |\mathcal{M}|^2 \tag{5.129}$$

where the Lorentz-invariant 3-body phase space is given by

$$d\Phi_3 \equiv \delta^4(p' + q + q' - p) \frac{d^3p'}{(2\pi)^3 2p'^0} \frac{d^3q}{(2\pi)^3 2q^0} \frac{d^3q'}{(2\pi)^3 2q'^0} \, . \tag{5.130}$$

Assuming that the parent particle is unpolarized, the 3-body phase space can be integrated over all variables except for E_1 and E_2, which are the energies of *any* two

of the three final-state particles in the C.M. frame of the parent, to obtain (left as an exercise)

$$\boxed{d\Phi_3 = \frac{dE_1 dE_2}{4(2\pi)^7}}, \tag{5.131}$$

which, together with (5.129), gives the differential decay rate in the C.M frame:

$$\boxed{d\Gamma = \frac{dE_1 dE_2}{64\pi^3\, m}\,|\mathcal{M}|^2} \quad \text{(3-body, parent unpolarized)} \tag{5.132}$$

where m is the mass of the parent ($p^0 = m$ in the C.M. frame). This formula gives the decay rate into an area element $dE_1 dE_2$ where E_1 and E_2 are the energies of any two of the three daughters. The probability density of the decay in the 2-dimensional space of E_1 vs E_2 is proportional to $|\mathcal{M}|^2$, and such density plot, called the *Dalitz plot*, provides a powerful tool to study the decay matrix element.

Exercise 5.3 *Three-body phase space.*
(*a*) *Take the expression of the 3-body Lorentz-invariant phase space (5.130) and show that it reduces to (5.131) when evaluated in the C.M. system* ($E_i \equiv p_i^0$). *The particles 1,2,3 have masses* m_1, m_2, m_3 *and 4-momenta* p_1, p_2, p_3, *respectively. Assume that the parent particle is spinless* (*or unpolarized*) *thus there is no special direction in the C.M. frame; namely, the matrix element does not depend on the direction of the first particle you pick. The matrix element, however, is in general a function of the angle between 1 and 2 among other variables.* [*hint: Namely, you can set* $d^3p_1 = 4\pi \breve{p}_1^2 d\breve{p}$, *but then, you cannot do the same for* $\vec{p}_2$; *it should be* $d^3p_2 = 2\pi \breve{p}_2^2 d\breve{p}_2 d\cos\theta_{12}$ ($\breve{p}_i \equiv |\vec{p}_i|$).]
(*b*) *Show that invariant masses of 2,3 and that of 3,1 are linearly related to* E_1 *and* E_2 *by*

$$E_1 = \frac{M^2 + m_1^2 - S_{23}}{2M}\,, \quad E_2 = \frac{M^2 + m_2^2 - S_{31}}{2M}\,,$$

where $S_{ij} = (p_i + p_j)^2$ *and* M *is the mass of the parent particle* (*or the invariant mass of the entire system in the case of scattering*). *Then rewrite* $d\Phi_3$ *in terms of* S_{23} *and* S_{31}. *Since* S_{23} *and* S_{31} *are Lorentz-invariant quantities, the resulting expression is valid in any frame.* [*hint: Energy-momentum conservation gives* $P - p_1 = p_{12}$ *with* $p_{12} = p_1 + p_2$. *Square both sides and use* $P = (M, \vec{0})$.]

The Lorentz-invariant matrix element given in (5.128) is for specific spin polarizations of the initial- and final-state particles. In calculating the decay rate, we sum $|\mathcal{M}|^2$ over all possible spins in the final state. Also, we assume that the parent is unpolarized, and thus we take the average over two possible spin states of the parent. Namely, we sum up all the spins of all initial and final state particles and then divide

by two. The result is the 'spin-averaged' matrix element squared - denoted as $\overline{|\mathcal{M}|^2}$:

$$\begin{aligned}
\overline{|\mathcal{M}|^2} &= \frac{1}{2}\sum_{\text{spins}}|\mathcal{M}|^2 \\
&= \frac{1}{2}\frac{G_F^2}{2}\sum_{\text{spins}}(\overline{u}_{\nu_\mu}\gamma_\alpha(1-\gamma_5)u_\mu)^*(\overline{u}_e\gamma^\alpha(1-\gamma_5)v_{\nu_e})^* \\
&\qquad\times(\overline{u}_{\nu_\mu}\gamma_\beta(1-\gamma_5)u_\mu)(\overline{u}_e\gamma^\beta(1-\gamma_5)v_{\nu_e}) \\
&= \frac{G_F^2}{4}\sum_{\substack{\mu,\nu_\mu\\ \text{spins}}}(\overline{u}_{\nu_\mu}\gamma_\alpha(1-\gamma_5)u_\mu)^*(\overline{u}_{\nu_\mu}\gamma_\beta(1-\gamma_5)u_\mu) \\
&\qquad\times\sum_{\substack{e,\nu_e\\ \text{spins}}}(\overline{u}_e\gamma^\alpha(1-\gamma_5)v_{\nu_e})^*(\overline{u}_e\gamma^\beta(1-\gamma_5)v_{\nu_e})\,. \qquad (5.133)
\end{aligned}$$

Noting that $\gamma^\alpha(1-\gamma_5)$ is self-adjoint:

$$\overline{\gamma^\alpha(1-\gamma_5)} = (1-\underbrace{\overline{\gamma_5}}_{-\gamma_5})\underbrace{\overline{\gamma^\alpha}}_{\gamma^\alpha} = (1+\gamma_5)\gamma^\alpha = \gamma^\alpha(1-\gamma_5)\,, \qquad (5.134)$$

and using the spin-sum formulas $\sum_{\text{spin}} u_{\nu_\mu}\overline{u}_{\nu_\mu} = \not{p}' + \not{m}_{\nu_\mu}$ ($m_{\nu_\mu}=0$) etc. as well as (5.102), the spin average leads to traces:

$$\begin{aligned}
\overline{|\mathcal{M}|^2} &= \frac{G_F^2}{4}\sum_{\substack{\mu,\nu_\mu\\ \text{spins}}}\overline{u}_\mu\gamma_\alpha(1-\gamma_5)\underbrace{u_{\nu_\mu}\overline{u}_{\nu_\mu}}_{\to\ \not{p}'}\gamma_\beta(1-\gamma_5)u_\mu \\
&\qquad\times\sum_{\substack{e,\nu_e\\ \text{spins}}}\overline{v}_{\nu_e}\gamma^\alpha(1-\gamma_5)\underbrace{u_e\overline{u}_e}_{\to\ \not{q}}\gamma^\beta(1-\gamma_5)v_{\nu_e} \\
&= \frac{G_F^2}{4}\text{Tr}\,(\not{p}+m)\gamma_\alpha(1-\gamma_5)\not{p}'\gamma_\beta(1-\gamma_5) \\
&\qquad\times\text{Tr}\,\not{q}'\gamma^\alpha(1-\gamma_5)\not{q}\gamma^\beta(1-\gamma_5) \qquad (5.135)
\end{aligned}$$

Since $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ contains four γ^μ's, the term linear in m in the first trace vanishes becuase of the odd number of γ^μ's in the trace. Noting that γ_5 commutes with $\not{p}'\gamma_\beta$, we have

$$\text{Tr}\,(\not{p}+\not{m})\gamma_\alpha\underbrace{(1-\gamma_5)\not{p}'\gamma_\beta}_{\not{p}'\gamma_\beta(1-\gamma_5)}(1-\gamma_5) = \text{Tr}\not{p}\gamma_\alpha\not{p}'\gamma_\beta\underbrace{(1-\gamma_5)^2}_{2(1-\gamma_5)}\,. \qquad (5.136)$$

Similarly,

$$\text{Tr}\,\not{q}\gamma^\alpha(1-\gamma_5)\not{q}'\gamma^\beta(1-\gamma_5) = 2\text{Tr}\not{q}'\gamma^\alpha\not{q}\gamma^\beta(1-\gamma_5)\,. \qquad (5.137)$$

Thus,

$$\overline{|\mathcal{M}|^2} = G_F^2 \mathrm{Tr} \not{p} \gamma_\alpha \not{p}' \gamma_\beta (1-\gamma_5) \times \mathrm{Tr} \not{q}' \gamma^\alpha \not{q} \gamma^\beta (1-\gamma_5)\,. \tag{5.138}$$

Using the trace theorems

$$\begin{aligned} \mathrm{Tr}\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\beta &= 4(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\nu}g_{\alpha\beta} + g_{\mu\beta}g_{\nu\alpha}) && (5.139)\\ \mathrm{Tr}\gamma_5\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\beta &= 4i\epsilon_{\mu\alpha\nu\beta}\,, && (5.140) \end{aligned}$$

we have

$$\begin{aligned} &\mathrm{Tr}\not{p}\gamma_\alpha\not{p}'\gamma_\beta(1-\gamma_5)\\ &\quad = p^\mu p'^\nu(\mathrm{Tr}\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\beta - \mathrm{Tr}\gamma_5\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\beta)\\ &\quad = 4p^\mu p'^\nu(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\nu}g_{\alpha\beta} + g_{\mu\beta}g_{\nu\alpha} - i\epsilon_{\mu\alpha\nu\beta})\,. \end{aligned} \tag{5.141}$$

Applying the same procedure to the second trace also, we obtain

$$\begin{aligned} \mathrm{Tr}\not{p}\gamma_\alpha\not{p}'\gamma_\beta(1-\gamma_5) &= 4(p_\alpha p'_\beta - p\cdot p' g_{\alpha\beta} + p_\beta p'_\alpha - ip^\mu p'^\nu \epsilon_{\mu\alpha\nu\beta}) && (5.142)\\ \mathrm{Tr}\not{q}'\gamma^\alpha\not{q}\gamma^\beta(1-\gamma_5) &= 4(q'^\alpha q^\beta - q'\cdot q\, g^{\alpha\beta} + q'^\beta q^\alpha - iq'_\rho q_\sigma \epsilon^{\rho\alpha\sigma\beta})\,. && (5.143) \end{aligned}$$

In each of the traces, the first three terms are symmetric under the exchange $\alpha \leftrightarrow \beta$ while the last term is antisymmetric. Thus, when we take the product of the two traces, the cross terms between symmetric and antisymmetric terms vanish; thus,

$$\begin{aligned} \overline{|\mathcal{M}|^2} \;=\; 16G_F^2\Big[&(p_\alpha p'_\beta - p\cdot p' g_{\alpha\beta} + p_\beta p'_\alpha)(q'^\alpha q^\beta - q'\cdot q\, g^{\alpha\beta} + q'^\beta q^\alpha)\\ &-p^\mu p'^\nu q'_\rho q_\sigma \epsilon_{\mu\alpha\nu\beta}\epsilon^{\rho\alpha\sigma\beta}\Big]\,. \end{aligned} \tag{5.144}$$

The first term is

$$\begin{aligned} &\Big[(p_\alpha p'_\beta + p_\beta p'_\alpha) - p\cdot p' g_{\alpha\beta}\Big]\Big[(q'^\alpha q^\beta + q'^\beta q^\alpha) - q'\cdot q\, g^{\alpha\beta}\Big]\\ &\quad = \underbrace{(p_\alpha p'_\beta + p_\beta p'_\alpha)(q'^\alpha q^\beta + q'^\beta q^\alpha)}_{2p\cdot q'\, p'\cdot q + 2p\cdot q\, p'\cdot q'} - 4p\cdot \cancel{p'}\cancel{q'}\cdot q + p\cdot \cancel{p'}\cancel{q'}\cdot q\, \underbrace{g_{\alpha\beta}g^{\alpha\beta}}_{4}\\ &\quad = 2(p\cdot q'\, p'\cdot q + p\cdot q\, p'\cdot q')\,, \end{aligned} \tag{5.145}$$

while the second term uses an identity for a product of two $\epsilon_{\mu\nu\alpha\beta}$'s:

$$\begin{aligned} -p^\mu p'^\nu q'_\rho q_\sigma \underbrace{\epsilon_{\mu\alpha\nu\beta}\epsilon^{\rho\alpha\sigma\beta}}_{\epsilon_{\mu\nu\alpha\beta}\epsilon^{\rho\sigma\alpha\beta} = -2(g^\rho_\mu g^\sigma_\nu - g^\sigma_\mu g^\rho_\nu)} &= 2(p\cdot q'\, p'\cdot q - p\cdot q\, p'\cdot q')\,. \end{aligned} \tag{5.146}$$

Finally, we obtain

$$\overline{|\mathcal{M}|^2} = 64G_F^2\, p\cdot q'\, p'\cdot q\,. \tag{5.147}$$

Using 4-momentum conservation $p = p' + q + q'$, we can relate $p' \cdot q$ and $p \cdot q'$:

$$(p - q')^2 = (p' + q)^2 \;\rightarrow\; \underbrace{p^2}_{m^2} + \underbrace{q'^2}_{0} - 2p \cdot q' = \underbrace{p'^2}_{0} + \underbrace{q^2}_{0} + 2p' \cdot q$$
$$\rightarrow\; p' \cdot q = \frac{m^2}{2} - p \cdot q' \,. \tag{5.148}$$

In the C.M. frame of the parent, we have

$$p = (m, \vec{0}\,) \quad \rightarrow \quad p \cdot q' = m E_{\bar{\nu}_e} \quad \text{(C.M. frame)}\,; \tag{5.149}$$

thus, the spin-averaged $|\mathcal{M}|^2$ can be written as

$$\overline{|\mathcal{M}|^2} = 64 G_F^2 \, m E_{\bar{\nu}_e} \Big(\frac{m^2}{2} - m E_{\bar{\nu}_e} \Big) \,. \tag{5.150}$$

We can now use the 3-body decay rate formula (5.132) to obtain the differential decay rate. The formula is valid for any two energies, E_1 and E_2, of the final state as long as the parent is unpolarized. It is reassuring that indeed the expression of $\overline{|\mathcal{M}|^2}$ obtained above does not depend on spacial direction: it depends only on the energy of the electron antineutrino. It is then natural to take it as one of the energies: $E_1 = E_{\bar{\nu}_e}$. What shall we take for the other energy? Actually, it does not matter; we could take $E_2 = E_{\nu_\mu}$ or $E_2 = E_e$ and both should give the correct distribution. This means that e^- and ν_μ have exactly the same energy distribution (in the limit of $m_e = 0$). Let's take $E_2 = E_e$. Then, we obtain

$$d\Gamma = \frac{G_F^2 m}{\pi^3} E_{\bar{\nu}_e} \Big(\frac{m}{2} - E_{\bar{\nu}_e} \Big) dE_{\bar{\nu}_e} dE_e \,. \tag{5.151}$$

When a particle of mass m at rest decays to three massless particles, the maximum energy of any of the daughters is $m/2$, which can be seen as follows: first, $p = p_1 + p_2 + p_3$ gives

$$\underbrace{(p - p_1)^2}_{m^2 - 2mE_1} = (p_2 + p_3)^2 \equiv s_{23} \quad \rightarrow \quad E_1 = \frac{m^2 - s_{23}}{2m} \,, \tag{5.152}$$

where the invariant mass squared of the particle 2 and 3, s_{23}, is given by

$$s_{23} = \underbrace{p_2^2 + p_3^2}_{0} + 2 p_2 p_3 = 2 E_2 E_3 (1 - \cos\theta_{23}) \,. \tag{5.153}$$

Namely, the energy of a given daughter (E_1) is larger when the invariant mass squared of the system recoiling against it (s_{23}) is smaller, and s_{23} is minimum, in fact zero,

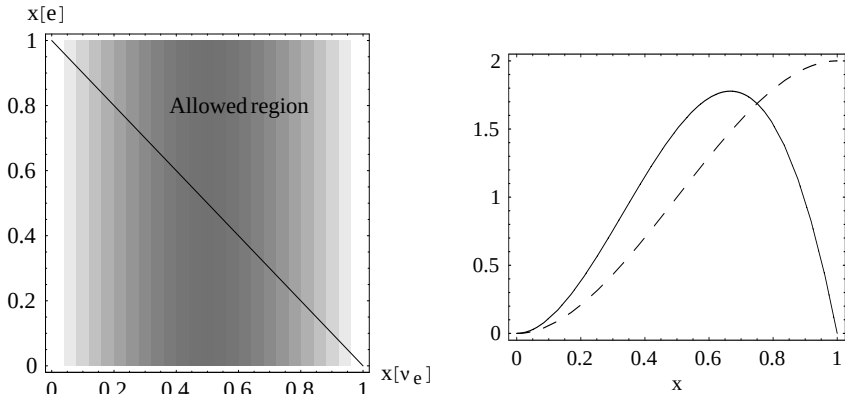


Figure 5.2: The probability density distribution (the Dalitz plot) of the decay $\mu^- \to \nu_\mu e^- \bar{\nu}_e$. The density is proportional to the decay probability. The kinematically allowed region is $x_{\bar{\nu}_e} + x_e \geq 1$. The plot on the right shows the projections of the allowed region on to $x_{\bar{\nu}_e}$ (solid line) and x_e (dashed line).

when the angle between the particles 2 and 3 is zero regardless of E_2 and E_3. Setting $s_{23} = 0$ in (5.152) the maximum energy is $E_1 = m/2$. Define the dimensionless parameters x_i $(i = \bar{\nu}_e, e, \nu_\mu)$ by

$$E_i \equiv \frac{m}{2} x_i \,, \qquad 0 \leq x_i \leq 1 \quad (i = \bar{\nu}_e, e, \nu_\mu) \,. \tag{5.154}$$

Then, the decay rate is written as

$$d\Gamma = \frac{G_F^2 m^5}{16\pi^3} x_{\bar{\nu}_e}(1 - x_{\bar{\nu}_e}) dx_{\bar{\nu}_e} dx_e \,, \tag{5.155}$$

which gives the probability distribution in the plane of $x_{\bar{\nu}_e}$ vs x_e (Figure 5.2).

Not all points in $0 \leq x_{\bar{\nu}_e} \leq 1$ and $0 \leq x_e \leq 1$ are kinematically allowed. When $x_{\bar{\nu}_e} = x_e = 0$, for example, all energy has to be carried by ν_μ to conserve energy, but it cannot have any energy without violating momentum conservation. Energy conservation gives the allowed region:

$$\begin{aligned} E_{\bar{\nu}_e} + E_e + E_{\nu_\mu} = m \;&\to\; x_{\bar{\nu}_e} + x_e + x_{\nu_\mu} = 2 \,, \\ &\to\; x_{\bar{\nu}_e} + x_e = 2 - \underbrace{x_{\nu_\mu}}_{\leq 1} \\ &\to\; x_{\bar{\nu}_e} + x_e \geq 1 \end{aligned} \tag{5.156}$$

which is shown in Figure 5.2. Taking only the allowed region, and projecting it on to $x_{\bar{\nu}_e}$ and x_e axes, we obtain the energy distributions for $\bar{\nu}_e$ and e^-, respectively. Since ν_μ should have the same energy distribution as e^- as discussed earlier, we see that

fermions (e^- and ν_μ) have a distribution that peaks at the maximum energy, and the antifermion ($\bar{\nu}_e$) has a distribution that peaks near the center. If the parent is an antifermion, the distributions of fermions and antifermions will be reversed.

The total decay rate is obtained by integrating (5.155) inside the allowed region. Figure 5.2 shows that the integral in the the allowed region is the same as that in the non-allowed region; thus,

$$\begin{aligned}\Gamma &= \int\limits_{x_{\bar{\nu}_e}+x_e>1} d\Gamma = \int\limits_{x_{\bar{\nu}_e}+x_e<1} d\Gamma = \frac{1}{2}\int d\Gamma \\ &= \frac{G_F^2 m^5}{32\pi^3} \underbrace{\int_0^1 dx_e}_{1} \underbrace{\int_0^1 dx_{\bar{\nu}_e} x_{\bar{\nu}_e}(1-x_{\bar{\nu}_e})}_{1/6}\,; \end{aligned} \tag{5.157}$$

thus, the total decay rate is

$$\Gamma(\mu^- \to \nu_\mu e^- \bar{\nu}_e) = \frac{G_F^2 m^5}{192\pi^3}\,. \tag{5.158}$$

The experimental lifetime τ_μ can be converted to the decay rate:

$$\begin{aligned} \tau_\mu &= 2.20\times 10^{-6}\ (\text{sec}) \\ \to \quad \Gamma &= \frac{\hbar}{\tau_\mu} = \frac{6.5822\times 10^{-25}\ (\text{GeV}\cdot\text{sec})}{2.20\times 10^{-6}\ (\text{sec})} = 2.99\times 10^{-19}\ (\text{GeV})\ . \end{aligned} \tag{5.159}$$

Together with $m = 0.106$ GeV, we then obtain $G_F = 1.15\times 10^{-5}$ GeV^{-2}. In order to obtain a more accurate value, one needs to include higher-order processes where photons are emitted from or absorbed by charged particles. The correction amounts to a 0.42% reduction in the theoretical decay rate. The most up-to-date value is

$$G_F = 1.16639(2)\times 10^{-5}\ (\text{GeV}^{-2})\,. \tag{5.160}$$

Spin polarizations

The interaction (5.124) is a product of two currents each of which has the form a vector current minus an axial vector current:

$$\bar{\psi}'\gamma^\alpha(1-\gamma_5)\psi = \underbrace{\bar{\psi}'\gamma^\alpha\psi}_{V} - \underbrace{\bar{\psi}'\gamma^\alpha\gamma_5\psi}_{A}\,. \tag{5.161}$$

This $V-A$ form of the interaction and the resulting Lorentz-invariant matrix element (5.128) tells us how the final-state particles are polarized. Let's write $\mathcal{M}$ as

$$\mathcal{M} \propto (\overline{u}_{\nu_\mu}\gamma_\alpha P_L u_\mu)(\overline{u}_e\gamma^\alpha P_L v_{\nu_e})\,, \tag{5.162}$$

where $P_L \equiv (1-\gamma_5)/2$ was defined in (3.370), where we have seen that P_L (P_R) acts as the helicity $-$ ($+$) projection operator for fermions and as the helicity $+$ ($-$) projection operator for antifermions in the massless limit or, equivalently, in the high-energy limit.

Suppose the spinor v_{ν_e} represents an antineutrino with negative helicity. Since P_R acts as the negative helicity projection operator for antifermions, v_{ν_e} should satisfy

$$v_{\nu_e} = P_R\, v_{\nu_e} \quad (\bar{\nu}_e \text{ helicity } -), \tag{5.163}$$

then $\mathcal{M}$ vanishes due to $P_L P_R = 0$ (3.352):

$$(\overline{u}_e \gamma^\alpha P_L \underbrace{v_{\nu_e}}_{P_R\, v_{\nu_e}}) = (\overline{u}_e \gamma^\alpha \underbrace{P_L P_R}_{0} v_{\nu_e}) = 0\,. \tag{5.164}$$

This means that $\bar{\nu}_e$ cannot be created with negative helicity. Since the spin of $\bar{\nu}_e$ should be either helicity $+$ or $-$, the vanishing amplitude for negative helicity indicates that $\bar{\nu}_e$ in the muon decay is in a pure helicity $+$ state. Now, suppose the spinor u_e represents an electron with positive helicity. Then, to the extent we can ignore the mass of electron,

$$u_e = P_R\, u_e \quad (e^- \text{ helicity } +). \tag{5.165}$$

Then, using $\bar{u}_e = \overline{P_R u_e} = \bar{u}_e \overline{P}_R = \bar{u}_e P_L$ and $P_L \gamma^\alpha = \gamma^\alpha P_R$, we see that $\mathcal{M}$ vanishes again:

$$(\underbrace{\overline{u}_e}_{\bar{u}_e P_L} \gamma^\alpha P_L v_{\nu_e}) = (\overline{u}_e \underbrace{P_L \gamma^\alpha P_L}_{\gamma^\alpha P_R P_L = 0} v_{\nu_e}) = 0\,. \tag{5.166}$$

Namely, the electron is created with negative helicity. Similarly, we see that the muon neutrino is purely left-handed. Thus, a massless fermion created by a $V-A$ current is always left-handed (i.e. negative helicity), and a massless antifermion created by a $V-A$ current is always right-handed (i.e. positive helicity).

One can use the helicity projection operator to calculate the decay rate where a given (massless) fermion has particular helicity. As an example, let's take the Higgs decay $H \to f\bar{f}$ where the mass of the fermion is small compared to its momentum, and calculate the decay rate where the antifermion, represented by $v_2 \equiv v_{\vec{p}_2,\vec{s}_2}$, is left-handed. We can still take advantage of the trace technique in the following way: First, assume that the spin quantization axis is taken as $\vec{s} = \hat{p}$ such that the two possible polarizations correspond to positive and negative helicities. In evaluating the spin sum of $|\mathcal{M}|^2 = \lambda^2 |\bar{u}_1 v_2|^2$ in (5.101), we can place P_R in front of v_2 and then sum over the spins of f and $\bar{f}$. Then the matrix element should vanish unless the v_2 represents a left-handed antifermion, and the resulting decay rate should correspond to the case where the antifermion is left-handed. With $u_1 \equiv u_{\vec{p}_1,\vec{s}_1}$ and ignoring the

fermion mass, we have

$$\begin{aligned}
&\sum_{\text{spins}} |\mathcal{M}|^2 \ (\bar{f} \text{ left-handed}) \\
&\qquad = \lambda^2 \sum_{\text{spins}} |\bar{u}_1 P_R \, v_2|^2 = \lambda^2 \sum_{\text{spins}} \bar{v}_2 P_L \underbrace{u_1 \bar{u}_1}_{\to \not{p}_1} P_R \, v_2 \\
&\qquad = \lambda^2 \mathrm{Tr} \not{p}_2 \underbrace{P_L \not{p}_1}_{\not{p}_1 P_R} P_R = \lambda^2 \mathrm{Tr} \not{p}_2 \not{p}_1 \underbrace{P_R^2}_{P_R} \\
&\qquad = \lambda^2 \mathrm{Tr} \not{p}_2 \not{p}_1 \frac{1+\gamma_5}{2} = \frac{\lambda^2}{2} \Big(\mathrm{Tr} \not{p}_2 \not{p}_1 + \underbrace{\mathrm{Tr} \not{p}_2 \not{p}_1 \gamma_5}_{0} \Big) \\
&\qquad = \frac{\lambda^2}{2} \mathrm{Tr} \not{p}_2 \not{p}_1 \, .
\end{aligned} \tag{5.167}$$

This is exactly one half of the sum without any helicity restriction (5.101) with $m = 0$. Thus, we see that $\bar{f}$ is left-handed 50% of the time. Similarly, one can see that it is right-handed 50% of the time. In addition, a technique similar to the muon decay case immediately tells us that the two daughters have to be both left-handed or both right-handed. For example, if f is left-handed and $\bar{f}$ is right-handed, we can replace u_1 by $P_L u_1$ and v_2 by $P_L v_2$, and the matrix element becomes

$$\mathcal{M} \propto \bar{u}_1 v_2 = \overline{P_L u_1} P_L v_2 = \bar{u}_1 P_R P_L v_2 = 0 \, . \tag{5.168}$$

Thus, each daughter is unpolarized when viewed individually, but there is a correlation between the polarizations of the two.

How can we calculate the rate where a fermion is polarized in some arbitrary direction, or if the fermion is heavy, for that matter? Fortunately, we have a spin projection operator for massive fermion or antifermion along any direction. All we need is then to place the spin projection operator (3.279) in front of the spinor in question

$$w_{\vec{p},\pm\vec{s}} \quad \to \quad \Sigma_\pm(s) w_{\vec{p},\pm\vec{s}} \, , \qquad \Big(\Sigma_\pm(s) \equiv \frac{1 \pm \gamma_5 \not{s}}{2} \text{ and } w_{\vec{p},\vec{s}} = u_{\vec{p},\vec{s}} \text{ or } v_{\vec{p},\vec{s}} \Big) \tag{5.169}$$

and then execute the spin sum, where s^μ is the boosted unit vector $(0, \vec{s})$ which defines the spin quantization axis in the rest frame of the particle, and the $\pm$ sign corresponds to the physical spin component along $\vec{s}$ in the rest frame of the particle. Note that the same projection operator works for both fermion and antifermion. For example, if we want the decay rate in which the antifermion is polarized as represented by s_2, we evaluate for a finite m

$$\sum_{\text{spins}} |\mathcal{M}|^2 \ = \ \lambda^2 \sum_{\text{spins}} |\bar{u}_1 \Sigma_+(s_2) \, v_2|^2$$

$$
\begin{aligned}
&= \lambda^2 \mathrm{Tr}(\not{p}_1 + m)\Sigma_+(s_2)(\not{p}_2 - m)\underbrace{\bar{\Sigma}_+(s_2)}_{\Sigma_+(s_2)} \\
(\,[\not{p}_2, \Sigma_+(s_2)] = 0 \to) \;&= \lambda^2 \mathrm{Tr}(\not{p}_1 + m)(\not{p}_2 - m)\underbrace{\Sigma_+^2(s_2)}_{\Sigma_+(s_2) = \frac{1+\gamma_5 \not{s}_2}{2}} \\
&= \frac{\lambda^2}{2}\mathrm{Tr}(\not{p}_1 + m)(\not{p}_2 - m)\,, \qquad (5.170)
\end{aligned}
$$

where in the last step we have noted, using Tr(odd number of γ) = 0 and $\mathrm{Tr}\not{a}\not{b}\gamma_5 = 0$,

$$
\mathrm{Tr}(\not{p}_1 + m)(\not{p}_2 - m)\gamma_5 \not{s}_2 = m(\mathrm{Tr}\not{p}_2\gamma_5\not{s}_2 - \mathrm{Tr}\not{p}_1\gamma_5\not{s}_2) = 0\,. \qquad (5.171)
$$

This is again exactly one half of (5.101). Thus, we see that in $H \to f\bar{f}$, the rate is always 1/2 of the total if the spin of f or $\bar{f}$ is restricted to one polarization, and this is so regardless of the spin direction and even when the daughters are massive. Namely, the fermion and the antifermion in $H \to f\bar{f}$ are unpolarized when examined individually regardless of the fermion mass.

5.5 Spin-1 Fields

We will now introduce spin-1 fields. Examples are the vector bosons $W^\pm$ and Z which mediate the weak interaction, and in the massless limit we have photon which mediates the electro-magnetic interaction. These are examples of *gauge bosons* which are responsible for so-called gauge interactions between fermions and sometimes refered to as 'force particles'. A pair of fermions can form a spin-1 bound state (as well as other integer spins), and this section applies to those bound-state particles also. As we will see, the simple massless limit for spin-1 particle encounters difficulties, and the rigorous treatment requires an understanding of a symmetry introduced by the masslessness called the *gauge invariance* which will be discussed in later chapters. Here, we will start from the Lagrangian formulation of free massive spin-1 field.

Lagrangian for a free massive spin-1 field

Let's start from searching for non-quantized fields that can represent a spin-1 particle. We have seen that single component real field can represent a single spin-0 particle corresponding to the fact that a spin-0 particle at rest has only one degree of freedom. A spin-1 particle at rest, on the other hand, have three degrees of freedom:

$$
|jm\rangle = |1,+1\rangle\,, \quad |1,0\rangle\,, \quad |1,-1\rangle\,, \qquad (5.172)
$$

where j is the absolute value of the spin and m is the component along some axis. Then, we expect that we need three components of real field to represent it. Also,

the three components have to transform under rotation in a way consistent with a spin-1 particle. We recall that under a rotation the Dirac field transformed as

$$\psi'(x') = U\psi(x) \qquad U = e^{-i\frac{\Sigma_i}{2}\theta_i}\,, \tag{5.173}$$

and the *generator* $\vec{J} = \vec{\Sigma}/2$ satisfied the commutation relations of angular momentum $[J_i, J_j] = i\epsilon_{ijk}J_k$ and $\vec{J}^2 = j(j+1)$ with $j = 1/2$ (3.160). In our case, we need three components that transform under rotation by $e^{-i\vec{\theta}.\vec{J}}$ where $[J_i, J_j] = i\epsilon_{ijk}J_k$ and $\vec{J}^2 = j(j+1) = 1(1+1) = 2$. Actually we already have such quantity - the space components of a 4-vector. In fact, we have seen that a rotation in the ordinary three-dimensional space can be written as $e^{-i\vec{\theta}.\vec{J}}$ with $(J_i)_{jk} = i(L_i)^j{}_k$ (3.162):

$$J_1 = i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_2 = i\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad J_3 = i\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{5.174}$$

and that $\vec{J}$ defined as such satisfy the commutation relations of angular momentum operator. Recalling that $(L_i)^j{}_k = -\epsilon_{ijk}$ (1.96), the components $(J_i)_{jk}$ is nothing but $(-i)$ times the structure consant ϵ_{ijk}:

$$(J_i)_{jk} = -i\epsilon_{ijk}\,. \tag{5.175}$$

It is a general feature of Lie algebra that a representation of generators can be consturcted directly out of the structure constants, and such representation is called the *adjoint representation*. For the absolute value of the spin, explicit evaluation using (5.174) indeed shows that it is spin one:

$$\vec{J}^2 = 2 = j(j+1) \quad \rightarrow \quad j = 1\,. \tag{5.176}$$

Thus, in order to represent a spin-1 particle, we take a *real* 4-vector field which transforms under Lorentz transformation as

$$\boxed{A'^\mu(x') = \Lambda^\mu{}_\nu A^\nu(x)} \quad (x' = \Lambda x)\,. \tag{5.177}$$

There are, however, four degrees of freedom for A^μ ($\mu = 0, 1, 2, 3$) while we need only three. We will try to remove the one extra degree of freedom by imposing a Lorentz-invariant condition

$$\boxed{\partial_\mu A^\mu(x) = 0}\,. \tag{5.178}$$

Let's try to see what this means by applying it to plane waves

$$A^\mu(x) \propto e^{\pm ip\cdot x}\,. \tag{5.179}$$

The condition $\partial_\mu A^\mu(x) = 0$ is then written as

$$p_\mu A^\mu(x) = 0\,. \tag{5.180}$$

If the plane wave A^μ represents a particle at rest, this condition becomes

$$p = (m, \vec{0}\,) \quad \rightarrow \quad mA^0(x) = 0 \quad \rightarrow \quad A^0(x) = 0 \text{ (if } m \neq 0\text{, at rest)}\,. \tag{5.181}$$

Thus, the condition $\partial_\mu A^\mu(x) = 0$ removes one degree of freedom out of the 4-vector such that for a particle at rest the time component A^0 is zero while keeping the 4-vector nature of the field intact. The condition $\partial_\mu A^\mu(x) = 0$ is called the *Lorentz condition* which is a transversality condition in the four-dimensional space-time.

How about the equation of motion and the Lagrangian density that leads to it? In order to be consistent with the relativistic energy-momentum relation $p^{0\,2} = \vec{p}^{\,2} + m^2$, each plane wave solution, and thus any free-field solution, should satisfy the Klein-Gordon equation

$$(\partial^2 + m^2)A^\mu(x) = 0 \qquad (\mu = 0, 1, 2, 3)\,. \tag{5.182}$$

If we can regard each component as independent, then the total Lagrangian density would be simply the sum of the Klein-Gordon Lagrangian density for each component:

$$\mathcal{L} \stackrel{?}{=} \sum_{\mu=0}^{3} \frac{1}{2}(\partial_\nu A^\mu \partial^\nu A^\mu - m^2 A^{\mu\,2})\,, \tag{5.183}$$

One problem is that this is not a Lorentz scaler. Since we have taken A^μ as a 4-vector, the paired μ indexes should be properly contracted in order to form a Lorentz scaler. And another problem is that this has four dynamical degrees of freedom instead of three.

One way to remove one dynamical degree of freedom is to eliminate the time derivative of one component so that the corresponding conjugate field vanishes. This can be accomplished in a Lorentz-invariant way by forming the antisymmetric combination

$$\boxed{F^{\mu\nu} \stackrel{\text{def}}{\equiv} \partial^\nu A^\mu - \partial^\mu A^\nu\,, \quad F^{\mu\nu} = -F^{\nu\mu}} \tag{5.184}$$

which has no $\dot{A}^0$ in it:

$$F^{00} = \partial^0 A^0 - \partial^0 A^0 = 0\,, \tag{5.185}$$

and try the following form

$$\boxed{\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{m^2}{2}A_\mu A^\mu\,.} \tag{5.186}$$

Note that the sign of the space part of the mass term $m^2 A^{i\,2}$ is the same as that in (5.183).

Let's derive the equation of motion from the above Lagrangian. First, the 'kinetic term' $F_{\mu\nu}F^{\mu\nu}$ can be written as

$$\begin{aligned} F_{\mu\nu}F^{\mu\nu} &= F_{\mu\nu}(\partial^\nu A^\mu - \partial^\mu A^\nu) \\ &= F_{\mu\nu}\partial^\nu A^\mu - \underbrace{F_{\mu\nu}}_{-F_{\nu\mu}}\partial^\mu A^\nu \\ &= 2F_{\mu\nu}\partial^\nu A^\mu = 2(\partial_\nu A_\mu - \partial_\mu A_\nu)\partial^\nu A^\mu\,; \end{aligned} \tag{5.187}$$

namely,

$$\mathcal{L} = \frac{1}{2}\Big[(\partial_\mu A_\nu - \partial_\nu A_\mu)\partial^\nu A^\mu + m^2 A_\mu A^\mu\Big] \tag{5.188}$$

The Euler-Lagrange equation of motion is

$$\frac{\partial\mathcal{L}}{\partial A_\alpha} = \partial_\beta \frac{\partial\mathcal{L}}{\partial(\partial_\beta A_\alpha)} \quad (\alpha = 0, 1, 2, 3)\,. \tag{5.189}$$

Using the Lagrangian (5.188), the left-hand side is

$$\frac{\partial\mathcal{L}}{\partial A_\alpha} = m^2 A^\alpha \tag{5.190}$$

and with some care,

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial(\partial_\beta A_\alpha)} &= \frac{1}{2}\frac{\partial}{\partial(\partial_\beta A_\alpha)}\Big[(\partial_\mu A_\nu - \partial_\nu A_\mu)\partial^\nu A^\mu\Big] \\ &= \frac{1}{2}\frac{\partial}{\partial(\partial_\beta A_\alpha)}\partial_\mu A_\nu \partial^\nu A^\mu - \frac{1}{2}\frac{\partial}{\partial(\partial_\beta A_\alpha)}\partial_\nu A_\mu \partial^\nu A^\mu \\ &= \partial^\alpha A^\beta - \partial^\beta A^\alpha \\ &= F^{\beta\alpha} = -F^{\alpha\beta}\,. \end{aligned} \tag{5.191}$$

Thus, we obtain

$$\boxed{\partial_\beta F^{\alpha\beta} + m^2 A^\alpha = 0}\,, \tag{5.192}$$

which is the equation of motion for a massive spin-1 particle and called the *Proca equation*. Taking ∂_α of this and assuming that m is non-zero,

$$\underbrace{\partial_\alpha\partial_\beta F^{\alpha\beta}}_{\to\ 0\ \text{since}\ \begin{cases}\partial_\alpha\partial_\beta = \partial_\beta\partial_\alpha \\ F^{\alpha\beta} = -F^{\beta\alpha}\end{cases}} + m^2\partial_\alpha A^\alpha = 0 \quad\to\quad \partial_\alpha A^\alpha = 0 \quad (m \neq 0)\,, \tag{5.193}$$

which is nothing but the Lorentz condition (5.178). Writing out $F^{\alpha\beta}$ in (5.192) and using $\partial_\alpha A^\alpha = 0$,

$$\begin{aligned} 0 &= \partial_\beta F^{\alpha\beta} + m^2 A^\alpha \\ &= \partial_\beta(\partial^\beta A^\alpha - \partial^\alpha A^\beta) + m^2 A^\alpha \\ &= \partial^2 A^\alpha - \partial_\alpha \underbrace{\partial_\beta A^\beta}_{0} + m^2 A^\alpha \\ &\rightarrow \quad (\partial^2 + m^2) A^\alpha = 0\,. \end{aligned} \tag{5.194}$$

Thus, the Lagrangian density (5.186) leads to the Klein-Gordon equation for each component and one degree of freedom is removed in a way consistent with the Lorentz condition. The Proca equation can then be thought of as a 4-component Klein-Gordon equation with the Lorentz condition built into it.

The fields conjugate to A^α can be obtained by simply setting $\beta = 0$ in (5.191):

$$\pi^\alpha \equiv \frac{\partial \mathcal{L}}{\partial \dot{A}_\alpha} = F^{0\alpha}\,, \tag{5.195}$$

or

$$\pi^0 = 0\,, \qquad \pi^i = F^{0i}\,. \tag{5.196}$$

Thus, as promised, there is no conjugate field for A^0. In fact, A^0 can be derived from the rest from the fields in the Hamiltonian formalism: setting $\alpha = 0$ in the Proca equation (5.192) and using $F^{00} = 0$, we have

$$m^2 A^0 = -\partial_i \underbrace{F^{0i}}_{\pi^i} = -\vec{\nabla} \cdot \vec{\pi}\,, \tag{5.197}$$

where we note that $\vec{\nabla}\cdot\vec{\pi}$ is the difference of neighboring values of $\vec{\pi}$ and not considered to be a new independent field.

Momentum expansion and quantization

Now we will momentum-expand an arbitrary solution of the Proca equation $A^\mu(x)$, namely, a real 4-component field that simultaneously satisfies the Klein-Gordon equation and the Lorentz condition. Following the same procedure as in the case of the real spin-0 field (4.173), each component of $A^\mu(x)$, which satisfies the Klein-Gordon equation, can be uniquely expanded using the normal-mode functions $e_{\vec{p}}(x)$ and $e^*_{\vec{p}}(x)$. Thus, we have

$$A^\mu(x) = \sum_{\vec{p}} \left(A^\mu_{\vec{p}} e_{\vec{p}}(x) + A^{\mu *}_{\vec{p}} e^*_{\vec{p}}(x) \right) \qquad (\mu = 0, 1, 2, 3)\,. \tag{5.198}$$

At this point, $A^\mu_{\vec{p}}$ is a complex 4-vector. The Lorentz condition $\partial_\mu A^\mu(x) = 0$ then becomes

$$\partial_\mu A^\mu(x) = -i \sum_{\vec{p}} \left(p_\mu A^\mu_{\vec{p}} e_{\vec{p}}(x) - p_\mu A^{\mu *}_{\vec{p}} e^*_{\vec{p}}(x) \right) = 0\,, \tag{5.199}$$

where $p^0 \equiv \sqrt{\vec{p}^2 + m^2}$ as before. Applying $\int d^3x e^*_{\vec{p}}(x) i \overset{\leftrightarrow}{\partial_0}$ and using the orthonormality (4.181), the Lorentz condition translates to

$$p_\mu A^\mu_{\vec{p}} = 0 \quad (\text{for all } \vec{p})\,. \tag{5.200}$$

The 4-component quantity $A^\mu_{\vec{p}}$ that satisfies this condition has three degrees of freedom, and we want to expand it using some three orthonormal 4-vectors. The situation is similar to the momentum expansion of the Dirac field where any complex 4-spinor was uniquely written in terms of the orthonormal set $(u_{\vec{p},\pm\vec{s}}, v_{-\vec{p},\pm\vec{s}})$.

We will find the unique expansion of $A^\mu_{\vec{p}}$ as follows: We first boost p^μ and A^μ into the frame where p^μ is at rest, namely $p^\mu = (m, \vec{0}\,)$. In that frame, the condition $p_\mu A^\mu_{\vec{p}} = 0$ becomes

$$m A^0_{\vec{p}} = 0 \quad \to \quad A^\mu_{\vec{p}} = (0, \vec{A}_{\vec{p}}) \quad (\text{rest frame})\,. \tag{5.201}$$

Then, the 3-vector $\vec{A}_{\vec{p}}$ can be expanded using three orthogonal unit vectors $\hat{\epsilon}_{\vec{p}1}$, $\hat{\epsilon}_{\vec{p}2}$, and $\hat{\epsilon}_{\vec{p}3}$. As in the case of the spin direction vector $\vec{s}$ of spin-1/2 field, the orientation of the basis vectors $\hat{\epsilon}_{\vec{p}1,2,3}$ are arbitrary (exept that the three vectors are orthogonal to each other, of course); it could be the x, y, z directions for all $\vec{p}$, or it can depend on $\vec{p}$. Since helicity is a useful quantity, however, let's take one of the basis vectors, say $\hat{\epsilon}_{\vec{p}3}$, to be in the $\vec{p}$ direction. To define the basis vectors uniquely, we may further define $\hat{\epsilon}_{\vec{p}2}$ to be in the direction of $\vec{p} \times \hat{z}$, and then $\hat{\epsilon}_{\vec{p}1}$ is taken so that the three unit vectors $\hat{\epsilon}_{\vec{p}i}$ $(i = 1, 2, 3)$ form a right-handed coordinate system. In the following, the specific way to define the azimuthal orientations of $\hat{\epsilon}_{\vec{p}1}$ and $\hat{\epsilon}_{\vec{p}2}$ is not important as long as $\hat{\epsilon}_{\vec{p}3}$ is in the $\vec{p}$ direction. Thus, $\vec{A}_{\vec{p}}$ in the rest frame can be uniquely expanded as

$$\vec{A}_{\vec{p}} = \sum_{\lambda=1}^{3} a_{\vec{p}\lambda} \hat{\epsilon}_{\vec{p}\lambda} \quad (\text{rest frame})\,, \tag{5.202}$$

or including the time component (which is zero),

$$A^\mu_{\vec{p}} = \sum_{\lambda=1}^{3} a_{\vec{p}\lambda} \epsilon^\mu_{\vec{p}\lambda}\,, \tag{5.203}$$

with

$$\epsilon^\mu_{\vec{p}\lambda} = (0, \hat{\epsilon}_{\vec{p}\lambda}) \quad (\lambda = 1, 2, 3\,,\ \text{rest frame})\,. \tag{5.204}$$

The original $A^\mu_{\vec{p}}$ is obtained by boosting (5.203) back to the original frame, which can be accomplished simply by boosting $\epsilon^\mu_{\vec{p}\lambda}$ in the expansion while keeping the same expansion coefficients $a_{\vec{p}\lambda}$. The boost is given by

$$\begin{aligned} \begin{pmatrix} \epsilon^{0\prime} \\ \epsilon'_\parallel \end{pmatrix} &= \begin{pmatrix} \gamma & \eta \\ \eta & \gamma \end{pmatrix} \begin{pmatrix} \epsilon^0 \\ \epsilon_\parallel \end{pmatrix}\,, \quad \text{with} \quad \eta^\mu \equiv (\gamma, \vec{\eta}) \equiv \Big(\frac{p^0}{m}, \frac{\vec{p}}{m}\Big)\,. \\ \vec{\epsilon}'_\perp &= \vec{\epsilon}_\perp \end{aligned} \tag{5.205}$$

In terms of $\epsilon^0, \epsilon_\parallel, and \vec{\epsilon}_\perp$, the three polarization vectors to be boosted are

$$\begin{array}{l} \epsilon^\mu_{\vec{p}1}: \ \epsilon^0 = 0, \ \ \epsilon_\parallel = 0, \ \ \vec{\epsilon}_\perp = \hat{\epsilon}_{\vec{p}1} \\ \epsilon^\mu_{\vec{p}2}: \ \epsilon^0 = 0, \ \ \epsilon_\parallel = 0, \ \ \vec{\epsilon}_\perp = \hat{\epsilon}_{\vec{p}2} \\ \epsilon^\mu_{\vec{p}3}: \ \epsilon^0 = 0, \ \ \epsilon_\parallel = 1, \ \ \ \vec{\epsilon}_\perp = \vec{0} \end{array} . \tag{5.206}$$

The boosted 4-vectors in the original frame are then (dropping the primes)

$$\begin{array}{l} \epsilon^\mu_{\vec{p}1} = (0, \hat{\epsilon}_{\vec{p}1}) , \\ \epsilon^\mu_{\vec{p}2} = (0, \hat{\epsilon}_{\vec{p}2}) , \\ \epsilon^\mu_{\vec{p}3} = (\eta, \gamma\, \hat{p}) , \end{array} \tag{5.207}$$

forming a basis called the *linear basis*. These 4-vectors are sometimes called the polarization 4-vectors.

The original $A^\mu_{\vec{p}}$ is thus uniquely expanded as (5.203) where $\epsilon^\mu_{\vec{p}\lambda}$ are taken to be the boosted polarization vectors given above. Then, the expansion of $A^\mu(x)$ (5.198) is now written as

$$A^\mu(x) = \sum_{\vec{p}\lambda} \Big(a_{\vec{p}\lambda} \epsilon^\mu_{\vec{p}\lambda}\, e_{\vec{p}}(x) + a^*_{\vec{p}\lambda} \epsilon^{\mu *}_{\vec{p}\lambda}\, e^*_{\vec{p}}(x) \Big) \tag{5.208}$$

or

$$\boxed{A^\mu(x) = \sum_{\vec{p}\lambda} \Big(a_{\vec{p}\lambda} h^\mu_{\vec{p}\lambda}(x) + a^\dagger_{\vec{p}\lambda} h^{\mu *}_{\vec{p}\lambda}(x) \Big)} \tag{5.209}$$

with

$$h^\mu_{\vec{p}\lambda}(x) \stackrel{\text{def}}{\equiv} \epsilon^\mu_{\vec{p}\lambda} e_{\vec{p}}(x) . \tag{5.210}$$

We have written $a^\dagger_{\vec{p}\lambda}$ instead of $a^*_{\vec{p}\lambda}$ anticipating the quantization of the field. At this point, the polarization vectors $\epsilon^\mu_{\vec{p}\lambda}$ are real and the their complex conjugation in the second term of (5.208) are irrelevant; however, they are needed for the helicity basis which we will now discuss.

Alternatively, one can define the *helicity basis* polarization 4-vectors as

$$\begin{array}{rcccl} \epsilon^\mu_{\vec{p}+} & \stackrel{\text{def}}{\equiv} & -\frac{1}{\sqrt{2}}(\epsilon^\mu_{\vec{p}1} + i\epsilon^\mu_{\vec{p}2}) & = & (0, -\frac{1}{\sqrt{2}}(\hat{\epsilon}_{\vec{p}1} + i\hat{\epsilon}_{\vec{p}2})) \\ \epsilon^\mu_{\vec{p}0} & \stackrel{\text{def}}{\equiv} & \epsilon^\mu_{\vec{p}3} & = & (\eta, \gamma\, \hat{p}) \\ \epsilon^\mu_{\vec{p}-} & \stackrel{\text{def}}{\equiv} & +\frac{1}{\sqrt{2}}(\epsilon^\mu_{\vec{p}1} - i\epsilon^\mu_{\vec{p}2}) & = & (0, +\frac{1}{\sqrt{2}}(\hat{\epsilon}_{\vec{p}1} - i\hat{\epsilon}_{\vec{p}2})) . \end{array} \tag{5.211}$$

In the helicity basis, the corresponding expansion coefficients become

$$\begin{array}{rcl} a_{\vec{p}+} & = & -\frac{1}{\sqrt{2}}(a_{\vec{p}1} - i a_{\vec{p}2}) \\ a_{\vec{p}0} & = & a_{\vec{p}3} \\ a_{\vec{p}-} & = & +\frac{1}{\sqrt{2}}(a_{\vec{p}1} + i a_{\vec{p}2}) , \end{array} \tag{5.212}$$

which can be verified as follows. In the momentum expansion (5.208), the sum over the linear basis becomes the same as the sum over the helicity basis:

$$a_{\vec{p}+}\epsilon^\mu_{\vec{p}+} + a_{\vec{p}-}\epsilon^\mu_{\vec{p}-} = a_{\vec{p}1}\epsilon^\mu_{\vec{p}1} + a_{\vec{p}2}\epsilon^\mu_{\vec{p}2}\,, \quad a_{\vec{p}0}\epsilon^\mu_{\vec{p}0} = a_{\vec{p}3}\epsilon^\mu_{\vec{p}3}\,,$$

$$\to \quad \sum_{\lambda=1,2,3} a_{\vec{p}\lambda}\epsilon^\mu_{\vec{p}\lambda} = \sum_{\lambda=+,0,-} a_{\vec{p}\lambda}\epsilon^\mu_{\vec{p}\lambda}\,. \tag{5.213}$$

Thus, the momentum expansion (5.209) is valid for $\lambda = +, 0, -$ (helicity basis) as well as for $\lambda = 1, 2, 3$ (linear basis). Also, the relations

$$\boxed{\epsilon_{\vec{p}\lambda}\cdot\epsilon^*_{\vec{p}\lambda'} = -\delta_{\lambda\lambda'}\,, \quad p\cdot\epsilon_{\vec{p}\lambda} = 0\,,} \tag{5.214}$$

hold for $\lambda = 1, 2, 3$ and $+, 0, -$, which can be trivially proven for the linear basis in the rest frame which can then be extended to the helicity basis using (5.211).

Using the orthonormality conditions of $e_{\vec{p}}(x)$ (4.181) and $\epsilon_{\vec{p}\lambda}\cdot\epsilon^*_{\vec{p}\lambda'} = -\delta_{\lambda\lambda'}$, it is strightforward to show the following orthonomality conditions for the spin-1 normal mode functions $h^\mu_{\vec{p}\lambda}(x)$:

$$\boxed{\begin{array}{rclrcl} \displaystyle\int d^3x\, h^{\mu*}_{\vec{p}\lambda}(x) i\overset{\leftrightarrow}{\partial}_0 h_{\vec{p}'\lambda'\mu}(x) &=& -\delta_{\vec{p}\vec{p}'}\delta_{\lambda\lambda'}\,, & \displaystyle\int d^3x\, h^\mu_{\vec{p}\lambda}(x) i\overset{\leftrightarrow}{\partial}_0 h_{\vec{p}'\lambda'\mu}(x) &=& 0\,, \\ \displaystyle\int d^3x\, h^\mu_{\vec{p}\lambda}(x) i\overset{\leftrightarrow}{\partial}_0 h^*_{\vec{p}'\lambda'\mu}(x) &=& \delta_{\vec{p}\vec{p}'}\delta_{\lambda\lambda'}\,, & \displaystyle\int d^3x\, h^{*\mu}_{\vec{p}\lambda}(x) i\overset{\leftrightarrow}{\partial}_0 h^*_{\vec{p}'\lambda'\mu}(x) &=& 0\,. \end{array}} \tag{5.215}$$

Exercise 5.4 *Prove the above orthonormal relations.*

We can see that the helicity-basis polarization vectors correspond to the angular momentum eigenstates $|j, m\rangle = |1, 1\rangle$, $|1, 0\rangle$, and $|1, -1\rangle$ in the rest frame, where the spin axis is taken as $\hat{\epsilon}_{\vec{p}3}$ (namely, J_3 represents the helicity). First we note that, in the rest frame of the particle, the space components of the polarization 4-vectors are (the time components are zero)

$$\begin{array}{rcll} \vec{\epsilon}_{\vec{p}+} &=& -\frac{1}{\sqrt{2}}(\hat{\epsilon}_{\vec{p}1} + i\hat{\epsilon}_{\vec{p}2}) & \\ \vec{\epsilon}_{\vec{p}0} &=& \hat{\epsilon}_{\vec{p}3} & \text{(rest frame : } \vec{p} = 0)\,. \\ \vec{\epsilon}_{\vec{p}-} &=& +\frac{1}{\sqrt{2}}(\hat{\epsilon}_{\vec{p}1} - i\hat{\epsilon}_{\vec{p}2}) & \end{array} \tag{5.216}$$

On the other hand, using the explicit expressions for $\vec{J}$ (5.174), we see that the eigenstate $|1, 0\rangle$ is given by $(0, 0, 1)$:

$$J_3\begin{pmatrix}0\\0\\1\end{pmatrix} = i\begin{pmatrix}0&-1&0\\1&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix} = 0 \quad \to \quad |1,0\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix}\,. \tag{5.217}$$

The states $|1,1\rangle$ and $|1,-1\rangle$ can be constructed from $|1,0\rangle$ by applying the raising and lowering operators:

$$J_\pm \equiv J_1 \pm iJ_2 = \begin{pmatrix} 0 & 0 & \mp 1 \\ 0 & 0 & -i \\ \pm 1 & i & 0 \end{pmatrix}, \tag{5.218}$$

whose action on $|j,m\rangle$ is given in general by

$$\sqrt{j(j+1) - m(m\pm 1)}|j, m\pm 1\rangle = J_\pm |j,m\rangle\,.$$

For $|j,m\rangle = |1,0\rangle$, we have $j(j+1) - m(m\pm 1) = 2$, and thus

$$\sqrt{2}|1,\pm 1\rangle = J_\pm |1,0\rangle = J_\pm \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \mp 1 \\ -i \\ 0 \end{pmatrix} \quad \to \quad |1,\pm 1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \mp 1 \\ -i \\ 0 \end{pmatrix}. \tag{5.219}$$

Namely, $|1,1\rangle$, $|1,0\rangle$, and $|1,-1\rangle$ are represented in the three-dimensional space by

$$|1,1\rangle = -\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}, \quad |1,0\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad |1,-1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}. \tag{5.220}$$

These are nothing but the polarization vectors $\hat{\epsilon}_{\vec{p}+}$, $\hat{\epsilon}_{\vec{p}0}$, and $\hat{\epsilon}_{\vec{p}-}$ given in (5.216) where the coordinate axes are $\hat{\epsilon}_{\vec{p}i}$ $(i = 1,2,3)$.

In dealing with massive spin-1 particles we often encounter a polarization sum of the form

$$\sum_\lambda \epsilon^\mu_{\vec{p}\lambda} \epsilon^{\nu *}_{\vec{p}\lambda}\,, \tag{5.221}$$

where the polarization vectors refer to a given particle (not different particles). This is the spin-1 equivalent of $\sum_{\vec{s}} u_{\vec{p},\vec{s}} \bar{u}_{\vec{p},\vec{s}}$ and $\sum_{\vec{s}} v_{\vec{p},\vec{s}} \bar{v}_{\vec{p},\vec{s}}$. To evaluate this, we first note that it is a Lorentz tensor; thus, we can evaluate it in the rest frame ($\vec{p} = 0$) and express it in a Lorentz-covariant form, then it will be valid in any frame. In the rest frame and using the linear basis, we have

$$\begin{array}{rcl} \mu & : 0,1,2,3 & \\ \eta^\mu_{\vec{0}} & = & (1,0,0,0) \\ \epsilon^\mu_{\vec{0}1} & = & (0,1,0,0) \\ \epsilon^\mu_{\vec{0}2} & = & (0,0,1,0) \\ \epsilon^\mu_{\vec{0}3} & = & (0,0,0,1)\,, \end{array} \tag{5.222}$$

where $\eta_{\vec{0}}$ is $\eta^\mu \equiv p^\mu/m$ evaluated in the rest frame. If we regard each column as a 4-vector and take the inner product of μ-th column and ν-th column, we see that the following expression holds numerically:

$$(\mu\text{−th column}) \cdot (\nu\text{−th column}) = \eta^\mu_{\vec{0}} \eta^\nu_{\vec{0}} - \sum_{\lambda=1,2,3} \epsilon^\mu_{\vec{0}\lambda} \epsilon^\nu_{\vec{0}\lambda} = \mathrm{g}^{\mu\nu}$$

$$\rightarrow \quad \sum_{\lambda=1,2,3} \epsilon^{\mu}_{\vec{0}\lambda}\epsilon^{\nu}_{\vec{0}\lambda} = -g^{\mu\nu} + \eta^{\mu}_{\vec{0}}\eta^{\nu}_{\vec{0}}\,. \tag{5.223}$$

Now the above equation on the right should be valid in any frame. Let's check it explicitly. We multiply both sides by $\Lambda^{\alpha}{}_{\mu}\Lambda^{\beta}{}_{\nu}$ and sum over μ and ν, where Λ is the boost that takes the rest mass m to p^{μ}:

$$\sum_{\lambda=1,2,3} \underbrace{\Lambda^{\alpha}{}_{\mu}\epsilon^{\mu}_{\vec{0}\lambda}}_{\epsilon^{\alpha}_{\vec{p}\lambda}} \underbrace{\Lambda^{\beta}{}_{\nu}\epsilon^{\nu}_{\vec{0}\lambda}}_{\epsilon^{\beta}_{\vec{p}\lambda}} = -\underbrace{\Lambda^{\alpha}{}_{\mu}\Lambda^{\beta}{}_{\nu}g^{\mu\nu}}_{g^{\alpha\beta}} + \underbrace{\Lambda^{\alpha}{}_{\mu}\eta^{\mu}_{\vec{0}}}_{\eta^{\alpha}} \underbrace{\Lambda^{\beta}{}_{\nu}\eta^{\nu}_{\vec{0}}}_{\eta^{\beta}}\,; \tag{5.224}$$

namely,

$$\sum_{\lambda=1,2,3} \epsilon^{\mu}_{\vec{p}\lambda}\epsilon^{\nu}_{\vec{p}\lambda} = -g^{\mu\nu} + \eta^{\mu}\eta^{\nu}\,. \tag{5.225}$$

Using the definition of the helicity basis (5.211), we can show that the helicity basis version is related to the linear basis version by

$$\sum_{\lambda=+,0,-} \epsilon^{\mu}_{\vec{p}\lambda}\epsilon^{\nu *}_{\vec{p}\lambda} = \sum_{\lambda=1,2,3} \epsilon^{\mu}_{\vec{p}\lambda}\epsilon^{\nu}_{\vec{p}\lambda}\,. \tag{5.226}$$

Then, together with $\eta^{\mu} = p^{\mu}/m$, the relation

$$\boxed{\sum_{\lambda} \epsilon^{\mu}_{\vec{p}\lambda}\epsilon^{\nu *}_{\vec{p}\lambda} = -g^{\mu\nu} + \frac{p^{\mu}p^{\nu}}{m^2} \qquad (\lambda = 1,2,3 \ \text{ or } \ +,0,-)} \tag{5.227}$$

works for both the linear basis and the helicity basis.

Now, we quantize the field by regarding the expansion coefficients as operators and introducing quantization conditions among them. Since the spin is an integer, we have to use commutators instead of anticommutators.

$$\boxed{\begin{array}{rcl} [a_{\vec{p}\lambda}, a^{\dagger}_{\vec{p}'\lambda'}] &=& \delta_{\vec{p},\vec{p}'}\delta_{\lambda,\lambda'}\,, \\ [a_{\vec{p}\lambda}, a_{\vec{p}'\lambda'}] &=& [a^{\dagger}_{\vec{p}\lambda}, a^{\dagger}_{\vec{p}'\lambda'}] = 0\,, \end{array}} \tag{5.228}$$

where one can use either linear or helicity bases, and both are consistent. With the help of the polarization sum formula above, this set of commutators leads to equal-time commutators among fields given by

$$\begin{array}{rcl} [A_i(t,\vec{x}), \pi^j(t,\vec{x}')] &=& i g_i{}^j \delta^3(\vec{x}-\vec{x}')\,, \\ [A_i(t,\vec{x}), A^j(t,\vec{x}')] &=& [\pi_i(t,\vec{x}), \pi^j(t,\vec{x}')] = 0\,. \end{array} \tag{5.229}$$

Exercise 5.5 *Quantization of massive spin-1 field.*
(a) Use the commutation relations of creation and annihilation operators (5.228) and the momentum expansion (5.209), where $\lambda = +,0,-$ or $1,2,3$, to derive the equal-time commutation relations (5.229).

(*b*) *Show that the commutation relation between* $A^\mu(x)$ *and* $A^\nu(x')$ *at two space-time points* x *and* x' *is given by*

$$[A^\mu(x), A^\nu(x')] = \Big(- g^{\mu\nu} - \frac{\partial^\mu_x \partial^\nu_x}{m^2} \Big) i\Delta(x - x') \,. \tag{5.230}$$

Is the microscopic causality satisfied?
(*hint: The conjugate field can be written as*

$$\pi^i = -i \sum_{\vec{p},\lambda} (a_{\vec{p},\lambda} F^i_{\vec{p},\lambda} - a^\dagger_{\vec{p},\lambda} F^{i*}_{\vec{p},\lambda}) \quad \textit{with} \quad F^i_{\vec{p},\lambda} \equiv p^i h^0_{\vec{p},\lambda} - p^0 h^i_{\vec{p},\lambda} \,. \tag{5.231}$$

You will need the polarization sum formula $\sum_\lambda \epsilon^\mu_{\vec{p}\lambda} \epsilon^{\nu *}_{\vec{p}\lambda} = -g^{\mu\nu} + p^\mu p^\nu / m^2$.)

Charged spin-1 field

Just as we combined two real scalar fields of same mass to form a single complex field that represents a 'charged' scalar particle, we can combine two real vector fields of same mass to form a single complex vector field that describes a charged spin-1 particle. Such a field can be used when particle and antiparticle are distinct such as the $W^\pm$ bosons. Thus, we combine two real vector fields of the same mass, A^1_μ and A^2_μ, as

$$A_\mu(x) = \frac{1}{\sqrt{2}} \Big(A^1_\mu(x) + i A^2_\mu(x) \Big) \,. \tag{5.232}$$

The Lagrangian density of the whole is simply the sum of the Lagrangian density of A^1_μ and that of A^2_μ:

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 = -\frac{1}{4} F^i_{\mu\nu} F^{i\mu\nu} + \frac{m^2}{2} A^i_\mu A^{i\mu} \tag{5.233}$$

where the sum over i is implied, and

$$F^i_{\mu\nu} \stackrel{\text{def}}{\equiv} \partial_\nu A^i_\mu - \partial_\mu A^i_\nu \quad (i = 1, 2) \quad (\text{real}) \,. \tag{5.234}$$

Defining the antisymmetric tensor $F_{\mu\nu}$ for the complex field A_μ in the same way,

$$F_{\mu\nu} \stackrel{\text{def}}{\equiv} \partial_\nu A_\mu - \partial_\mu A_\nu \quad (\text{complex}) \,, \tag{5.235}$$

we have

$$F_{\mu\nu} = \frac{1}{\sqrt{2}} \Big[\partial_\nu (A^1_\mu + i A^2_\mu) - \partial_\mu (A^1_\nu + i A^2_\nu) \Big] = \frac{1}{\sqrt{2}} (F^1_{\mu\nu} + i F^2_{\mu\nu}) \,. \tag{5.236}$$

The kinetic term can then be written as

$$F^*_{\mu\nu} F^{\mu\nu} = \frac{1}{2} (F^1_{\mu\nu} - i F^2_{\mu\nu})(F^{1\mu\nu} + i F^{2\mu\nu}) = \frac{1}{2} F^i_{\mu\nu} F^{i\mu\nu} \,. \tag{5.237}$$

Similarly, we have

$$A^*_\mu A^\mu = \frac{1}{2}(A^1_\mu - iA^2_\mu)(A^{1\mu} + iA^{2\mu}) = \frac{1}{2}A^i_\mu A^{i\mu}\,. \tag{5.238}$$

Thus, the total Lagrangian density can be written using the complex field as

$$\boxed{\mathcal{L} = -\frac{1}{2}F^*_{\mu\nu}F^{\mu\nu} + m^2 A^*_\mu A^\mu}\,. \tag{5.239}$$

The momentum expansion of the complex (namely, non-hermitian) vector field proceeds similarly to the charged scalar field case. We define

$$\begin{aligned} a_{\vec{p}\lambda} &\stackrel{\text{def}}{\equiv} \frac{1}{\sqrt{2}}(a_{1\vec{p}\lambda} + ia_{2\vec{p}\lambda}) \\ b_{\vec{p}\lambda} &\stackrel{\text{def}}{\equiv} \frac{1}{\sqrt{2}}(a_{1\vec{p}\lambda} - ia_{2\vec{p}\lambda}) \end{aligned} \tag{5.240}$$

where $a_{i\vec{p}\lambda}$ are the annihilation operators for the vector field $A^i_\mu(x)$. Then, using the expansion (5.209), the non-hermitian field A^μ is written as

$$\boxed{A^\mu(x) = \sum_{\vec{p}\lambda}\left(a_{\vec{p}\lambda}h^\mu_{\vec{p}\lambda}(x) + b^\dagger_{\vec{p}\lambda}h^{\mu*}_{\vec{p}\lambda}(x)\right)}\,, \tag{5.241}$$

where the normal-mode functions $h^\mu_{\vec{p}\lambda}(x)$ have been defined in (5.210). Assuming that operators belonging to different fields commute, the commutation relations among the creation and annihilation operators of the real fields are

$$\begin{gathered} [a^i_{\vec{p}\lambda}, a^{j\dagger}_{\vec{p}'\lambda'}] = \delta_{ij}\delta_{\vec{p},\vec{p}'}\delta_{\lambda,\lambda'}\,, \\ \text{all others } = 0\,. \end{gathered} \tag{5.242}$$

From this, one obtains

$$\boxed{\begin{gathered} [a_{\vec{p}\lambda}, a^\dagger_{\vec{p}'\lambda'}] = [b_{\vec{p}\lambda}, b^\dagger_{\vec{p}'\lambda'}] = \delta_{\vec{p},\vec{p}'}\delta_{\lambda,\lambda'}\,, \\ \text{all others } = 0\,. \end{gathered}} \tag{5.243}$$

As in the case of a charged scalar field, $a^\dagger_{\vec{p}\lambda}$ is regarded as a creation operator of a particle and $b^\dagger_{\vec{p}\lambda}$ as that of its antiparticle. This interpretation is justified by the conserved quantity Q corresponding to the invariance of the Lagrangian density under the phase transformation

$$A'_\mu(x) = e^{i\theta}A_\mu(x) \quad \text{and} \quad A'^*_\mu(x) = e^{-i\theta}A^*_\mu(x)\,, \tag{5.244}$$

where all 4 components of A^μ are phase-rotated by the same amount. Following the same procedure that lead to the conserved current for a single complex field (4.260), the conserved current for 4 complex fields A^μ is

$$\partial_\mu j^\mu = 0\,, \quad j^\mu \equiv i\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A^*_\alpha)}A^*_\alpha - \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\alpha)}A_\alpha\Big)\,. \tag{5.245}$$

The conserved quantity Q is then given by the space integral of the time component of the conserved current:

$$\dot{Q} = 0\,, \quad Q = \int d^3x\, j^0\,. \tag{5.246}$$

Using the Lagrangian (5.239) and the expansion of A^μ (5.241), one obtains (see the exercise below)

$$Q = \sum_{\vec{p}\lambda}(a^\dagger_{\vec{p}\lambda}a_{\vec{p}\lambda} - b^\dagger_{\vec{p}\lambda}b_{\vec{p}\lambda}) \tag{5.247}$$

which shows that an a-particle carries charge $+1$ and a b-particle carries charge -1 regardless of momentum and spin. Again, only the relative sign is important and the overall sign is arbitrary at this point.

Exercise 5.6 *Conserved charge of a complex vector field.*
(*a*) *Show that the conserved current (5.245) for the Lagrangian (5.239) is given by*

$$j^\mu = i(F^{\alpha\mu *}A_\alpha - F^{\alpha\mu}A^*_\alpha)\,.$$

(*b*) *Then, show that the conserved quantity Q can be written as*

$$Q = \int d^3x\,[A^*_\alpha i\overset{\leftrightarrow}{\partial}{}^0 A^\alpha - i\partial_k(A^0A^{*k} - A^{0*}A^k)]\,,$$

where we note that the second term becomes a surface integral and vanishes.
(*hint: Recall the Lorentz condition $\partial_\mu A^\mu = 0$.*)
(*c*) *Use the momentum expansion of A^μ and the orthonormality relations (5.215) to verify $Q = -\sum_{\vec{p}\lambda}(a^\dagger_{\vec{p}\lambda}a_{\vec{p}\lambda} - b^\dagger_{\vec{p}\lambda}b_{\vec{p}\lambda})$ (normal ordering on Q is implicit).*

W-fermion coupling

Weak decays are mostly caused by couplings between the massive charged spin-1 particle $W^\pm$ and fermion currents. Decays such as $W^+ \to e^+\nu_e$, $\pi^+ \to \mu^+\nu_\mu$, and $\mu^- \to \nu_\mu e^-\bar{\nu}_e$ are some examples. Often, W-fermion couplings are hidden inside effective couplings. For example, even though π^+ is a spin-0 particle, quarks (which are fermions) inside couple to W^+ which then creates the $\mu^+\nu_\mu$ pair. In $\mu^- \to \nu_\mu e^-\bar{\nu}_e$, the current $\mu^- \to \nu_\mu$ couples to W^- which then creates the $e^-\bar{\nu}_e$ pair.

The W^+ particle can be described by the charged spin-1 field introduced in the previous section. The $W - e\nu_e$ coupling is a $V - A$ coupling given by

$$\mathcal{L}_{\rm int} = \frac{g}{\sqrt{2}}\big(\bar{\nu}_e\gamma_\mu P_L e\big)W^\mu + h.c. \qquad \Big(P_L \equiv \frac{1-\gamma_5}{2}\Big)\,, \tag{5.248}$$

where $W^\mu(x)$ is the charged vector field for $W^\pm$ (W^+ = particle, W^- = antiparticle) and $\nu_e(x)$ and $e(x)$ are the short hands for $\psi_{\nu_e}(x)$ and $\psi_e(x)$, respectively. The constant g is the universal coupling constant $g \sim 0.65$.

In the standard model, leptons appear in three 'generations':

$$\begin{array}{ll} Q=0 & \begin{pmatrix}\nu_e\\ e^-\end{pmatrix} \quad \begin{pmatrix}\nu_\mu\\ \mu^-\end{pmatrix} \quad \begin{pmatrix}\nu_\tau\\ \tau^-\end{pmatrix}, \\ Q=-1 & \end{array} \tag{5.249}$$

where Q is the electrical charge and all particles shown are fermions as opposed to antifermions (namely, each has its own antifermion). They have exactly the same types of interactions, including weak interactions, with only difference being the charged lepton masses

$$m_e = 0.000511\text{GeV}\,, \quad m_\mu = 0.106\text{GeV}\,, \quad m_\tau = 1.78\text{GeV}\,. \tag{5.250}$$

At this time, we assume that all neutrinos are massless. In a later chapter, we will discuss the case of massive neutrinos. Thus, W-lepton coupling in the standard model is written as

$$\boxed{\mathcal{L}_{\text{int}} = \frac{g}{\sqrt{2}}\big(\bar\nu_i\gamma_\mu P_L \ell_i\big)W^\mu + h.c. \quad (i=1,2,3)}\,, \tag{5.251}$$

with

$$(\nu_1,\nu_2,\nu_3) \equiv (\nu_e,\nu_\mu,\nu_\tau)\,, \qquad (\ell_1,\ell_2,\ell_3) \equiv (e,\mu,\tau)\,, \tag{5.252}$$

and the sum over $i=1,2,3$ is implied. There is no off-diagonal couplings such as $(\bar\nu_e\gamma_\mu(1-\gamma_5)\mu)W^\mu$, which is a consequance of the assumed masslessness of neutrinos (strictly speaking, that they have the same mass).

Quarks also come in three generations,

$$\begin{array}{ll} Q=+2/3 & \begin{pmatrix}u\\ d\end{pmatrix} \quad \begin{pmatrix}c\\ s\end{pmatrix} \quad \begin{pmatrix}t\\ b\end{pmatrix}, \\ Q=-1/3 & \end{array} \tag{5.253}$$

and couple to W in a similar way:

$$\boxed{\mathcal{L}_{\text{int}} = \frac{g}{\sqrt{2}}V_{ij}\big(\bar U_i\gamma_\mu P_L D_j\big)W^\mu + h.c. \quad (i=1,2,3)}\,, \tag{5.254}$$

where

$$(U_1,U_2,U_3) \equiv (u,c,t)\,, \qquad (D_1,D_2,D_3) \equiv (d,s,b)\,, \tag{5.255}$$

and sum over $i,j=1,2,3$ is implied. The set of constants V_{ij} is called the Cabibbo-Kobayashi-Masukawa matrix (the CKM matrix) and the element V_{ij} specifies the strength of coupling between the U_i-D_j current and W in units of the W-$e\nu_e$ coupling. It turns out that V is unitary (the standard model requires it theoretically, but it is

also consistent with experiment) and that the diagonal elements V_{ii} are nearly unity while cross-generational couplings are suppressed but finite. Approximately, the sizes of $|V_{ij}|$ are given by

$$|V_{ij}| \sim \begin{pmatrix} 1 & \lambda & \lambda^3 \\ \lambda & 1 & \lambda^2 \\ \lambda^3 & \lambda^2 & 1 \end{pmatrix}, \tag{5.256}$$

where $\lambda \sim 0.22$ is called the Cabibbo factor. Also, it is believed that there are relative complex phases between V_{ij} which result in violation of CP symmetry as will be discussed in a later chapter. For example, the W-tb coupling is given by

$$\mathcal{L}_{\rm int} = \frac{g}{\sqrt{2}} \underbrace{V_{tb}}_{\sim 1} \Big(\bar{t}\gamma_\mu P_L b\Big) W^\mu + h.c. \tag{5.257}$$

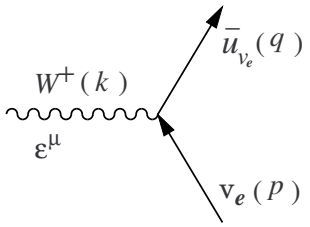
Actually, quarks come in three colors, and the same interaction above repeats for each color. That is, the current $\bar{U}_i\gamma_\mu P_L D_j$ is understood to be

$$\bar{U}_i\gamma_\mu P_L D_j \equiv \sum_a \bar{U}_i^a \gamma_\mu P_L D_j^a \tag{5.258}$$

where the sum is over a = red, blue, and green. The W-quark coupling does not change color, namely there is no coupling with color-changing currents.

W decays

We will first calculate the decay $W^+ \to e^+\nu_e$. Other decays such as $W^+ \to \mu^+\nu_\mu$ are treated in essentially the same way. The interaction responsible is given by (5.248) :



$$\mathcal{L}_{\rm int} = \frac{g}{\sqrt{2}}(\bar{\nu}_e\gamma_\mu P_L e)W^\mu\,. \tag{5.259}$$

The 4-momenta of W^+, e^+, and ν_e are k, p and q, respectively, and ϵ^μ represents the polarization of W^+. The masses of e^+ and ν_e are assumed to be zero. What is the Lorentz-invariant matrix element? The situation is nearly identical to that of $H \to b\bar{b}$. This time we have

$$\langle f|\mathcal{L}_{\rm int}|i\rangle = \langle 0|b_e a_{\nu_e} \frac{g}{\sqrt{2}}(\bar{\nu}_e\gamma_\mu P_L e)W^\mu a_W^\dagger|0\rangle \tag{5.260}$$

where the subscripts e, ν_e, and W for the creation or annihilation operators indicate that they are for the corresponding particles of the initial and final states. The rule

for the final state fermion or antifermion stays the same; namely, ν_e will pick up $\bar{u}_{\nu_e}$ and e^+ will pick up v_e. When the vector field W^μ is expanded as

$$W^\mu = \sum_{\vec{p}\lambda} \left(a_{\vec{p}\lambda} h^\mu_{\vec{p}\lambda} + b^\dagger_{\vec{p}\lambda} h^{\mu *}_{\vec{p}\lambda} \right), \tag{5.261}$$

the annihilation operator $a_{\vec{k}\lambda}$ that matches $a^\dagger_W$ ($\equiv a^\dagger_{\vec{k}\lambda}$) of the initial state comes with the factor $h^\mu_{\vec{k}\lambda}(x) = \epsilon^\mu_{\vec{k}\lambda} e_{\vec{k}}(x)$. The $e_{\vec{k}}(x)$ factor will become part of the delta function for energy-momentum conservation and the normalization in the definition of $\mathcal{M}$ and $\epsilon^\mu_{\vec{k}\lambda}$ will be included in $\mathcal{M}$:

$$S_{fi} = i \int d^4x \langle f | \mathcal{L}_{\rm int} | i \rangle = \frac{(2\pi)^4 \delta^4(p+q-k)}{\sqrt{(2p^0V)(2q^0V)(2k^0V)}} \mathcal{M} \tag{5.262}$$

with

$$\mathcal{M} = i \frac{g}{\sqrt{2}} (\bar{u}_{\nu_e} \gamma_\mu P_L v_e) \epsilon^\mu \tag{5.263}$$

where $\epsilon^\mu \equiv \epsilon^\mu_{\vec{k}\lambda}$. If the initial state is W^-, then the matching annihilation operator $b_{\vec{k}\lambda}$ is in $W^{\mu\dagger}$ which appears in the hermitian conjugate term, and the normal-mode function associated is again $h^\mu_{\vec{k}\lambda}(x)$. We note that regardless of particle or antiparticle, an annihilation operator is associated with $h^\mu_{\vec{p}\lambda}$ and a creation operator with $h^{\mu *}_{\vec{p}\lambda}$. Thus, the rule is to include in $\mathcal{M}$ the factor ϵ^μ for a spin-1 particle in the initial state, and the factor $\epsilon^{\mu *}$ for the final state regardless of particle or antiparticle:

	initial state	final state
spin-1	$\epsilon^\mu_{\vec{p}\lambda}$	$\epsilon^{\mu *}_{\vec{p}\lambda}$

(for particle or antiparticle) . (5.264)

Let's proceed to calculate the decay rate of $W^+ \to e^+ \nu_e$. Since the $e^+\nu_e$ pair is created by a $V-A$ current, e^+ is right-handed (to the extent that m_e is small) and ν_e is left-handed. We will, however, sum over the spins of e^+ and ν_e to take advantage of the trace techniques. We assume that the initial state W^+ is unpolarized, and thus take average over the three possible helicity states or equivalently the three linear polarizations. Thus, we will evaluate the sum of $|\mathcal{M}|^2$ over all spins and divide by three to obtain the spin-averaged (unpolarized) $|\mathcal{M}|^2$: with $m_W \equiv m$ and $m_e = m_{\nu_e} = 0$,

$$\begin{aligned} \overline{|\mathcal{M}|^2} &= \frac{1}{3} \sum_{\rm spins} |\mathcal{M}|^2 \\ &= \frac{g^2}{3 \cdot 2} \sum_{\rm spins} (\bar{u}_{\nu_e} \gamma_\mu P_L v_e \, \epsilon^\mu)^* (\bar{u}_{\nu_e} \gamma_\nu P_L v_e \, \epsilon^\nu) \end{aligned}$$

$$= \frac{g^2}{6}\sum_{\text{spins}} \bar{v}_e P_R \gamma_\mu u_{\nu_e} \bar{u}_{\nu_e} \gamma_\nu P_L v_e \underbrace{\sum_{\text{spins}} \epsilon^{\mu *}\epsilon^{\nu}}_{-g^{\mu\nu}+\frac{k^\mu k^\nu}{m^2}}$$

$$= \frac{g^2}{6}\Big[-\sum_{\text{spins}} \bar{v}_e P_R \gamma_\mu u_{\nu_e} \bar{u}_{\nu_e} \gamma^\mu P_L v_e + \frac{1}{m^2}\sum_{\text{spins}} \bar{v}_e P_R \not{k} u_{\nu_e} \bar{u}_{\nu_e} \not{k} P_L v_e \Big] . \tag{5.265}$$

Now, the second term vanishes as follows: the Dirac equations in momentum space for u_{ν_e} and v_e are (with $m_e = m_{\nu_e} = 0$)

$$\not{p} v_e = 0\,, \quad \not{q} u_{\nu_e} = 0\,; \tag{5.266}$$

thus, using $k = p + q$, we have

$$\bar{v}_e P_R \not{k} u_{\nu_e} = \bar{v}_e P_R (\not{p} + \underbrace{\not{q}}_{0}) u_{\nu_e} = \bar{v}_e \underbrace{P_R \not{p}}_{\not{p} P_L} u_{\nu_e} = \underbrace{\overline{\not{p} v_e}}_{0} P_L u_{\nu_e} = 0\,. \tag{5.267}$$

Incidentally, a similar calculation shows that

$$[\bar{w}_{\vec{p},\vec{s}} \gamma_\mu (a + b\gamma_5) w'_{\vec{p}',\vec{s}'}] k^\mu = 0 \quad (w, w' = u \text{ or } v, \text{ massless}) \tag{5.268}$$

where a and b are arbitrary constants and k is any linear combination of p and p'. Thus, discarding the second term in (5.265) and executing the spin sums, we have

$$\overline{|\mathcal{M}|^2} = -\frac{g^2}{6}\sum_{\text{spins}} \bar{v}_e P_R \gamma_\mu \underbrace{u_{\nu_e}\bar{u}_{\nu_e}}_{\to \not{q}} \gamma^\mu P_L v_e = \frac{g^2}{3} \underbrace{\text{Tr}\not{p} \underbrace{P_R \not{q}}_{\not{q} P_L} P_L}_{\not{q} P_L}$$

$$= \frac{g^2}{6} \underbrace{\text{Tr}\not{p}\not{q}}_{4p\cdot q} (1 - \not{\gamma}_5) = \frac{2g^2}{3} p \cdot q\,, \tag{5.269}$$

where we have used $\gamma_\mu \not{a} \gamma^\mu = -2\not{a}$, $P_L^2 = P_L$, $\text{Tr}\not{a}\not{b}\gamma_5 = 0$, and $\text{Tr}\not{a}\not{b} = 4a \cdot b$.

Squaring $k = p + q$ and using $m_e = m_\nu = 0$, we get $m^2 = 2p \cdot q$, or

$$p \cdot q = \frac{m^2}{2}\,. \tag{5.270}$$

Using the 2-body decay rate formula $\Gamma = (|\vec{p}|/8\pi m^2)\overline{|\mathcal{M}|^2}$, we obtain,

$$\Gamma(W^+ \to e^+ \nu_e) = \frac{g^2}{48\pi} m\,. \tag{5.271}$$

With $g = 0.65$ and $m = 80$ GeV, we obtain $\Gamma(W^+ \to e^+\nu_e) = 0.224$ GeV. The total decay rate of W is obtained by summing up the partial decay rates over all possible final states, which are

$$\begin{array}{rccccccc} \text{mode}: & e^+\nu_e & \mu^+\nu_\mu & \tau^+\nu_\tau & u\bar{d} & u\bar{s} & u\bar{b} & c\bar{d} & c\bar{s} & c\bar{b} \\ \Gamma: & 1 & 1 & 1 & |V_{ud}|^2 & |V_{us}|^2 & |V_{ub}|^2 & |V_{cd}|^2 & |V_{cs}|^2 & |V_{cb}|^2 \end{array} \tag{5.272}$$

where the decay rates are given in the unit of $\Gamma(W^+ \to e^+\nu_e)$. The W boson can also couple to $t\bar{d}$, $t\bar{s}$, and $t\bar{b}$; the mass of t (~ 175 GeV), however, is heavier than that of W, and thus such decays are prohibited. We have assumed that the fermion masses are small, which is a good assumption since the heaviest in the list is $m_b \sim 5$ GeV which is still much smaller than $m_W \sim 80$ GeV. Recalling that quarks come in three colors, and using the unitarity of the CKM matrix

$$\sum_i V_{ij}^* V_{ik} = \delta_{jk}\,, \tag{5.273}$$

the total decay rate is

$$\begin{aligned} \Gamma_{\text{tot}} &= \Gamma(W^+ \to e^+\nu_e)\Big(\underset{\substack{\uparrow \\ e,\mu,\tau}}{3} + 3(\underbrace{|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2}_{1} + \underbrace{|V_{cd}|^2 + |V_{cs}|^2 + |V_{cb}|^2}_{1})\Big) \\ &= 2.02 \text{ GeV}\,. \end{aligned} \tag{5.274}$$

The experimental value is $\Gamma_{\text{tot}} = 2.08 \pm 0.07$ GeV. Since the value of g is determined from elsewhere - as we will determine from the decay $\mu^- \to \nu_\mu e^- \bar{\nu}_e$ later - the agreement is quite remarkable. This also supports that quarks indeed come in three colors.

5.6 $\nu_\mu e^- \to \mu^- \nu_e$ scattering

Next, we will consider another W-mediated interaction: the scattering of an muon neutrino on an atomic electron $\nu_\mu e^- \to \mu^- \nu_e$. Two main purposes of this exercise are to introduce *scattering cross section* and *W-propagator*. Let's start from the first.

Scattering cross section

Consider a general scattering interaction $a + b \to 1 + 2 + \ldots + n$, where a is the projectile particle, b is the target particle, and $1, 2, \ldots, n$ are the final-state particles. The concept of scattering cross section is just about the simplest way to define the 'likelihood' that a projectile interacts with a target. When single projectile particle is traveling with velocity v in a uniform target medium where the density of the target is n particles per unit volume, the probability to 'encounter' a target particle would

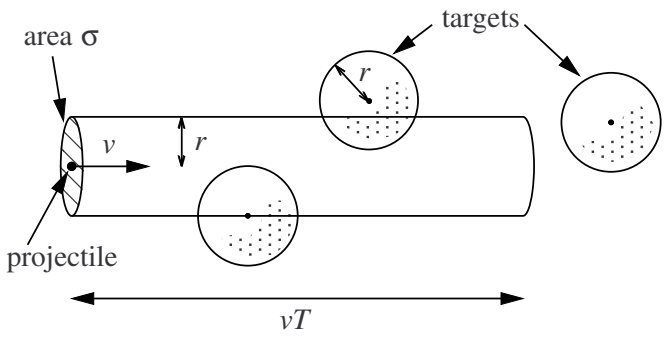


Figure 5.3: When a target center is located within the cylinder of cross section $\sigma = \pi r^2$ and length vT centered around the trajectory of the projectile, the point projectile will hit the target.

be proportional to the time duration T, to the velocity v, and to the density of the target n (assuming that the probability is much less than unity). When these trivial factors are removed, then what is left as the measure of the rate of the interaction is the scattering cross section which happens to have the dimension of area.

As a simple example, consider the classical case where each target is a sphere of radius r which is at rest in the laboratory frame, and the radius of the projectile is small enough to be neglected. The probability that the projectile will hit a target in time T is the probability that a center of target is located inside the cylinder of cross section $\sigma = \pi r^2$ and length vT centered around the trajectory of the projectile (Figure 5.3). If the probability is small enough, we can neglect the chance of finding multiple targets in the cylinder. Then, the probability, or the expected number of collisions N^0, is given by

$$N^0 = n \times (\text{volume of the cylinder}) = n\,\sigma v T\,. \tag{5.275}$$

This formula is valid even when the target medium is also moving as long as the velocity of the projectile ($\vec{v}_a$) and that of target ($\vec{v}_b$) are parallel to each other. In such case, n is still measured in the laboratory frame, and the velocity v is understood to be the relative velocity of projectile and target measured in the laboratory frame:

$$v \stackrel{\text{def}}{\equiv} |\vec{v}_a - \vec{v}_b|\,. \tag{5.276}$$

Now, suppose that there are ρ projectiles in unit volume measured in the laboratory frame. Then, there are ρV projectiles in a volume V on average, and for each projectile, the probability to interact in time T is $N^0 = n\,\sigma v T$. Thus, the probability N to see an interaction in the volume V in the time duration T is

$$N = \rho V N^0 = \rho V n\,\sigma v T\,. \tag{5.277}$$

For a general scattering $a + b \to 1 + 2 + \ldots + n$ where the target does not have a classical target area, the relation (5.277) defines the cross section σ. In the classical example above, the cross section did not depend on the relative velocity v; in general, however, cross section may depend on v or the momenta of the particles involved.

The differential cross section $d\sigma$ for the final-state particles to scatter into the momentum ranges $\vec{p}_f \in d^3p_f$ $(f = 1, \ldots, n)$ is then defined by

$$dN = \rho V n \, d\sigma v T \,, \tag{5.278}$$

where dN is the probability to find the final state in the specified momentum ranges. On the other hand, the same quantity dN is given by summing $|S_{fi}|^2$ over the corresponding final states. Since the initial state $|i\rangle = a_a^\dagger a_b^\dagger |0\rangle$ corresponds to having one projectile partile and one target particle in the entire volume V, we have

$$\rho = \frac{1}{V} : \text{ projectile density} \,, \quad n = \frac{1}{V} : \text{ target density} \,. \tag{5.279}$$

Thus,

$$dN = \sum_{\vec{p}_f \in d^3p_f} |S_{fi}|^2 = \rho V n \, d\sigma \, vT = d\sigma \, \frac{vT}{V} \,. \tag{5.280}$$

Using the definition of the Lorentz-invariant matrix element $\mathcal{M}$ (5.109)

$$S_{fi} \equiv \frac{(2\pi)^4 \delta^4(p_a + p_b - \sum_f p_f)}{\sqrt{(2p_a^0 V)(2p_b^0 V) \prod_f (2p_f^0 V)}} \, \mathcal{M} \,, \tag{5.281}$$

and the identity (5.89), we have

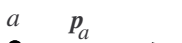
$$\begin{aligned} d\sigma &= \frac{V}{vT} \sum_{\vec{p}_f \in d^3p_f} |S_{fi}|^2 \\ &\qquad \overbrace{(2\pi)^4 \delta^4(p_a + p_b - \textstyle\sum_f p_f) VT}^{\text{by (5.87)}} \\ &= \frac{V}{vT} \frac{\overbrace{[(2\pi)^4 \delta^4(p_a + p_b - \sum_f p_f)]^2}}{(2p_a^0 V)(2p_b^0 V) \prod_f (2p_f^0 V)} \, |\mathcal{M}|^2 \prod_f \frac{V}{(2\pi)^3} d^3p_f \\ &= \frac{(2\pi)^4}{2p_a^0 2p_b^0 v} |\mathcal{M}|^2 \delta^4(p_a + p_b - \sum_f p_f) \prod_f \frac{d^3p_f}{(2\pi)^3 2p_f^0} \,, \end{aligned} \tag{5.282}$$

or

$$\boxed{d\sigma = \frac{(2\pi)^4}{4E_a E_b \, v} |\mathcal{M}|^2 d\Phi_n} \tag{5.283}$$

where the n-body Lorentz-invariant phase space is given by (5.111) with $P = p_a + p_b$ (the total 4-momentum of the system), and v is the relative velocity between projectile and target as defined in (5.276).

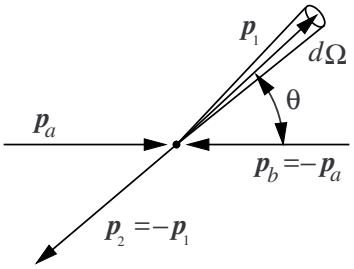
The factor $E_a E_b v$ is sometimes called the 'flux factor', and they can be written in two typical cases as



$$(b \text{ at rest}) \quad E_a E_b v = \overbrace{E_b}^{m_b} \overbrace{E_a v}^{|\vec{p}_a|} = m_b |\vec{p}_a| \tag{5.284}$$

$$\begin{aligned} (\text{C.M.}) \quad E_a E_b v &= E_a E_b (|v_a| + |v_b|) = \overbrace{E_a |v_a|}^{|\vec{p}_a|} E_b + E_a \overbrace{E_b |v_b|}^{|\vec{p}_b|} \\ &= \overbrace{(E_a + E_b)}^{M \,:\, \text{invariant mass}} |\vec{p}| = M |\vec{p}| \,. \end{aligned} \tag{5.285}$$

Following the calculation of the two-body phase space in the C.M. frame (5.98), but this time without integrating over the direction of $\vec{p}_1$, we obtain (left as an exercise)



$$d\Phi_2 = \frac{|\vec{p}_1|}{(2\pi)^6 4M} d\Omega \quad (\text{C.M.}) \,, \tag{5.286}$$

where $M = E_a + E_b$ is the invariant mass or the total energy in the C.M. frame, and $d\Omega = d\phi d\cos\theta$ is the angular element of $\vec{p}_1$ where (θ, ϕ) is the polar coordinates of $\vec{p}_1$ with respect to $\vec{p}_a$. Together with the cross section formula (5.283) and $E_a E_b v = M|\vec{p}_a|$, we have

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{|\mathcal{M}|^2}{(8\pi M)^2} \frac{|\vec{p}_1|}{|\vec{p}_a|} \quad (a + b \to 1 + 2, \text{ C.M.}) \,.} \tag{5.287}$$

If the above cross section does not depend on the azimuthal angle ϕ, then it is a function only of θ only, or

$$t \stackrel{\text{def}}{\equiv} (p_1 - p_a)^2 = m_1^2 + m_a^2 - 2(E_1 E_a - |\vec{p}_1||\vec{p}_a| \cos\theta) \,. \tag{5.288}$$

Changing the variable from θ to t, we obtain (left as an exercise)

$$\boxed{\frac{d\sigma}{dt} = \frac{|\mathcal{M}|^2}{16\pi\lambda(M^2, m_a^2, m_b^2)} \quad (a + b \to 1 + 2)} \tag{5.289}$$

with

$$\lambda(x,y,z) \stackrel{\text{def}}{\equiv} x^2+y^2+z^2-2xy-2yz-2zx\,, \tag{5.290}$$

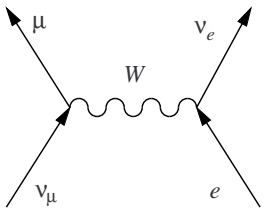
and this formula is valid in any frame. For completeness, we give a general 2-body formula for the case b is at rest:

$$\boxed{\frac{d\sigma}{d\Omega_1} = \frac{|\vec{p}_1||\mathcal{M}|^2}{64\pi^2 m_b|\vec{p}_a|\Big[m_b + E_a(1-\frac{\beta_a}{\beta_1}\cos\theta_1)\Big]} \quad (a+b\to 1+2,\ b \text{ at rest})\,.} \tag{5.291}$$

where $d\Omega_1 = d\phi_1 d\cos\theta_1$ is the angular element of $\vec{p}_1$ with respect to the direction of $\vec{p}_a$.

Exercise 5.7 *Verify (5.286), (5.289), and (5.291).*

Let's get back to the scattering $\nu_\mu e^- \to \mu^- \nu_e$. This process occurs through W exchange between $e\nu_e$ current and $\nu_\mu\mu$ current. The responsible interaction is a part of the general W-lepton couplings (5.251):



$$\begin{aligned}\mathcal{L}_{\text{int}} &= \frac{g}{\sqrt{2}}\Big(i_\alpha(x)+j_\alpha(x)\Big)W^\alpha(x) + h.c.\\ &\text{with } \begin{cases} i_\alpha = \bar{\nu}_e\gamma_\alpha P_L e \\ j_\alpha = \bar{\nu}_\mu\gamma_\alpha P_L\,\mu \end{cases},\end{aligned} \tag{5.292}$$

where we have defined i_α to be the $e\nu_e$ current and j_α the $\nu_\mu\mu$ current. The initial and final states are

$$|i\rangle = a_e^\dagger a_{\nu_\mu}^\dagger|0\rangle\,, \quad |f\rangle = a_{\nu_e}^\dagger a_\mu^\dagger|0\rangle\,; \tag{5.293}$$

thus, the non-vanishing term in $S_{fi} = \langle f|S|i\rangle$ should contain the product of four creation and annihilation operators $a_\mu^\dagger a_{\nu_e}^\dagger a_{\nu_\mu} a_e$. When the fields are momentum-expanded in $\mathcal{L}_{\text{int}}$, however, each term contains a product of only two fermion operators; so the first-order term in the Dyson series vanishes. Let's try the second-order term

$$\begin{aligned} S &= \frac{(-i)^2}{2}\int dt\,dt'\,T(h(t)h(t'))\\ &= -\frac{1}{2}\int dt\,dt'\,T\Big(\int d^3x\mathcal{H}_{\text{int}}(t,\vec{x})\int d^3x'\mathcal{H}_{\text{int}}(t',\vec{x}')\Big)\\ &\quad \text{(using the linearity of } T\text{-product)}\\ &= -\frac{1}{2}\int d^4x\,d^4x'\,T(\mathcal{H}_{\text{int}}(x)\mathcal{H}_{\text{int}}(x'))\,. \end{aligned} \tag{5.294}$$

Since there is no time derivatives in $\mathcal{L}_{\rm int}$, we have $\mathcal{L}_{\rm int} = -\mathcal{H}_{\rm int}$; then the S-matrix element is

$$S_{fi} = -\frac{1}{2}\int d^4x\, d^4x' \langle f|T(\mathcal{L}_{\rm int}(x)\mathcal{L}_{\rm int}(x')|i\rangle \quad \text{(second order)}\,. \tag{5.295}$$

Using the interaction Lagrangian (5.292) and writing out the hermitian conjugate parts, we have

$$\begin{aligned}
&\langle f|T(\mathcal{L}_{\rm int}\mathcal{L}'_{\rm int})|i\rangle \\
&\quad = \frac{g^2}{2}\langle 0|a_\mu a_{\nu_e} T\Big(\Big[(\boxed{i_\alpha} + j_\alpha)W^\alpha + (i^\dagger_\alpha + \boxed{j^\dagger_\alpha})W^{\alpha\dagger}\Big] \\
&\qquad\qquad \times\Big[(\boxed{i'_\beta} + j'_\beta)W^{\beta\prime} + (i'^\dagger_\beta + \boxed{j'^\dagger_\beta})W^{\beta\prime\dagger}\Big]\Big) a^\dagger_e a^\dagger_{\nu_\mu}|0\rangle\,,
\end{aligned} \tag{5.296}$$

where the primed quantities are understood to be functions of x':

$$i_\alpha \equiv i_\alpha(x)\,, \quad j'_\beta \equiv j_\beta(x') \quad \text{etc.} \tag{5.297}$$

There are many terms, but which will survive? Out of the required $a^\dagger_\mu a^\dagger_{\nu_e} a_{\nu_\mu} a_e$, the pair $a^\dagger_{\nu_e} a_e$ is found in $i_\alpha = \bar\nu_e\gamma_\alpha P_L e$, and the pair $a^\dagger_\mu a_{\nu_\mu}$ is found in $j^\dagger_\alpha = \bar\mu\gamma_\alpha P_L\,\nu_\mu$. Thus, the surviving terms should contain both i_α and $j^\dagger_\alpha$ where they could be functions of x or x' and the indices could be α or β. There are two such terms:

$$\begin{aligned}
&\langle f|T(\mathcal{L}_{\rm int}\mathcal{L}'_{\rm int})|i\rangle \\
&\quad = \frac{g^2}{2}\langle 0|a_\mu a_{\nu_e}\Big[\underbrace{T(i_\alpha W^\alpha j'^\dagger_\beta W^{\beta\prime\dagger})}_{(a)} + \underbrace{T(j^\dagger_\alpha W^{\alpha\dagger} i'_\beta W^{\beta\prime})}_{(b)}\Big] a^\dagger_e a^\dagger_{\nu_\mu}|0\rangle\,.
\end{aligned} \tag{5.298}$$

Now, the two T-products gives identical results when integrated over x and x', which can be seen as follows: with $f(x) \equiv i_\alpha W^\alpha$ and $g(x) \equiv j^\dagger_\alpha W^{\alpha\dagger}$,

$$(a) = T\Big(f(x)g(x')\Big) \overset{x\leftrightarrow x'}{\longrightarrow} T\Big(f(x')g(x)\Big) = T\Big(g(x)f(x')\Big) = (b)\,, \tag{5.299}$$

where we noted that the ordering of $f(x')$ and $g(x)$ inside the T-product is irrelevant since the ordering is uniquely defined by which of x^0 and $x^{0\prime}$ is larger after all. Thus, (a) and (b) are related by the exchange $x \leftrightarrow x'$, and thus give the same value when integrated over x and x'. We will thus evaluate the first term only and then multiply the result by two which will cancel the factor 1/2 of (5.295).

Remark

The n-th order term of the Dyson series is (by the same derivation that led to the second order expression)

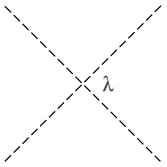
$$S = \frac{(-i)^n}{n\,!}\int d^4x_1 \ldots d^4x_n T\Big(\mathcal{H}_{\rm int}(x_1)\ldots\mathcal{H}_{\rm int}(x_n)\Big)\,, \tag{5.300}$$

where terms related by permutation of $(x_1, \ldots, x_n)$ give identical results as we have seen for the second order case. There are $n\,!$ such terms and it cancels the factor $1/n\,!$ in the expression above. Thus, we can ignore the factor $1/n\,!$ of the Dyson series if we evaluate only one term out of those related by the permutation of $(x_1, \ldots, x_n)$. When same field appears m times ($m > 1$) in $\mathcal{L}_{\rm int}$, where ϕ and $\phi^\dagger$ of a charged field are considered different since they contain different creation and annihilation operators, then this protocol (usually) under-counts the number of combinations by factor of $m\,!$ for each space-time integration. For example, if the interaction is $\mathcal{L}_{\rm int} = \lambda\phi^4$, then the first order matrix element for $1 + 2 \to 3 + 4$ would be

$$S_{fi} = i\lambda \int d^4x \, \langle 0 | a_{\vec{p}_3} a_{\vec{p}_4} \phi^4(x) a^\dagger_{\vec{p}_1} a^\dagger_{\vec{p}_2} | 0 \rangle \tag{5.301}$$

where there are $m\,!$ ways to match $a_{\vec{p}_3} a_{\vec{p}_4}$ and $a^\dagger_{\vec{p}_1} a^\dagger_{\vec{p}_2}$ with those in ϕ^4 each giving identical result. Such a factor is usually taken into account by re-defining (augumenting) the coupling constant by the factor $m\,!$: e.g.

$$\mathcal{L}_{\rm int} = \frac{1}{4\,!}\lambda\phi^4 \,. \qquad \tag{5.302}$$



Then, if we take the vertex factor to be simply λ and proceed with the Feynman rules, the factor $4\,!$ above will be properly accounted for. There are exceptions, however, which occur when there are extra symmetries in the diagram. Such symmetry factors will be dealt with when we encounter them. ∎

Note that if one operator, say $A_k(t_k)$, commutes with all others in a time-ordered product $T(A_1(t_1) \ldots A_n(t_n))$, then it can come out of the time-ordered product:

$$\begin{aligned} T\Big(A_1(t_1) \ldots A_n(t_n)\Big) &= A_{i_1}(t_{i_1}) \ldots A_{i_n}(t_{i_n}) \qquad\qquad (t_{i_1} \geq \ldots \geq t_{i_n}) \\ &= A_k(t_k)\, A_{i_1}(t_{i_1}) \ldots \cancel{A_k(t_k)} \ldots A_{i_n}(t_{i_n}) \qquad (t_{i_1} \geq \ldots \geq t_{i_n}) \\ &= A_k(t_k)\, T\Big(A_{i_1}(t_{i_1}) \ldots \cancel{A_k(t_k)} \ldots A_{i_n}(t_{i_n})\Big) \,. \end{aligned} \tag{5.303}$$

In $T(i_\alpha W^\alpha j'^\dagger_\beta W^{\beta\prime\dagger})$, i_α and $j'^\dagger_\beta$ commute with each other and with W fields; thus, i_α and $j'^\dagger_\beta$ can come out of the T-product. Then, the first term in (5.298) is

$$\begin{aligned} &\langle 0 | a_\mu a_{\nu_e} i_\alpha j'^\dagger_\beta T(W^\alpha W^{\beta\prime\dagger}) a^\dagger_e a^\dagger_{\nu_\mu} | 0 \rangle \\ &\quad = \langle 0 | T(W^\alpha \overbrace{W^{\beta\prime\dagger})}^{1 = \sum_i |i\rangle\langle i|} a_\mu a_{\nu_e} i_\alpha j'^\dagger_\beta a^\dagger_e a^\dagger_{\nu_\mu} | 0 \rangle \\ &\quad = \langle 0 | T(W^\alpha W^{\beta\prime\dagger}) | 0 \rangle \langle 0 | a_\mu a_{\nu_e} i_\alpha j'^\dagger_\beta a^\dagger_e a^\dagger_{\nu_\mu} | 0 \rangle \,. \end{aligned} \tag{5.304}$$

We have inserted $1 = \sum_i |i\rangle\langle i|$, where i runs over all basis states, of which only $|0\rangle\langle 0|$ survives as can be seen as follows: There are only spin-1 fields to its left (call it A) and only fermion fields to its right (call it B). Then, in the matrix element

$$\langle 0|A|i\rangle\langle i|B|0\rangle\,, \tag{5.305}$$

if $|i\rangle$ contains any spin-1 particle, the corresponding annihilation operator in $\langle i|$ faces the vacuum to its right and the matrix element vanishes. Similarly, if $|i\rangle$ contains any fermion, then the corresponding creation operator faces the vacuum to its left and the matrix element vanishes. Thus, the state $|i\rangle$ cannot contain any spin-1 particle nor fermion leaving the vacuum state as the only non-zero possibility. By the same technique, we can write the second factor in (5.304) as a product of two currents:

$$\begin{aligned}
\langle 0|a_\mu a_{\nu_e} i_\alpha j'^\dagger_\beta a^\dagger_e a^\dagger_{\nu_\mu}|0\rangle &= \langle 0|a_{\nu_e}\overbrace{i_\alpha a^\dagger_e a_\mu}^{1=\sum_i |i\rangle\langle i|} j'^\dagger_\beta a^\dagger_{\nu_\mu}|0\rangle \\
&= \langle 0|a_{\nu_e} i_\alpha a^\dagger_e|0\rangle\langle 0|a_\mu j'^\dagger_\beta a^\dagger_{\nu_\mu}|0\rangle \\
&= \langle \nu_e|i_\alpha|e\rangle\langle \mu|j'^\dagger_\beta|\nu_\mu\rangle\,,
\end{aligned} \tag{5.306}$$

where we have written

$$|\nu_\mu\rangle \stackrel{\text{def}}{\equiv} a^\dagger_{\nu_\mu}|0\rangle\,, \quad |\mu\rangle \stackrel{\text{def}}{\equiv} a^\dagger_\mu|0\rangle\,, \quad \text{etc.} \tag{5.307}$$

Thus, S_{fi} is now

$$S_{fi} = -\frac{g^2}{2}\int d^4x\, d^4x' \langle 0|T(W^\alpha W^{\beta\prime\dagger})|0\rangle\, \langle \nu_e|i_\alpha|e\rangle\, \langle \mu|j'^\dagger_\beta|\nu_\mu\rangle\,. \tag{5.308}$$

As we will see below, this form of transition amplitude has an intuitive physical interpretation. If we divide the W field into the creation and annihilation parts

$$W^\alpha = W^\alpha_a + W^{\alpha\dagger}_b \quad \text{with} \quad \begin{cases} W^\alpha_a \equiv \sum_{\vec{p}\lambda} a_{\vec{p}\lambda} h^\alpha_{\vec{p}\lambda} \\ W^\alpha_b \equiv \sum_{\vec{p}\lambda} b_{\vec{p}\lambda} h^\alpha_{\vec{p}\lambda} \end{cases}, \tag{5.309}$$

then, using the definition of the time-ordered product,

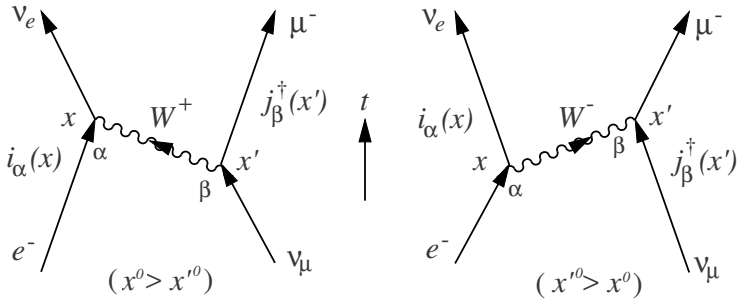
$$\begin{aligned}
&\langle 0|T(W^\alpha W^{\beta\prime\dagger})|0\rangle \\
&\quad = \langle 0|W^\alpha W^{\beta\prime\dagger}|0\rangle\theta(x^0 - x^{0\prime}) + \langle 0|W^{\beta\prime\dagger} W^\alpha|0\rangle\theta(x^{0\prime} - x^0) \\
&\quad = \langle 0|W^\alpha_a W^{\beta\prime\dagger}_a|0\rangle\theta(x^0 - x^{0\prime}) + \langle 0|W^{\beta\prime}_b W^{\alpha\dagger}_b|0\rangle\theta(x^{0\prime} - x^0)\,,
\end{aligned} \tag{5.310}$$

where in the last line, we have kept only the terms in which creation operators faces the vacuum on their right or annihilation operators faces the vacuum on their left.

Recalling that, in the case of scalar field, $\langle 0|\phi_a(x')\phi_a^\dagger(x)|0\rangle$ can be interpreted as the amplitude for the particle to be created at x and found at x', we then naturally interpret $\langle 0|T(W^\alpha W^{\beta\prime\dagger})|0\rangle$ as the amplitude to

$$\begin{pmatrix} \text{create } \beta\text{-component of } W^+ \text{ at } x' \text{ and} \\ \text{annihilate } \alpha\text{-component of } W^+ \text{ at } x \end{pmatrix} \text{ if } x^0 > x^{0\prime}, \\ \begin{pmatrix} \text{create } \alpha\text{-component of } W^- \text{ at } x \text{ and} \\ \text{annihilate } \beta\text{-component of } W^- \text{ at } x' \end{pmatrix} \text{ if } x^{0\prime} > x^0. \tag{5.311}$$

Then, the amplitude S_{fi} can be graphically interpreted as below:



(5.312)

One sees that when $x^0 > x^{0\prime}$, W^+ is emitted from the $\nu_\nu \to \mu^-$ current at x' and absorbed by the $e^- \to \nu_e$ current at x. When $x^0 < x^{0\prime}$, W^- is emitted from the $e^- \to \nu_e$ current at x and absorbed by the $\nu_\mu \to \mu^-$ current at x'. Note that at each vertex, the charge is conserved. Also, the α-component of the $e^- \to \nu_e$ current couples to the α-component of W, and the β-component of the $\nu_\mu \to \mu^-$ current couples to the β-component of W. The amplitude is then summed over α and β and integrated over x and x'. The propagation amplitude $\langle 0|T(W^\alpha(x)W^{\beta\dagger}(x'))|0\rangle$ is called the *Feynman propagator* for a spin-1 particle.

Let's evaluate the Feynman propagator $\langle 0|T(W^\alpha(x)W^{\beta\dagger}(x'))|0\rangle$ so that we can actually calculate the transition amplitude S_{fi}. The first term in (5.310) (apart from the θ function) is

$$\begin{aligned} \langle 0|W_a^\alpha(x)W_a^{\beta\dagger}(x')|0\rangle &= \langle 0|\sum_{\vec{p}\lambda} a_{\vec{p}\lambda} h^\alpha_{\vec{p}\lambda}(x) \sum_{\vec{p}'\lambda'} a^\dagger_{\vec{p}'\lambda'} h^{\beta *}_{\vec{p}'\lambda'}(x')|0\rangle \\ &= \sum_{\substack{\vec{p}\lambda \\ \vec{p}'\lambda'}} h^\alpha_{\vec{p}\lambda}(x) h^{\beta *}_{\vec{p}'\lambda'}(x') \underbrace{\langle 0|a_{\vec{p}\lambda} a^\dagger_{\vec{p}'\lambda'}|0\rangle}_{\delta_{\vec{p},\vec{p}'}\delta_{\lambda,\lambda'}} \\ &= \underbrace{\sum_{\vec{p}}}_{\frac{V}{(2\pi)^3}\int d^3p} \overbrace{\sum_\lambda \epsilon^\alpha_{\vec{p}\lambda}\epsilon^{\beta *}_{\vec{p}\lambda}}^{-g^{\alpha\beta}+\frac{p^\alpha p^\beta}{m^2}} \underbrace{e_{\vec{p}}(x)e^*_{\vec{p}}(x')}_{\frac{e^{-ip\cdot(x-x')}}{2p^0V}} \end{aligned}$$

$$= \frac{1}{(2\pi)^3}\int \frac{d^3p}{2p^0}\Big(-g^{\alpha\beta}+\frac{p^\alpha p^\beta}{m^2}\Big)e^{-ip\cdot(x-x')}\,. \qquad (5.313)$$

We obtain the second term by the exchanges $a \leftrightarrow b$, $\alpha \leftrightarrow \beta$, and $x \leftrightarrow x'$, which amounts to the simple sign change of the exponent:

$$\langle 0|W_b^{\beta\prime}W_b^{\alpha\dagger}|0\rangle = \frac{1}{(2\pi)^3}\int \frac{d^3p}{2p^0}\Big(-g^{\alpha\beta}+\frac{p^\alpha p^\beta}{m^2}\Big)e^{ip\cdot(x-x')}\,. \qquad (5.314)$$

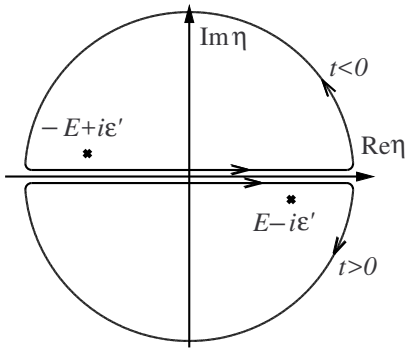
Thus, $\langle 0|T\Big(W^\alpha(x)W^{\beta\dagger}(x')\Big)|0\rangle$ is a function of $z \equiv x-x'$, which we denote by $iD_F^{\alpha\beta}(z)$. Combining (5.313) and (5.314),

$$\begin{aligned} iD_F^{\alpha\beta}(z) &\overset{\text{def}}{\equiv} \langle 0|T\Big(W^\alpha(x)W^{\beta\dagger}(x')\Big)|0\rangle \qquad (5.315)\\ &= \int \frac{d^3p}{(2\pi)^3 2p^0}\Big[\Big(-g^{\alpha\beta}+\frac{p^\alpha p^\beta}{m^2}\Big)e^{-ip\cdot z}\theta(z^0) + \underbrace{\Big(-g^{\alpha\beta}+\frac{p^\alpha p^\beta}{m^2}\Big)e^{ip\cdot z}}\,\theta(-z^0)\Big]\\ &\qquad\qquad \text{relabel } \vec{p}\to-\vec{p}: \ \Big(-g^{\alpha\beta}+\frac{p_\alpha p_\beta}{m^2}\Big)e^{ip^0z^0+i\vec{p}\cdot\vec{z}}\\ &= \int \frac{d^3p}{(2\pi)^3}\frac{e^{i\vec{p}\cdot\vec{z}}}{2p^0}\Big[\Big(-g^{\alpha\beta}+\frac{p^\alpha p^\beta}{m^2}\Big)e^{-ip^0z^0}\theta(z^0) + \Big(-g^{\alpha\beta}+\frac{p_\alpha p_\beta}{m^2}\Big)e^{ip^0z^0}\theta(-z^0)\Big]\,. \end{aligned}$$

Note that the indeces α and β changed from superscripts to subscripts on $p^\alpha p^\beta$ upon relabeling $\vec{p}\to-\vec{p}$ and that p^0 is constrained to $\sqrt{\vec{p}^2+m^2}$. The θ functions can be nicely taken care of using complex analysis. To do so, we will first prove the identity

$$I \equiv \int_{-\infty}^{\infty}\frac{d\eta}{2\pi}\frac{f(\eta)e^{-i\eta t}}{\eta^2-E^2+i\epsilon} = \frac{-i}{2E}\Big(f(E)e^{-iEt}\theta(t)+f(-E)e^{iEt}\theta(-t)\Big) \qquad (5.316)$$

where t, E, ϵ are real, $E, \epsilon > 0$, ϵ is small, and η is an integration variable to which we will now apply the technique of contour integral. We assume that $f(E)$ is a polynomial of E and does not have poles. Then, the integrand has two poles: $(E-i\epsilon')$ and $-(E-i\epsilon')$:



$$\begin{aligned} &\frac{1}{\eta^2-E^2+i\epsilon} = \frac{1}{\eta^2-(E-i\epsilon')^2}\\ &= \frac{1}{(\eta-(E-i\epsilon'))(\eta+(E-i\epsilon'))}\,, \end{aligned} \qquad (5.317)$$

where $\epsilon' \sim \epsilon/2E > 0$. What we want is the integral on the real axis from $-\infty$ to ∞. If we choose the contour such that the integral on the semicircle vanish, then the valule of the countour integral becomes the integration along the real axis. We note that

$$e^{-i\eta t} = e^{-i(\mathrm{Re}\,\eta)t} e^{(\mathrm{Im}\,\eta)t} \to 0 \text{ when } \begin{cases} t > 0 \text{ and } \mathrm{Im}\,\eta \to -\infty, \text{ or} \\ t < 0 \text{ and } \mathrm{Im}\,\eta \to +\infty\,. \end{cases} \tag{5.318}$$

Thus, when $t > 0$ we have to take the lower loop which picks up the pole at $E - i\epsilon'$, and when $t > 0$, we have to take the upper loop which picks up the pole at $-E + i\epsilon'$. The clockwise (anti-clockwise) contour integral of some function $F(\eta)$ around a first-order pole η_0 is $-(+)2\pi i\,\mathrm{Res}(\eta_0)$ where the residue is given by

$$\mathrm{Res}(\eta_0) = F(\eta)(\eta - \eta_0)\Big|_{\eta=\eta_0}\,. \tag{5.319}$$

For our case, the function $F(\eta)$ is

$$F(\eta) = \frac{1}{2\pi} \frac{f(\eta) e^{-i\eta t}}{(\eta - (E - i\epsilon'))(\eta + (E - i\epsilon'))}\,. \tag{5.320}$$

The integral I is then

$$\begin{aligned} (t > 0)\ \ -2\pi i\,\mathrm{Res}(E - i\epsilon') &= -2\pi i \frac{1}{2\pi} \frac{f(\eta) e^{-i\eta t}}{\eta + (E - i\epsilon')}\Big|_{\eta = E - i\epsilon'} = \frac{-i}{2E} f(E) e^{-iEt}\,, \\ (t < 0)\ \ 2\pi i\,\mathrm{Res}(-E + i\epsilon') &= 2\pi i \frac{1}{2\pi} \frac{f(\eta) e^{-i\eta t}}{\eta - (E - i\epsilon')}\Big|_{\eta = -E + i\epsilon'} = \frac{-i}{2E} f(-E) e^{iEt}\,, \end{aligned} \tag{5.321}$$

which establishes (5.316).

Now, we define

$$f(p^0) \stackrel{\mathrm{def}}{\equiv} -g^{\alpha\beta} + \frac{p^\alpha p^\beta}{m^2}\,, \tag{5.322}$$

which is a function of p^0 when $\alpha = 0$ or $\beta = 0$, where p^i $(i = 1, 2, 3)$ are considered to be constant. Then, the following holds for all values of α and β:

$$f(-p^0) = -g^{\alpha\beta} + \frac{p_\alpha p_\beta}{m^2}\,, \tag{5.323}$$

where we have lowered the indexes α and β in the second term, which can be verified explicitly for each value of α and β. Using this definition of $f(p^0)$ with $E = p^0$ and $t = z^0$, the identity (5.316) can then be applied to the Feynman propagator (5.315):

$$iD_F^{\alpha\beta}(z) \;=\; \int \frac{d^3p}{(2\pi)^3} \frac{e^{i\vec{p}\cdot\vec{z}}}{2p^0} \Big(f(p^0) e^{-ip^0 z^0} \theta(z^0) + f(-p^0) e^{ip^0 z^0} \theta(-z^0) \Big)$$

$$= \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot\vec{z}}\, i \int_{-\infty}^{\infty} \frac{d\eta}{2\pi} \frac{f(\eta) e^{-i\eta z^0}}{\eta^2 - \underbrace{p^{0^2}}_{\vec{p}^{\,2} + m^2} + i\epsilon}$$

$$= i \int \frac{d^3p\, d\eta}{(2\pi)^4} \frac{f(\eta) e^{-i\eta z^0} e^{i\vec{p}\cdot\vec{z}}}{\eta^2 - \vec{p}^{\,2} - m^2 + i\epsilon}\,. \tag{5.324}$$

In the second line above, note that p^0 is a constant as long as the integration with respect to η is concerned, and happens to have the value defined by $p^0 = \sqrt{\vec{p}^2 + m^2}$. At this point, there is no p^0 in sight, not even in $f(\eta)$ where p^0 has been replaced by η in the definition (5.322). Since η is a dummy integration variable, we can call it any way we wish. We choose to call it p^0, which is *no longer constrained to* $p^0 = \sqrt{\vec{p}^2 + m^2}$. Then,

$$\begin{aligned} iD_F^{\alpha\beta}(z) &= i \int \frac{d^3p\, dp^0}{(2\pi)^4} \frac{f(p^0) e^{-ip^0 z^0} e^{i\vec{p}\cdot\vec{z}}}{p^{0^2} - \vec{p}^{\,2} - m^2 + i\epsilon} \\ &= i \int \frac{d^4p}{(2\pi)^4} \frac{-g^{\alpha\beta} + \frac{p^\alpha p^\beta}{m^2}}{p^2 - m^2 + i\epsilon}\, e^{-ip\cdot z}\,, \end{aligned} \tag{5.325}$$

where we have used the definitions

$$d^4p \stackrel{\text{def}}{\equiv} d^3p\, dp^0 \quad p^2 \stackrel{\text{def}}{\equiv} p^{0^2} - \vec{p}^{\,2}\,. \tag{5.326}$$

Thus, the spin-1 Feynman propagator is now

$$\boxed{\begin{array}{c} iD_F^{\alpha\beta}(x - x') \stackrel{\text{def}}{\equiv} \langle 0|T\Big(W^\alpha(x) W^{\beta\dagger}(x')\Big)|0\rangle = \displaystyle\int \frac{d^4p}{(2\pi)^4} iD_F^{\alpha\beta}(p) e^{-ip\cdot(x-x')} \\ \text{with} \quad iD_F^{\alpha\beta}(p) = i \frac{-g^{\alpha\beta} + \frac{p^\alpha p^\beta}{m^2}}{p^2 - m^2 + i\epsilon}\,. \end{array}} \tag{5.327}$$

We emphasize again that p^0 is an integration variable and not constrained to $p^0 = \sqrt{\vec{p}^2 + m^2}$. The numerator of the propagator in momentum space $iD_F^{\alpha\beta}(p)$ came from the spin sum over the propagating W particle, and the denominator is a measure of how off-mass-shell[1] the 4-moomentum of the propagating particle is: the more off-mass-shell the smaller the corresponding amplitude. As we will see, this is a general feature of propagators (up to gauge invariance as we will see in the next chapter).

We can now use this spin-1 Feynman propagator to calculate S_{fi} (5.308) for the scattering $\nu_\mu e^- \to \mu^- \nu_e$ and extract the Lorentz-invariant matrix element $\mathcal{M}$.

[1] A particle of mass m is said to be on mass shell when its 4-momentum p satisfies $p^2 - m^2 = 0$. Thus, $p^2 - m^2$ is a measure of how close the particle is to the mass shell.

Roughly speaking, what happens is as follows: after collecting exponentials from the fermion normal-mode functions and from the Feynman propagator, the integration over x gives a delta function for the 4-momentum conservation at the W-$e\nu_e$ vertex, and the integration over x' results in another delta function for the 4-momentum conservation at the W-$\nu_\mu\mu$ vertex. Upon performing the integration over the 4-momentum of the W [which is in $iD_F(x-x')$], the two delta functions becomes one that corresponds to the 4-momentum conservation of initial and final states. Then, the propagator in momentum space $D_F^{\alpha\beta}(p)$ will survive into $\mathcal{M}$ together with the u, v spinors for the external fermion legs.

Using the explicit expressions for the currents i_α and j_β (5.292), we have

$$\begin{array}{rclcl} \langle \nu_e | i_\alpha | e \rangle & = & \langle 0 | a_{\nu_e} (\bar{\nu}_e \gamma_\alpha P_L e) a_e^\dagger | 0 \rangle & = & \bar{f}_{\nu_e} \gamma_\alpha P_L f_e \\ \langle \mu | j_\beta'^\dagger | \nu_\mu \rangle & = & \langle 0 | a_\mu (\bar{\mu}' \gamma_\beta P_L \nu_\mu') a_{\nu_\mu}^\dagger | 0 \rangle & = & \bar{f}_\mu' \gamma_\beta P_L f_{\nu_\mu}' \end{array} \; . \tag{5.328}$$

Then, assigning the 4-momenta as in Figure 5.4, S_{fi} (5.308) is now written as

$$\begin{aligned} S_{fi} &= -\frac{g^2}{2} \int d^4x\, d^4x' \overbrace{\langle 0 | T(W^\alpha W^{\beta\prime\dagger}) | 0 \rangle}^{} \overbrace{(\bar{f}_{\nu_e} \gamma_\alpha P_L f_e)(\bar{f}_\mu' \gamma_\beta P_L f_{\nu_\mu}')}^{} \\ &\qquad \int \frac{d^4p}{(2\pi)^4} i D_F^{\alpha\beta}(p) e^{-ip\cdot(x-x')} \frac{e^{ik'\cdot x}}{\sqrt{2k'^0 V}} \frac{e^{-iq\cdot x}}{\sqrt{2q^0 V}} \frac{e^{iq'\cdot x'}}{\sqrt{2q'^0 V}} \frac{e^{-ik\cdot x'}}{\sqrt{2k^0 V}} \\ &\qquad \times (\bar{u}_{\nu_e} \gamma_\alpha P_L u_e)(\bar{u}_\mu \gamma_\beta P_L u_{\nu_\mu}) \\ &= \int \frac{d^4p}{(2\pi)^4} \frac{\int d^4x\, e^{-i(p+q-k')\cdot x} \int d^4x' e^{i(p+q'-k)\cdot x'}}{\sqrt{2k'^0 V 2q^0 V 2q'^0 V 2k^0 V}} \\ &\qquad \times \underbrace{\frac{-g^2}{2} (\bar{u}_{\nu_e} \gamma_\alpha P_L u_e) i D_F^{\alpha\beta}(p) (\bar{u}_\mu \gamma_\beta P_L u_{\nu_\mu})}_{\equiv \mathcal{M}} \\ &= \int \frac{d^4p}{(2\pi)^4} \underbrace{(2\pi)^4 \delta^4(p+q-k')}_{x \text{ vertex}} \underbrace{(2\pi)^4 \delta^4(p+q'-k)}_{x' \text{ vertex}} \frac{\mathcal{M}}{\sqrt{2k'^0 V 2q^0 V 2q'^0 V 2k^0 V}} \\ &= \frac{(2\pi)^4 \delta^4(k+q-k'-q')}{\sqrt{2k'^0 V 2q^0 V 2q'^0 V 2k^0 V}} \mathcal{M} \, , \end{aligned} \tag{5.329}$$

which has the standard form of the definition of $\mathcal{M}$ (5.281) and allows us to use the cross section formulas we have derived. The Lorentz-invariant matrix element $\mathcal{M}$

$$\mathcal{M} \stackrel{\text{def}}{\equiv} \left(\frac{ig}{\sqrt{2}}\right)^2 (\bar{u}_{\nu_e} \gamma_\alpha P_L u_e) i \frac{-g^{\alpha\beta} + \frac{p^\alpha p^\beta}{m^2}}{p^2 - m^2 + i\epsilon} (\bar{u}_\mu \gamma_\beta P_L u_{\nu_\mu}) \tag{5.330}$$

can be obtained directly from the Feynman diagram shown in Figure 5.4 by following the same rules for the external fermions and vertexes as before and assigning the factor $iD_F^{\alpha\beta}(p)$ for the W propagator. Note that the value of p in the propagator is constrained to $p = k' - q = k - q'$ by the delta functions.

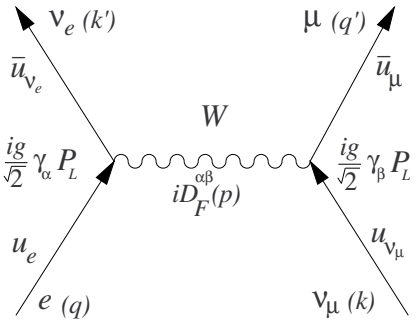


Figure 5.4: The Feynman diagram for the scattering $\nu_\mu e^- \to \mu^- \nu_e$.

Let's evaluate the spin-averaged $|\mathcal{M}|^2$. We will assume that the initial-state electron is unpolarized, but the muon neutrino is assumed to be left-handed since the $V-A$ current is the only known source of neutrinos; namely, when we have a beam of neutrinos, it is a good assumption that they are left-handed. On the other hand, the matrix element above vanishes for a right-handed ν_μ due to the factor $P_L u_{\nu_\mu}$. Consequently, if we simply sum over the spin of ν_μ *without* taking average by dividing by two, it will properly evaluate the cross section for left-handed ν_μ. Thus, we will sum over all spins and divide by two to account for the unpolarized initial-state electron. Also, we will ignore the masses of the fermions assuming a high-energy scattering. Then, the $p^\alpha p^\beta$ term of the W-propagator vanishes by the same reason as does the $p^\alpha p^\beta$ term of the W spin sum in the $W^+ \to e^+ \nu_e$ decay (5.268). The properly spin-averaged matrix element squared is then

$$\overline{|\mathcal{M}|^2} = \frac{1}{2}\Big(\frac{g^2}{2}\Big)^2 \frac{1}{|p^2 - m^2 + i\epsilon|^2} \underbrace{\sum_{\text{spins}} |(\bar{u}_{\nu_e}\gamma_\alpha P_L u_e)(\bar{u}_\mu \gamma^\alpha P_L u_{\nu_\mu})|^2}_{*} \tag{5.331}$$

We have already performed exactly the same spin sum (*) for the $\mu^- \to \nu_\mu e^- \bar{\nu}_e$ decay. The only difference is that we had a v spinor for ν_e before instead of a u spinor in this case; when summed over spin, however, there is no difference between u and v spinors as long as the particle is massless:

$$\sum_{\vec{s}} v_{\vec{p},\vec{s}}\, \bar{v}_{\vec{p},\vec{s}} = \not{p} - \cancel{m} = \not{p} + \cancel{m} = \sum_{\vec{s}} u_{\vec{p},\vec{s}}\, \bar{u}_{\vec{p},\vec{s}} \qquad \text{(massless)}\,. \tag{5.332}$$

Comparing with the complex conjugate of (5.128), we see that all that is needed is to substitute

$$p \to q'(\mu)\,, \quad q \to q(e)\,, \quad p' \to k(\nu_\mu)\,, \quad q' \to k'(\nu_e)\,, \tag{5.333}$$

in (5.147) and adjust for the spin average factor of 1/2 used there and the difference between P_L and $(1-\gamma_5)$. Thus, we obtain

$$(*) = 16(q'\cdot k')(k\cdot q) \tag{5.334}$$

The $i\epsilon$ term in the propagator has no effect in this case, and $\overline{|\mathcal{M}|^2}$ is now

$$\overline{|\mathcal{M}|^2} = \frac{1}{2}\Big(\frac{g^2}{2}\Big)^2 \frac{1}{(p^2-m^2)^2} 16(q'\cdot k')(k\cdot q) = 2g^4 \frac{(q'\cdot k')(k\cdot q)}{(p^2-m^2)^2}\,. \tag{5.335}$$

Let's define the Lorentz-invariant parameters s and t as

$$\begin{aligned} s &\equiv (k+q)^2 = (k'+q')^2 = 2k\cdot q = 2k'\cdot q' \\ t &\equiv (k'-q)^2 = (k-q')^2 = p^2 \end{aligned} \tag{5.336}$$

where we have used the 4-momentum conservation $k+q = k'+q'$ and $k^2 = q^2 = k'^2 = q'^2 = 0$. Note that s is the invariant mass of the system squared that we called M^2 when we derived the cross section formulas, and t is the same 4-momentum transfer squared that we defined in (5.288). Using the formula $d\sigma/dt = \overline{|\mathcal{M}|^2}/16\pi\lambda(M^2, m_a^2, m_b^2)$ with $\lambda(M^2, m_a^2, m_b^2) = s^2$, we have

$$\frac{d\sigma}{dt} = \frac{1}{16\pi s^2} 2g^4 \frac{(s/2)^2}{(t-m^2)^2} = \frac{g^4}{32\pi(t-m^2)^2} \sim \frac{g^4}{32\pi m^4}\,, \tag{5.337}$$

where the last approximation used $|t| \ll m^2$ which is a good approximation for most neutrino beams generated by present-day high-energy accelerators as we will see shortly. The distribution is essentially flat in t. The range of t can be easily obtained in the C.M. system where all particles in the initial and final states have the same energy $E = M/2$. If the angle between the incoming ν_μ and the outgoing μ^- is θ, then together with $s = 4E^2$,

$$\begin{aligned} t = (k-q')^2 = -2k\cdot q' = -2(E^2 - E^2\cos\theta) = -\frac{s}{2}(1-\cos\theta) \\ \to \quad 0 \ge t \ge -s\,; \end{aligned} \tag{5.338}$$

namely, t distributes uniformly from 0 to $-s$. Thus, the total cross section is (recalling that m is the W mass)

$$\sigma(\nu_\mu e^- \to \mu^- \nu_e) = \frac{g^4}{32\pi m_W^4} s \qquad (|t| \ll m_W^2, \quad \text{massless fermions})\,. \tag{5.339}$$

Let's work out the number of interaction for a hypothetical case of 10^{13} ν_μ's at 200 GeV hitting an iron target of thickness 10 m. With $g = 0.65$, $m_W = 80$ GeV, and $m_e = 5.11 \times 10^{-4}$ GeV, we have

$$s = 2k\cdot q = 2E_{\nu_e} m_e = 0.204 \text{ GeV}^2$$

$$\rightarrow \quad \sigma(\nu_\mu e^- \to \mu^- \nu_e) = 8.8 \times 10^{-12}\ \mathrm{GeV}^{-2}\,. \tag{5.340}$$

This value of s calculated above is much smaller than $m_W^2 \sim 80^2$ (GeV^2), thus, as promised, $|t| \leq s \ll m_W^2$. The expected number of interaction per projectile is

$$N = n(\mathrm{cm}^{-3})\,\sigma(\mathrm{cm}^2) L(\mathrm{cm})\,, \tag{5.341}$$

where n and L are the density and the length of the target. Thus, we have to convert the unit of cross section from GeV^{-2} to cm^2. Since σ was evaluated in the unit system where $\hbar = c = 1$, and $\hbar c$ has the value

$$\hbar c = 1.9732 \times 10^{-14}\ \mathrm{GeV} \cdot \mathrm{cm}\,, \tag{5.342}$$

if we pick the unit of energy to be GeV, then the unit of length (GeV^{-1}) should correspond to 1.9732×10^{-14} cm in order to make $\hbar c$ unity. Thus, the unit conversion is

$$\sigma(\mathrm{cm}^2) = \sigma(\mathrm{GeV}^{-2}) \times (1.9732 \times 10^{-14})^2 = 3.4 \times 10^{-39} \mathrm{cm}^2\,. \tag{5.343}$$

The electron density of the iron is $n = 2.2 \times 10^{24}/\mathrm{cm}^3$; then, the total number of interactions is

$$\begin{aligned} (\#\nu_\mu) N &= (\#\nu_\mu) n \sigma L \\ &= 10^{13} \cdot 2.2 \times 10^{24}(\mathrm{cm}^{-3}) \cdot 3.4 \times 10^{-39}(\mathrm{cm}^2) \cdot 10^3(\mathrm{cm}) \sim 75\,, \end{aligned} \tag{5.344}$$

which is not a bad number to start planning the experiment!

The muon decay revisited - extraction of g

The muon decay $\mu^- \to \nu_\mu e^- \nu_e$ that we have studied earlier using the effective interaction (5.124) occurs through an exchange of W as shown in Figure 5.5, from which the Lorentz-invariant matrix element can be read off:

$$\mathcal{M} \stackrel{\mathrm{def}}{\equiv} \Big(\frac{ig}{\sqrt{2}}\Big)^2 (\bar{u}_{\nu_\mu} \gamma_\alpha P_L u_\mu) i \frac{-g^{\alpha\beta} + \frac{k^\alpha k^\beta}{m_W^2}}{k^2 - m_W^2 + i\epsilon} (\bar{u}_e \gamma_\beta P_L v_{\nu_e}) \tag{5.345}$$

Since $\mathcal{M}$ is Lorentz-invariant, let's roughly evaluate the sizes of parameters in the muon rest frame. The 4-momentum $k = q + q'$ is the 4-momentum of the $e^- \bar{\nu}_e$ system, and the maximum value of $|k^\alpha|$ occurs when the momentum of ν_μ is zero and all energy of μ^- has to be carried by the $e^- \bar{\nu}_e$ system; namely, $k^0 = m_\mu$ in this configuration and it is the maximum size that any component of k can take. Thus,

$$\left|\frac{k^\alpha k^\beta}{m_W^2}\right|\,,\ \left|\frac{k^2}{m_W^2}\right| \leq \Big(\frac{m_\mu}{m_W}\Big)^2 \sim 10^{-5}\,. \tag{5.346}$$

Thus, to a good accuracy,

$$\frac{-g^{\alpha\beta} + \frac{k^\alpha k^\beta}{m_W^2}}{k^2 - m_W^2 + i\epsilon} \approx \frac{g^{\alpha\beta}}{m_W^2}\,; \tag{5.347}$$

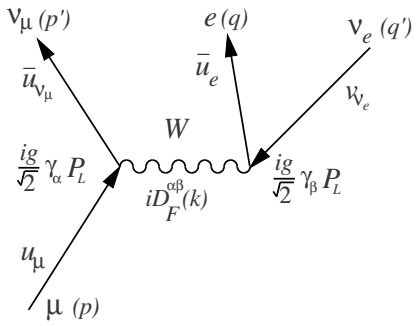


Figure 5.5: The Feynman diagram for the muon decay $\mu^- \to \nu_\mu e^- \bar{\nu}_e$ occurring through a W exchange.

namely,

$$\mathcal{M} = \frac{ig^2}{2m_W^2}(\bar{u}_{\nu_\mu}\gamma_\alpha P_L u_\mu)(\bar{u}_e\gamma^\alpha P_L v_{\nu_e})\,. \tag{5.348}$$

You may be feeling uncomfortable since a Lorentz-invariant quantity is formed only after the propagator is combined with the currents; namely, we have to compare

$$(\bar{u}_{\nu_\mu}\gamma_\alpha P_L u_\mu)g^{\alpha\beta}(\bar{u}_e\gamma_\beta P_L v_{\nu_e}) \quad \text{and} \quad (\bar{u}_{\nu_\mu}\gamma_\alpha P_L u_\mu)\frac{k^\alpha k^\beta}{m_W^2}(\bar{u}_e\gamma_\beta P_L v_{\nu_e})\,,$$

in order to claim that the term $k^\alpha k^\beta/m_W^2$ can be ignored compared to the term $g^{\alpha\beta}$ in the numerator of the propagator. Even though such worry is justified, the approximation (5.347) works in general unless there is some accidental cancellation. Actually, an accidental cancellation does happen here since $(\bar{u}_e \not{k} P_L v_{\nu_e})$ happens to be zero according to (5.268). The effect of the cancellation, however, is to make the approximation (5.347) more accurate.

Now, the approximate matrix element (5.348) obtained for $m_\mu \ll m_W$ is exactly the same matrix element obtained from the effective interaction (5.124) with the identification

$$\boxed{\frac{g^2}{8m_W^2} = \frac{G_F}{\sqrt{2}}}\,. \tag{5.349}$$

Since we know the value of the Fermi coupling constant G_F from the muon life time, we can extract the value of the universal coupling constant g:

$$G_F = 1.1664 \times 10^{-5}(\text{GeV}^{-2}) \quad \to \quad g = 0.65 \tag{5.350}$$

which is the value we have been using.

Propagators for spin-1/2 and spin-0 particles

Just as in the case of spin-1 particles, when a spin-1/2 or spin-0 particle is created and then absorbed in the course of a process - namely, when it does not appear as an external leg - it has an effect of propagating a particle from some space-time point x' to another space-time point x when $x^0 > x^{0\prime}$ and propagating an antiparticle in the opposite direction if $x^{0\prime} > x^0$. They are again expressed as vacuum expectation values of time-ordered products of fields as in the spin-1 case and called the Feynman propagators for spin-1/2 and spin-0 particles, respectively. Following the same procedure as in the spin-1 case, the spin-1/2 Feynman propagator is

$$\boxed{\begin{array}{c} iS_{Fnm}(x-x') \stackrel{\text{def}}{\equiv} \langle 0|T\Big(\psi_n(x)\bar{\psi}_m(x')\Big)|0\rangle = \int \frac{d^4p}{(2\pi)^4} iS_{Fnm}(p)e^{-ip\cdot(x-x')} \\ \text{with} \quad iS_{Fnm}(p) = i\frac{(\not{p}+m)_{nm}}{p^2-m^2+i\epsilon}\,, \end{array}} \tag{5.351}$$

where p is the 4-momentum that would be carried by the fermion propagating forward in time (not the anti-fermion). As in the case of the spin-1 propagator, the numerator $(\not{p}+m)$ arises from the spin sum of the propagating fermion. The spin-0 Feynman propagator is

$$\boxed{\begin{array}{c} i\Delta_F(x-x') \stackrel{\text{def}}{\equiv} \langle 0|T\Big(\phi(x)\phi^\dagger(x')\Big)|0\rangle = \int \frac{d^4p}{(2\pi)^4} i\Delta_F(p)e^{-ip\cdot(x-x')} \\ \text{with} \quad i\Delta_F(p) = \frac{i}{p^2-m^2+i\epsilon}\,, \end{array}} \tag{5.352}$$

which is valid for both hermitian and non-hermitian (charged) fields.

Exercise 5.8 *Feynman propagators.*
(a) For a hermitian spin-0 field $\phi(x)$, show that the vacuum expectation value of the time-ordered product of $\phi(x)$ and $\phi(y)$ (the Feynman propagator for spin-0 particle) can be written as

$$\begin{aligned} i\Delta_F(x-y) &\equiv \langle 0|T(\phi(x)\phi(y))|0\rangle \\ &= \int \frac{d^3p}{(2\pi)^3 2p^0}\Big[e^{-ip\cdot(x-y)}\theta(x^0-y^0) + e^{ip\cdot(x-y)}\theta(y^0-x^0)\Big] \end{aligned} \tag{5.353}$$

where $p^0 \equiv \sqrt{\vec{p}^2+m^2}$. Then use the identity

$$\frac{1}{2E}\Big(f(E)e^{-iEt}\theta(t) + f(-E)e^{iEt}\theta(-t)\Big) = \int_{-\infty}^{\infty}\frac{d\eta}{2\pi}\frac{i\,f(\eta)e^{-i\eta t}}{\eta^2-E^2+i\epsilon} \tag{5.354}$$

to write it in the form given in (5.352) with $\phi^\dagger = \phi$. The Feynman propagator for a hermitian spin-0 field is the same as that for a charged spin-0 field.
(*b*) *Propagator for Dirac field: Verify*

$$\begin{aligned} \langle 0|\psi_j(x)\bar{\psi}_k(y)|0\rangle &= \int \frac{d^3p}{(2\pi)^3 2p^0}(\not{p}+m)_{jk}e^{-ip\cdot(x-y)} \\ \langle 0|\bar{\psi}_k(y)\psi_j(x)|0\rangle &= \int \frac{d^3p}{(2\pi)^3 2p^0}(\not{p}-m)_{jk}e^{ip\cdot(x-y)} \end{aligned} \tag{5.355}$$

Then, show that the Feynman propagator for a spin-1/2 particle is given by

$$\begin{aligned} iS_{F\,jk}(x-y) &\equiv \langle 0|T(\psi_j(x)\bar{\psi}_k(y))|0\rangle \\ &= \langle 0|\psi_j(x)\bar{\psi}_k(y)|0\rangle\theta(x^0-y^0) - \langle 0|\bar{\psi}_k(y)\psi_j(x)|0\rangle\theta(y^0-x^0) \\ &= \int \frac{d^4p}{(2\pi)^4}\frac{i(\not{p}+m)_{jk}}{p^2-m^2+i\epsilon}e^{-ip\cdot(x-y)}\,. \end{aligned} \tag{5.356}$$

Note the minus sign for the T-product for femion fields.
(*c*) *Verify*

$$(\not{p}-m)^{-1} = \frac{\not{p}+m}{p^2-m^2}\,; \tag{5.357}$$

thus, the spin-1/2 propagator in momentum space is often written as

$$iS_F(p) = \frac{i}{\not{p}-m+i\epsilon} \stackrel{\text{def}}{\equiv} i\frac{\not{p}+m}{p^2-m^2+i\epsilon}\,. \tag{5.358}$$

The exponential factor $e^{-ip\cdot(x-x')}$ is common to the propagators for spin-0, spin-1/2, and spin-1 particle, and when integrated over x and x', it becomes part of the delta functions that represent 4-momentum conservation at 'each end' of the propagator. Thus, the rule is that when we have an internal line in Feynman diagram (i.e. propagator), we assign the following factor:

$$\begin{aligned} &\text{spin-0:} && \bullet\!\!-\!\!-\!\!-\!\!-\!\!-\!\!-\!\!-\!\!\bullet && i\Delta_F(p) = \frac{i}{p^2-m^2+i\epsilon}\,, \\ &\text{spin-1/2:} && \bullet\!\!\longrightarrow\!\!\bullet && iS_F(p) = i\frac{\not{p}+m}{p^2-m^2+i\epsilon}\,, \\ &\text{spin-1:} && \bullet\!\sim\!\sim\!\sim\!\sim\!\sim\!\bullet && iD_F^{\alpha\beta}(p) = i\frac{-g^{\alpha\beta}+\frac{p^\alpha p^\beta}{m^2}}{p^2-m^2+i\epsilon}\,. \end{aligned} \tag{5.359}$$

Since the 4-momentum is to be conserved at each end of the propagator, what value to take for the 4-momentum p in the propagators in momentum space is usually quite obvious - except for the sign which depends on whether the propagator is thought

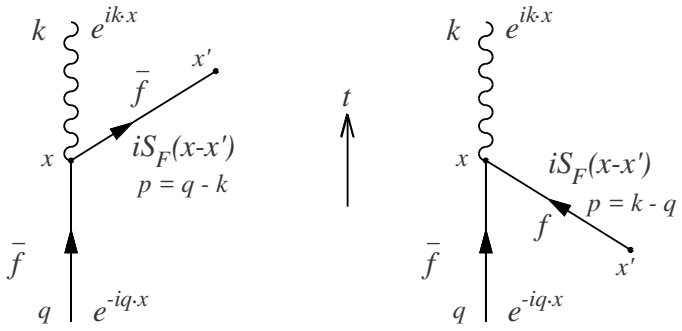


Figure 5.6: The sign of the 4-momentum in a fermion propagator is defined to be the one carried by fermion propagating forward in time, not by anti-fermion.

of as a particle propagating *forward* in time or as an anti-particle propagating again *forward* in time. The sign of p does not matter for the spin-0 and spin-1 propagators which are symmetric under $p \to -p$ as can be seen in (5.359). On the other hand, the sign does matter for the spin-1/2 propagator since the relative sign of p and m in the numerator $\not{p} + m$ makes a difference.

As an example, consider a case where initial state anti-fermion emits a vector meson at x and connects there to a fermion propagator (Figure 5.6). The other end of the propagator (x') would connect to a fermion line and a vector, etc., but it is irrelevant for us now. The propagator $iS_F(x - x')$ as defined in (5.351) propagates an anti-fermion from x to x' when $x'^0 > x^0$ or propagates a fermion from x' to x when $x'^0 < x^0$ which is exactly what is needed in this case. If the incoming particle were a fermion instead of anti-fermion, we would have used $iS_F(x' - x)$ which would propagate a fermion from x to x' when $x'^0 > x^0$. What p to use in $iS_F(p)$ may be determined by the delta function that arises when integrated over x. The incoming antifermion should be matched with $b_{\vec{q},\vec{s}}\,\bar{g}_{\vec{q},\vec{s}}(x)$, thus it comes with $e^{-iq\cdot x}$. Similarly, the outgoing vector comes with $e^{ik\cdot x}$, and $iS_F(x - x')$ contains $e^{-ip\cdot x}$. Thus we obtain $\delta^4(q - k + p)$ upon integration over x; namely, we should use $p = k - q$ which is the 4-momentum of the *fermion* propagating forward in time (the diagram on the right of Figure 5.6). If the incoming particle were a fermion, then the only modification in the exponential factors above comes from $iS_F(x' - x)$ which now has $e^{ip\cdot x}$ instead of $e^{-ip\cdot x}$. Thus, the 4-momentum to use in $iS_F(p)$ would be $p = q - k$ which again is the 4-momentum of the fermion propagating forward in time (the diagram on the left of Figure 5.6 where $\bar{f}$ is changed to f). The rule is then the 4-momentum of a fermion propagator should be the one carried by a fermion propagating forward in time.

In the massless limit, the spin-0 and spin-1/2 propagators in (5.359) do not encounter diffculties; the spin-1 propagator, however, requires some care since the term

$p^\alpha p^\beta/m^2$ in the numerator diverges. As we will see later, only for certain type of theories (where the spin-1 particle couples to a conserved current) the term $p^\alpha p^\beta/m^2$ is strictly zero and thus the spin-1 propagator is well-defined in the massless limit. We will come back to this problem later.

One important feature to note is that even though the time-ordered product is defined in a given Lorentz frame, the expressions above for the Feynman propagators appear to be Lorentz-invariant. To see this more clearly, let's write down the Feynman propagator for the real spin-0 particle as

$$\begin{aligned}\langle 0|T\Big(\phi(x)\phi(x')\Big)|0\rangle &= \langle 0|\phi(x)\phi(x')|0\rangle\theta(x^0-x^{0\prime}) + \langle 0|\phi(x')\phi(x)|0\rangle\theta(x^{0\prime}-x^0) \\ &= \Delta_+(x-x')\theta(x^0-x^{0\prime}) + \Delta_+(x'-x)\theta(x^{0\prime}-x^0) \\ &= \Delta_+(x-x')\theta(x^0-x^{0\prime}) + \Delta_+^*(x-x')\theta(x^{0\prime}-x^0)\,, \qquad (5.360)\end{aligned}$$

where we have used (4.274) and $\Delta_+(-z) = \Delta_+^*(z)$. Namely, the Feynman propagator is obtained from $\Delta_+(z)$ by taking the complex conjuagate when $z^0 < 0$. Then, a natural question is whether one obtains a different result if the complex conjugation is taken in a different frame. If for example we boost in the x-direction by a boost factor γ, then the function $\Delta_+(z)$ will still be identical since it depends only on z^2, and the line (actually a plane) $z^0 = 0$ in the original frame corresponds to

$$z^0 = \gamma z^{0\prime} - \eta z^{1\prime} = 0\,, \qquad (5.361)$$

which is a straight line going through the origin whose slope angle never exceeds $\pi/4$. Thus, the time-ordering in the original frame amounts to taking the complex conjugate of $\Delta_+(z)$ below the tilted line in the boosted frame. Since $\Delta_+(z)$ in the space-like region is real (see Figure 4.3), the result does not depend on the slope. Thus, the Feynman propagator is independent of the frame in which the time-ordering is taken. This may be surprising since a propagation of a particle from 0 to z in original frame becomes a propagation of antiparticle from z to 0 in another frame when the sign of z^0 changes under the Lorentz transformation. One sees that the reason why this does not make any difference is because $\Delta_+(z)$ is real in the time-like region.

Before ending this chapter, let's consider what is the meaning of the sign of $i\epsilon$. If one sets the small imaginary part to be $-i\epsilon$ in (5.316) and follow the derivation of the identity, one sees that it only changes the sign of t in the two theta functions, which traces back to the theta functions in the time-ordered product (5.310). Thus, it corresponds to propagating a particle from x' to x if x' is later in time than x and propagating an antiparticle from x to x' if x is later in time than x'; namely, if the sign of $i\epsilon$ is flipped, it will propagate particle or antiparticle backward in time.

Problems

5.1 *Charged Scaler Field with Electromagnetic Interaction.*
In this problem, we will rigorously follow the procedure of canonical quantization to obtain the interaction Hamiltonian of scalar-photon coupling. We start from the free field Lagrangian

$$\mathcal{L}_0 = \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi$$

where ϕ is a complex field corresponding to a charged scaler particle which we denote as H^+.
(a) We will treat the fields as classical for now (i.e. not operators). Introduce the electromagnetic interaction by the minimal substitution

$$\partial_\mu \to \partial_\mu + ieA_\mu$$

where the photon field A_μ is real and e is the electric charge of H^+ ($e > 0$). Write it in the form

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\rm int}.$$

and express $\mathcal{L}_{\rm int}$ in terms of ϕ, $\phi^\dagger$, A_μ and their space-time derivatives. In principle, we should also have the free field part of the photon field, but we can ignore it for our purpose now.
(b) Regard the fields ϕ and $\phi^\dagger$ as independent fields and obtain the corresponsing conjugate fields π and $\pi^\dagger$ using the definition of conjugate fields

$$\pi_r \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}_r}$$

where r is a general field index. Of course, the conjugate field of ϕ and that of $\phi^\dagger$ are not necessarily complex conjugate of each other, but you will see that it happens to be true in this case.
(c) Apply the Euler-Lagrange's equation of motion

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \frac{\partial \mathcal{L}}{\partial \phi}$$

to obtain the euation of motion for $\phi^\dagger$ in the form $(\partial^2 + m^2)\phi^\dagger = xxx$.
(d) Construct the hamiltonian density following the definition

$$\mathcal{H} \equiv \sum_r \pi_r \dot{\phi}_r - \mathcal{L},$$

and express it in terms of π,ϕ,$\nabla\phi$, A_μ and their complex conjugates. Then write it as the sum of the free field part and the interaction part:

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\rm int},$$

where $\mathcal{H}_0$ has the same form as the free field case when expressed in terms of the above fields.
(e) At this point, we move to quantized fields; now all the fields are operators in the Heisenberg picture. The commutation relations between fields are also imposed as usual:

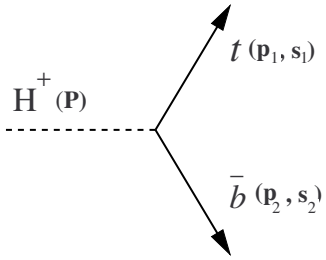
$$[\phi(t,\vec{x}),\pi(t,\vec{x}')] = i\delta^3(\vec{x}-\vec{x}'), \quad [\phi^\dagger(t,\vec{x}),\pi^\dagger(t,\vec{x}')] = i\delta^3(\vec{x}-\vec{x}')$$
$$(\textit{all others} = 0).$$

Obtain the time derivative of ϕ and π from the Heisenberg's equation of motion

$$i\dot{O} = [O,H], \qquad H = \int d^3x \mathcal{H}.$$

The former should give the expression for $\pi^\dagger$, and the latter should give the equation of motion for $\phi^\dagger$. Are they the same form as the ones obtained classically above?
(f) Next, we take the interaction picture. Do the fields $\phi^{(\dagger)}$ and $\pi^{(\dagger)}$ still satisfy the same commutaion relations as above? The time dependences of the operators are now governed by $\mathcal{H}_0$. Write down the expressions for $\dot{\phi}$ and $\dot{\pi}$. The latter gives the equation of motion; is it the free field Klein-Gordon equation?
(g) Finally, we will find what interaction term to use for the calculation of S matrix. Write down the interaction Hamiltonian density $\mathcal{H}_{\rm int}$ in the interaction picture; express it in terms of ϕ, $\phi^\dagger$, A_μ and their space-time derivatives (i.e. eliminate $\pi^{(\dagger)}$). Is it the same as $-\mathcal{L}_{\rm int}$?

5.2 *Charged Higgs decay to top and bottom quarks.*
The charged Higgs particle (H^+) is predicted in a large number of models which are extensions of the standard model. Such particle may decay to a top quark (t) and a anti-bottom quark ($\bar{b}$). The charged Higgs is represented by a complex (non-hermitian) Klein-Gordon field ϕ, and t and b quarks are represented by Dirac fields ψ_t and ψ_b, respectively. Assume that the interaction responsible for the decay is given by



$$\mathcal{L}_{\rm int} = c\phi\,\bar{\psi}_t\psi_b + c\phi^\dagger\,\bar{\psi}_b\psi_t\,,$$

where c is a real constant and the second term is simply the hermitian conjugate of the first, making the whole $\mathcal{L}_{\rm int}$ hermitian. The strength of the coupling depends on

models, but a reasonable guess would be

$$c = \frac{g\, m_b}{m_W}\,,$$

where m_b is the b-quark mass ($5\ GeV$), m_W is the W-boson mass ($80\ GeV$), and g is the so-called universal coupling constant of the standard model (0.65). The top quark mass m_t is 175 GeV, and assume that the mass of the charged Higgs m_H is 400 GeV.
(a) Start from the first-order term of the Dyson series

$$S_{fi} = i \int d^4x \langle f|\mathcal{L}_{\rm int}|i\rangle\,,$$

and use the momentum expansions of the fields

$$\begin{aligned}
\phi &= \textstyle\sum_{\vec{p}} (a_{\vec{p}}\, e_{\vec{p}} + b^\dagger_{\vec{p}}\, e^*_{\vec{p}}) \\
\psi_t &= \textstyle\sum_{\vec{p},\vec{s}} (a_{\vec{p},\vec{s}} f_{\vec{p},\vec{s}} + b^\dagger_{\vec{p},\vec{s}}\, g_{\vec{p},\vec{s}}) \\
\psi_b &= \textstyle\sum_{\vec{p},\vec{s}} (a'_{\vec{p},\vec{s}} f'_{\vec{p},\vec{s}} + b'^\dagger_{\vec{p},\vec{s}}\, g'_{\vec{p},\vec{s}})
\end{aligned}$$

to evaluate the S matrix element S_{fi}. Note that $f_{\vec{p},\vec{s}}$ and $f'_{\vec{p},\vec{s}}$ are different functions since they correspond to different masses which appear in $p^0 = \sqrt{\vec{p}^2 + m^2}$. The initial state (Higgs) has momentum k, and the final state has momentum and spin $\vec{p}_1, \vec{s}_1$ and $\vec{p}_2, \vec{s}_2$ for t and $\bar{b}$ quarks, respectively.

Show that the Lorentz-invariant matrix element $\mathcal{M}$ is given by

$$\mathcal{M} = ic\, \bar{u}_{\vec{p}_1,\vec{s}_1} v'_{\vec{p}_2,\vec{s}_2}$$

where $v'_{\vec{p}_2,\vec{s}_2}$ is the v-spinor of the $\bar{b}$ quark, and $u_{\vec{p}_1,\vec{s}_1}$ is the u-spinor of the t quark. (hint: One of the terms of the Lagrangian does not contribute since it does not have creation and annihilation operators to match the initial and final states.)
(b) Square the matrix element $\mathcal{M}$, sum over the spins, and use the 2-body decay rate formula to show that the decay rate is given by

$$\Gamma = 3\frac{c^2|\vec{p}|}{4\pi}\Big[1 - \Big(\frac{m_b + m_t}{m_H}\Big)^2\Big]$$

where $\vec{p}$ is the momentum of the t quark in the C.M. system of H^+, and the factor 3 is added to account for the three colors of the quarks.
(c) Numerically calculate the decay rate in unit of GeV.

5.3 *The weak decay of charged pion.*
The decay $\pi^+ \to \mu^+\nu$, where ν is the muon neutrino, is caused by the effective Hamiltonian density given by

$$\mathcal{H}_{\rm int} = \frac{G_F}{\sqrt{2}} f(\partial_\alpha \phi)(\bar{\psi}_\nu \gamma^\alpha (1 - \gamma_5)\psi_\mu) + h.c.$$

where ϕ is the π^+ field (a charged spin-0 field), ψ_μ and ψ_ν are the muon and neutrino fields respectively, G_F is the Fermi coupling constant, and f is a constant with dimension of energy (called the pion decay constant: $f = 0.132$ GeV). We take μ^- and ν as fermions and μ^+ and $\bar{\nu}$ as anti-fermions. The branching fraction of this mode is essentially 100 %. Masses and 4-momenta of the particles are

	mass (GeV)	*4-momentum*
π^+	$M = 0.140$	q
μ^+	$m = 0.106$	p
ν	0	p'

(a) Use the momentum expansions of the fields in the first-order expression of the S matrix to show that the Lorentz invariant matrix element is given by

$$\mathcal{M} = \frac{G_F}{\sqrt{2}} f q_\alpha (\bar{u}_\nu \gamma^\alpha (1 - \gamma_5) v_\mu)$$

where u_ν and v_μ are the u, v spinors for the neutrino and muon, respectively. [Namely, the Feynman rule for the derivative coupling of a scalar is to replace ∂_μ by the 4-momentum of the particle and add it to the vertex factor.]
(b) Square the matrix element, sum over the spins of muon and neutrino to obtain

$$\sum_{\text{spins}} |\mathcal{M}|^2 = 4 G_F^2 f^2 m^2 (p \cdot p')$$

(c) Show that the total decay rate of $\pi^+ \to \mu^+ \nu$ is given by

$$\Gamma = \frac{G_F^2 f^2}{8\pi} M m^2 \Big(1 - \frac{m^2}{M^2}\Big)^2 .$$

Numerically evaluate the lifetime of π^+ in seconds.
(d) Discuss the helicities of the muon and neutrino. Note that the muon is rather heavy relative to its momentum in the C.M. of the pion; thus, the helicity projection operator P_L does not act as the pure left-handed helicity operator. What happens if μ^+ is massless?

5.4 *Top quark decay.*
The decay of top quark is dominated by $t \to bW^+$ caused by the interaction term given by

$$\mathcal{L}_{\text{int}}(x) = \frac{g}{2\sqrt{2}} V_{td}\, \bar{\psi}_t(x) \gamma_\mu (1 - \gamma_5) \psi_b(x)\, W^\mu(x) + h.c.$$

where ψ_t and ψ_b are the Dirac fields representing top and bottom quarks, respectively, and W^μ is the vector field representing the W boson. The universal coupling constant

g is approximately 0.65, and $V_{tb} = 1$ *to a good accuracy. Assume that t, b, and* W^+ *are particles as opposed to anti-particles. The masses, momenta, and spins of the particles are denoted as*

	mass	*momentum*	*spin*
t	m	p	s
b	m'	q	s'
W	M	k	ϵ_μ

(*a*) *Write down the Feynman diagram and directly apply the Feynman rules to obtain the Lorentz-invariant matrix element*

$$\mathcal{M} = i\frac{g}{\sqrt{2}}(\bar{u}_b\gamma^\mu P_L u_t)\epsilon^*_\mu \,.$$

(*b*) *Square* $\mathcal{M}$ *and sum over final state spins, average over the initial state spin (the top quark is assumed to be unpolarized) to obtain*

$$\overline{|\mathcal{M}|^2} = \frac{g^2}{2}\Big(p\cdot q + \frac{2}{M^2}p\cdot k\, q\cdot k\Big)$$

(*c*) *Show that the decay rate is given by*

$$\Gamma = \frac{g^2|\vec{p}|}{32\pi}\Big[1 + x - 2\omega + \frac{(1-x)^2}{\omega}\Big]$$

where

$$x \equiv \frac{m'^2}{m^2}\,, \quad \omega \equiv \frac{M^2}{m^2}\,,$$

and $|\vec{p}|$ *is the momentum of* W *(or* b*) in the C.M. system of* t*. Numerically evaluate* Γ *in unit of GeV. Use* $m = 175$ *GeV,* $m' = 5$ *GeV, and* $M = 80$ *GeV.*

5.5 *Decay of a polarized top quark.*
In the decay $t(p) \to b(q)W^+(k)$*, the top quark at rest is assumed to be polarized in the* $\vec{s}$ *direction. The spins of the final state partcles are not measured. A convenient way to evaluate the decay rate for the polarized top quark is to place the corresponding spin projection operator*

$$\Sigma_+(s) = \frac{1 + \gamma_5 \not{s}}{2}$$

in front of the top quark u spinor, and then execute the spin sum on the top quark spin as well as other spins. In this way, one can take advantage of the powerful trace techniques. In the top quark rest frame, the polarization 4-vector is $s = (0, \vec{s})$*.*
(*a*) *Show that the two-body differential decay rate (angular distribution) in the parent rest frame is given by*

$$\frac{d\Gamma}{d\Omega} = \frac{|\vec{p}|}{32\pi^2 m^2}|\mathcal{M}|^2$$

where $\vec{p}$ is the momentum of one of the daughters, m is the mass of the parent, and $d\Omega = d\phi\, d\cos\theta$ is the angular element of the direction of one of the daughters.
(b) Write down the Lorentz-invariant matrix element with the spin projection operator inserted, then evaluate the spin-averaged $\overline{|\mathcal{M}|^2}$. Does one have to divide by two for the top quark spin? One should obtain

$$\overline{|\mathcal{M}|^2} = \frac{g^2}{2}\Big[2(p\cdot q - m\, q\cdot s) + \frac{m^2}{M^2}(p\cdot q + m\, q\cdot s)\Big]$$

where m is the top quark mass, M is the W mass, the bottom quark mass is assumed to be zero, and $g = 0.65$ is the universal coupling constant. (hint: note that the $k^\alpha k^\beta$ term of the W spin sum does not vanish in this case. The basic idea is to reduce the number of gamma's in the traces by variety of tricks we have learned: move P_L's next to each other and use $P_L^2 = P_L$, and similarly for $\Sigma_+(s)$. Take advantage of $\not{p}\not{p} = m^2$, $\not{q}\not{q} = 0$. Also, you may want to use $p\cdot s = 0$, $\gamma_\mu \not{a} \gamma_\mu = -2\not{a}$, $\mathrm{Tr}\not{a}\not{b}\gamma_5 = 0$, and above all $\mathrm{Tr}(\textit{odd number of } \gamma' s) = 0$.)
(c) Obtain the decay distribution $d\Gamma/d\Omega$ in the t rest frame and express it in terms of $\omega \stackrel{\text{def}}{\equiv} (M/m)^2$, g, m and θ which is the angle between the top quark spin and the flight direction of the b quark.
(d) Integrate over the angles and verify that it becomes the unpolarized decay rate obtained in the previous problem. Does it make sense that the polarized decay rate is the same as the unpolarized decay rate?

5.6 *Polarization of W in top quark decay.*
In this problem, we again consider the decay $t(p) \to b(q)W^+(k)$ but this time paying attention to the polarization of W.
(a) Obtain the matrix element squared for each of the three helicity states $+1, 0, -1$ of W (summed/averaged over spins of t and b) as defined in the top quark rest frame. Evaluate them using the energy-momenta of particles in the W rest frame where t and b have the same spatial momentum (by the conservation of momentum). Namely, express them in terms of g, E_t, E_b, and $|\vec{p}|$. Don't assume m_b is small.
(b) Evaluate the matrix element squared for each W helicity in the limit $m_b \to 0$. Does it make sense?
(c) Numerically obtain the fraction of the helicity 0 component over the total decay rate. (hint: use the explicit expression of the polarization vector ϵ for each polarization. By going to the W rest frame, the ϵ becomes a little simpler.)

5.7 *The $\nu_\mu e^-$ scattering by neutral current.*
The scattering $\nu_\mu e^- \to \nu_\mu e^-$ occurs through coupling of the neutral vector boson Z^0 to electron (e^-) and muon neutrino (ν_μ). The relevant interaction is given by

$$\mathcal{L} = -\frac{g}{\cos\theta_W} Z^\mu \Big[\bar{\nu}\gamma_\mu s_L^\nu P_L \nu + \bar{e}\gamma_\mu (s_L^e P_L + s_R^e P_R) e\Big]$$

where Z^μ is the real spin-1 field for the Z^0 boson, ν and e are the Dirac fields representing muon neutrino and electron, respectively. The constant g is the universal coupling constant (~ 0.65), and θ_W is called the Weinberg angle and related to the ratio of W^+ and Z^0 masses:

$$\cos\theta_W = \frac{m_W}{m_Z} \sim 0.88\,.$$

The coefficients s_L^ν and $s_{R,L}^e$ are real constants that give strength of left-handed and right-handed couplings of ν and e to Z. they are given by

$$s_L^\nu = \frac{1}{2}\,, \quad s_R^e = \sin^2\theta_W\,, \quad s_L^e = -\frac{1}{2} + \sin^2\theta_W\,.$$

The propagator of Z is the same as that of W (with different mass).
(a) Show that $\mathcal{L}$ is hermitian.
(b) Write down the relevant lowest-order Feynman diagram for

$$\nu_\mu(p)e^-(q) \to \nu_\mu(p')e^-(q')$$

and the Lorentz-invariant matrix element $\mathcal{M}$. Then assume that the 4-momenta involved are small compared to the Z mass; namely, $k^2 \ll m_Z^2$ and all other Lorentz-invariant expressions such as $p\cdot q$, $p\cdot q' \cdots \ll m_Z^2$, where k is the 4-momentum carried by Z. In this limit the propagator acts as a constant. Then use the Fermi coupling constant $G_F = g^2/(4\sqrt{2}m_W^2)$ to simplify $\mathcal{M}$.
(c) Assume that the incoming muon neutrino is left-handed, then calculate the differential cross section $d\sigma/dt$ separately for the cases the final-state electron is right-handed and left-handed, where $t \equiv (p'-p)^2$ is the invariant 4-momentum transfer. The initial-state electron is unpolarized. Assume that the mass of electron m_e is much smaller than the energies and momenta involved. Express the results in terms of G_F, s, t, and $\sin^2\theta_W$, where $s \equiv (p+q)^2$ is the C.M. total energy squared. The approximation of (b) also applies.
(d) Suppose the energy of the neutrino beam is 200 GeV and the electron target is at rest. Is the approximation we have made in (b) reasonable? What is the angle of the final-state electron with respect to the neutrino beam direction when the final-state electron has the maximum transverse momentum which is the component of $\vec{p}$ perpendicular to the neutrino beam direction? Give the answer in radian. This is not the maximum angle, but one can see that the scattering angle is in general very small.
(e) Finally, let's find out the energy distribution of the recoil electron in the lab. frame for a fixed beam energy. Re-write the two differential cross sections, separately for the two helicities of the recoil electron, in terms of the energy of the final-state electron in unit of the neutrino beam energy; call it x which is dimensionless. Plot $d\sigma/dx$ in unit of $G_F^2 s/\pi$. You can use your favorite plotting program or you can hand-draw it on a graph paper or equivalent.

5.8 *Polarized W decay.*
A polarized W^+ with polarization vector ϵ_μ decays to μ^+ and ν_μ:

$$W^+(k,\epsilon) \to \mu^+(p,s)\nu_\mu(q,s')$$

where the initial state is at rest. Assume the muon and neutrino to be massless.
(a) W^+ is polarized with its spin pointing to $+z$ direction; namely, the component of spin in z direction is $+1$. Write down the explicit polarization vector ϵ_μ in the laboratory coordinate system.
(b) Calculate the differential decay rate $d\Gamma/d\Omega$ in the C.M. system, where Ω is the direction of the muon with respect to the z axis. Express it in terms of the weak coupling constant g, W mass M, and θ, ϕ of the muon direction. (hint: You may have to evaluate the term with $\epsilon_{\alpha\beta\gamma\delta}$ explicitly.)
(c) Does the muon tend to get emitted forward or backward with respect to the W^+ spin direction? What if the parent is W^-?

Chapter 6

Quantum Electrodynamics

As far as we know, the mass of photon is zero, and the main purpose of this chapter is to deal with subtleties that arise as the consequence. And when we do so, we will have a remarkably successul theory of photon and charged fermion called quantum electrodynamics (QED). The story is roughly as follows: first, the helicity-0 polarization vector diverges in the massless limit, and an attempt to restore it results in an unphysical state leaving only two degrees of freedom corresponding to helicity ± 1. Then, we will see that simply banishing the unphysical states is incompatible with the 4-vector nature of the photon field $A^\mu(x)$; or equivalently, even if $A^\mu(x)$ happens to have no unphysical component, a Lorentz transformation can generate it. It then follows that in order for the theory to be Lorentz-invariant, the unphysical component should interact with other fields in such a way that it has no observable consequences. Such is the case if photon field couples to a conserved current, and this then leads to the concept of gauge invariance.

It is a non-trivial question how such gauge symmetry in classical field theory translates to features in the corresponding quantized theory. Quantizing a theory with massless spin-1 particle encounters obstacles because the field A^μ has 4-components while photon has only two physical degrees of freedom. In the case of massive spin-1 field, the Lorentz condition $\partial_\mu A^\mu = 0$ removed one degree of freedom in a Lorentz-invariant way. Upon momentum expanding the field under the constraint, introducing standard commutators for the resulting creation and annihilation operators gave a satisfactory quantized field. In the case of massless spin-1 field, we have another degree of freedom to remove, and this cannot be done easily without explicitly breaking Lorentz-invariance. One approach is to impose additional condition that explicitly restricts the number of degrees of freedom to two (e.g. $\vec{\nabla} \cdot \vec{A} = 0$) accepting that the procedure is no longer Lorentz-invariant. The Lorentz invariance will be restored in the end when S matrices are calculated. Another approach is to keep the formulation Lorentz-invariant while allowing unphysical states to enter into the quantization procedure (the Gupta-Bleuler method). This results in fictitious particles (referred, in

general, as 'ghosts') that cancel in the evaluations of observable effects. In this book, after briefly describing these methods, we will take the third approach: we will start from a theory with massive photon coupling to a conserved current, and take the limit of massless photon. Yet another way, which is widely used to directly obtain Feynman rules for more complicated theories, is the path integral method. This technique, in spite of its importance and elegance, is not strictly required for the quantization of quantum electrodynamics and we will leave it to many other excellent textbooks.

6.1 Massless spin-1 particle and gauge invariance

We will start from trying to find the appropriate momentum expansion of a massless spin-1 field within the framework of non-quantized field. For a free massive spin-1 field A^μ which we assume to be real, the momentum expansion that satisfies the Lorentz condition $\partial_\mu A^\mu(x) = 0$ and the Klein-Gordon equation $(\partial^2 + m^2)A^\mu(x) = 0$ (which together are equivalent to the Proca equation) is

$$A^\mu(x) = \sum_{\vec{p},\lambda=1,2,3} a_{\vec{p}\lambda}\epsilon^\mu_{\vec{p}\lambda}e_{\vec{p}}(x) + c.c. \qquad \text{with} \quad e_{\vec{p}}(x) \equiv \frac{e^{-ip\cdot x}}{\sqrt{2p^0 V}}\,. \tag{6.1}$$

where $p^0 \stackrel{\text{def}}{\equiv} \sqrt{\vec{p}^{\,2} + m^2}$ with m being the mass of the spin-1 particle. The polarization vectors are given in the linear basis by

$$\begin{array}{rcl} \epsilon_{\vec{p}1} &=& (0, \hat{e}_{\vec{p}1}) \\ \epsilon_{\vec{p}2} &=& (0, \hat{e}_{\vec{p}2}) \\ \epsilon_{\vec{p}3} &=& (\eta, \gamma\hat{e}_{\vec{p}3}) \end{array} \qquad \left(\eta \equiv \frac{|\vec{p}|}{m}\,,\ \ \gamma \equiv \frac{p^0}{m}\,,\ \ \hat{e}_{\vec{p}3} = \hat{p}\right), \tag{6.2}$$

where $\hat{e}_{\vec{p}1}$ and $\hat{e}_{\vec{p}2}$ are unit vectors perpendicular to $\hat{p}$ and to each other.

Upon taking the limit $m \to 0$, one immediately encounters a problem; namely, the polarization vector for the helicity-0 component $\epsilon_{\vec{p}3}$ diverges. Then, how can $A^\mu(x)$ be expanded under $\partial_\mu A^\mu(x) = 0$ and $\partial^2 A^\mu(x) = 0$ (the Klein-Gordon equation with $m = 0$)? For this, we have to start from the beginning. Any 4-component real function that satisfies $(\partial^2 + m^2)A^\mu(x) = 0$ can be uniquely expanded as

$$A^\mu(x) = \sum_{\vec{p}} A^\mu_{\vec{p}} e_{\vec{p}}(x) + c.c. \tag{6.3}$$

where $A^\mu_{\vec{p}}$ is a complex number uniquely defined for a given $A^\mu(x)$. The expansion works fine as before; the only difference is that we now have $m = 0$ and thus $p^0 \equiv |\vec{p}|$ (p^0 appears in $e_{\vec{p}}(x)$). Because of the orthonormality of the normal-mode functions $e_{\vec{p}}(x)$, the Lorentz condition $\partial_\mu A^\mu(x) = 0$ leads to

$$p_\mu A^\mu_{\vec{p}} = 0 \qquad (\text{for all } \vec{p}) \tag{6.4}$$

If m were non-zero, we could move to the rest frame where $p^\mu = (m, \vec{0})$ and expand $A^\mu_{\vec{p}}$ in terms of $\hat{e}_{\vec{p},\lambda=1,2,3}$. This time, however, we cannot go to the rest frame since $m = 0$. We thus have to expand the coefficient $A^\mu_{\vec{p}}$ without moving to the rest frame. In the coordinate system defined by $\hat{e}_{\vec{p}1,2,3}$, the 4-momentum is

$$p^\mu = (p^0, 0, 0, p^0)\,, \tag{6.5}$$

and then the Lorentz condition $p^\mu A^\mu_{\vec{p}} = 0$ becomes

$$p^0 A^0_{\vec{p}} - p^0 A^3_{\vec{p}} = 0 \quad \text{or} \quad A^0_{\vec{p}} = A^3_{\vec{p}}. \tag{6.6}$$

Namely, if $A^\mu_{\vec{p}}$ has only 1 and/or 2 component (namely, the transverse component), the Lorentz condition is trivially satisfied, and if there is nonzero 3 component (namely, the component parallel to $\vec{p}$) the 0 component should have the same value. Thus, $A^\mu_{\vec{p}}$ can be uniquely expanded as

$$A^\mu_{\vec{p}} = \sum_{\lambda=1,2,3} a_{\vec{p}\lambda} \varepsilon^\mu_{\vec{p}\lambda} \tag{6.7}$$

with

$$\begin{array}{rcl} \varepsilon^\mu_{\vec{p}1} &=& (0,1,0,0) \\ \varepsilon^\mu_{\vec{p}2} &=& (0,0,1,0) \\ \varepsilon^\mu_{\vec{p}3} &=& (1,0,0,1) \end{array}\,, \tag{6.8}$$

where we have used curly epsilon ε instead of ϵ for the polarization vectors to distinguish them from the massive case. Note that $\varepsilon^\mu_{\vec{p}1}$ and $\varepsilon^\mu_{\vec{p}2}$ are the same as the massive case (6.2), and that the third polarization is proportional to the 4-momentum:

$$\varepsilon^\mu_{\vec{p}3} \propto p^\mu\,, \tag{6.9}$$

which can be regarded as the massless limit of $\epsilon^\mu_{\vec{p}3} = (\eta, 0, 0, \gamma)$ where we have $\eta \to \gamma$ asymptotically. Substituting (6.7) in (6.3), a massless spin-1 field that satisfies $\partial_\mu A^\mu = 0$ and $\partial^2 A^\mu = 0$ can be uniquely expanded as

$$A^\mu(x) = \sum_{\vec{p},\lambda=1,2,3} a_{\vec{p}\lambda} \varepsilon^\mu_{\vec{p}\lambda} e_{\vec{p}}(x) + c.c. \tag{6.10}$$

Now, the problem is that the third polarization $\varepsilon^\mu_{\vec{p}3}$ has zero norm:

$$\varepsilon^{\mu *}_{\vec{p}3} \varepsilon_{\vec{p}3\mu} = 0\,. \tag{6.11}$$

This effectively results in such state carrying no energy or momentum. This can be seen classically by evaluating the corresponding electric and magnetic fields which are seen to vanish. In fact, for the plane wave of the third polarization

$$A^\mu(x) = \varepsilon^\mu_{\vec{p}3}\, e^{-ip\cdot x} + c.c.\,, \tag{6.12}$$

$\vec{E}$ and $\vec{B}$ are

$$\vec{E} \equiv -\partial^0 \vec{A} - \vec{\nabla} A^0 = (i \underbrace{p^0 \vec{\varepsilon}_{\vec{p}3}}_{\vec{p}} - i\vec{p} \underbrace{\varepsilon^0_{\vec{p}3}}_{1}) e^{-ip\cdot x} + c.c. = 0 \tag{6.13}$$

$$\vec{B} \equiv \vec{\nabla} \times \vec{A} = i\vec{p} \times \vec{\varepsilon}_{\vec{p}3}\, e^{-ip\cdot x} + c.c. = 0\,. \tag{6.14}$$

Thus, such state does not carry any energy or momentum which are given by

$$H = \frac{1}{2}\int d^3x (\vec{E}^2 + \vec{B}^2)\,, \qquad \vec{P} = \int d^3x \vec{E} \times \vec{B}\,. \tag{6.15}$$

This is for classical fields, but the situation is similar if we were to quantize A^μ by introducing the standard commutators for $a^{(\dagger)}_{\vec{p}\lambda}$ ($\lambda = 1, 2, 3$) in the expansion (6.10) and evaluate H and $\vec{P}$ defined as above: the number operators corresponding to the third polarization drop out leaving only $\lambda = 1$ and 2.

Thus, we declare that the third polarization to be unphysical and postulate that the polarization vectors $\varepsilon^\mu_{\vec{p}}$ and $\varepsilon^\mu_{\vec{p}} + c\varepsilon^\mu_{\vec{p}3}$ represent the same physical state where $\varepsilon^\mu_{\vec{p}}$ is any polarization vector and c is an arbitrary constant. Then, when the theory is quantized and the Feynman rules are derived, the Lorentz-invariant matrix element $\mathcal{M}$ should have the same value under the replacement (using $\varepsilon^\mu_{\vec{p}3} \propto p^\mu$)

$$\varepsilon^\mu_{\vec{p}} \to \varepsilon^\mu_{\vec{p}} + cp^\mu\,; \tag{6.16}$$

namely,

$$\mathcal{M}(\ldots, \varepsilon_{\vec{p}}, \ldots) = \mathcal{M}(\ldots, \varepsilon_{\vec{p}} + cp^\mu, \ldots)\,. \tag{6.17}$$

Such postulate would be unnecessary if the unphysical components can be banished from the theory and never appear again. This, however, is not the case since a Lorentz transformation can generate the banished unphysical polarization. To see this clearly, consider a Lorentz transformation that is a small boost along $\hat{e}_{\vec{p}1}$ followed by a rotation around $\hat{e}_{\vec{p}2}$ that rotates $\vec{p}$ back to the original direction (Figure 6.1). The corresponding 4×4 matrix defined in the the space coordinate axes $(\hat{e}_{\vec{p}1}, \hat{e}_{\vec{p}2}, \hat{e}_{\vec{p}3} = \hat{p})$ is

$$\begin{aligned}
\Lambda &= 1 + \alpha(K_1 - L_2) = 1 + \alpha \left\{ \left(\begin{array}{c|ccc} 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) - \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right) \right\} \\
&= 1 + \alpha \left(\begin{array}{c|ccc} 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right) \qquad (\alpha\text{: real and small})\,.
\end{aligned} \tag{6.18}$$

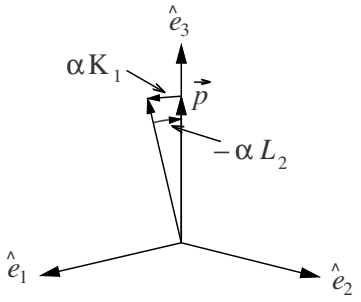


Figure 6.1: A rotation that keeps a massless p^μ invariant. It first boosts along $\hat{e}_1$ by a velocity α and then rotates it back around $\hat{e}_2$ by $-\alpha$ radian.

We can verify that it indeed leaves the momentum direction invarirant: since $p^\mu = (p^0, 0, 0, p^0)$,

$$p' = \Lambda p = p + \alpha \underbrace{\left(\begin{array}{c|ccc} 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right) \begin{pmatrix} p^0 \\ 0 \\ 0 \\ p^0 \end{pmatrix}}_{0} = p\,. \tag{6.19}$$

We see that this transformation leaves invariant not only the direction of $\vec{p}$ but also the whole 4-momentum p^μ.[1] Under this Λ, $\varepsilon_{\vec{p}2}$ is also invariant:

$$\Lambda\varepsilon_{\vec{p}2} = \varepsilon_{\vec{p}2} + \alpha \left(\begin{array}{c|ccc} 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right) \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \varepsilon_{\vec{p}2}\,, \tag{6.20}$$

but $\varepsilon_{\vec{p}1}$ develops a component proportional to $\varepsilon_{\vec{p}3}$:

$$\Lambda\varepsilon_{\vec{p}1} = \varepsilon_{\vec{p}1} + \alpha \left(\begin{array}{c|ccc} 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right) \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \varepsilon_{\vec{p}1} + \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \varepsilon_{\vec{p}1} + \alpha\varepsilon_{\vec{p}3}\,. \tag{6.21}$$

[1]The group formed by the Lorentz transformations that keep the 4-momentum of a particle invariant is called the Wigner's little group, and plays an important role in rigorous treatment of Lorentz transformation of states in the Hilbert space. For a massive particle, the relevant transformation boosts to the rest frame of the particle rotate it and then boost back such that the particle has the same 4-momentum as before.

Thus, even if one eliminates the unphysical component in one frame, it shows up in other frames. Similarly, one can verify that the transformation $\Lambda = 1 + \alpha(K_2 + L_1)$ generates the unphysical component out of $\varepsilon_{\vec{p}2}$ while keeping p^μ invariant.

Getting back to the condition (6.17), we first note that a polarization vector in general appears linearly in $\mathcal{M}$ as

$$\mathcal{M}(\ldots, \varepsilon_{\vec{p}}, \ldots) = \varepsilon^\mu_{\vec{p}}\, I_\mu\,, \tag{6.22}$$

where I_μ is whatever $\varepsilon^\mu_{\vec{p}}$ is multiplied to in forming $\mathcal{M}$. Then the condition (6.17) becomes

$$\begin{aligned} \varepsilon^\mu_{\vec{p}}\, I_\mu &= (\varepsilon^\mu_{\vec{p}} + c\, p^\mu) I_\mu \\ \to\ p^\mu\, I_\mu &= 0\,. \end{aligned} \tag{6.23}$$

Namely, if the polarization vector of a photon is replaced by its 4-momentum, the matrix element vanishes. Even though the matrix element is evaluated in the framework of quantized field theory, let's for the moment try to guess what it means in terms of the classical field theory. Applying the condition (6.16) each of the three basis polarization vectors, the replacement can be written as

$$\varepsilon^\mu_{\vec{p}\lambda} \to \varepsilon^\mu_{\vec{p}\lambda} + c_{\vec{p}\lambda} p^\mu \quad (\lambda = 1, 2, 3)\,, \tag{6.24}$$

where the subscripts on $c_{\vec{p}\lambda}$ indicates that it is in general a function of $\vec{p}$ and λ. In terms of the field $A_\mu(x)$, this replacement translates to

$$\begin{aligned} A^\mu(x) &= \sum_{\vec{p},\lambda=1,2,3} a_{\vec{p}\lambda} \varepsilon^\mu_{\vec{p}\lambda} e_{\vec{p}}(x) + c.c. \\ \to &\sum_{\vec{p},\lambda=1,2,3} a_{\vec{p}\lambda} \varepsilon^\mu_{\vec{p}\lambda} e_{\vec{p}}(x) + \sum_{\vec{p},\lambda=1,2,3} \overbrace{p^\mu}^{i\partial^\mu} a_{\vec{p}\lambda} c_{\vec{p}\lambda} e_{\vec{p}}(x) + c.c. \\ &= A^\mu(x) + \partial^\mu \Lambda(x) \end{aligned} \tag{6.25}$$

with $\Lambda(x) = i \sum_{\vec{p},\lambda} a_{\vec{p}\lambda} c_{\vec{p}\lambda} e_{\vec{p}}(x) + c.c.$. Thus, it seems reasonable to demand that the theory is invariant under the transformation

$$A^\mu(x) \to A^\mu(x) + \partial^\mu \Lambda(x) \tag{6.26}$$

where $\Lambda(x)$ is an arbitrary differentiable real function of x. As we will see below, this is realized if A^μ is linearly coupled to a conserved current,

$$\mathcal{L}_{\rm int} = -e A^\mu(x) j_\mu(x)\,, \quad \partial^\mu j_\mu(x) = 0\,. \tag{6.27}$$

In the later sections, we will show that, upon quantization, such theory (the quantum electrodynamics) indeed leads to matrix elements that are invariant under the substitution $\varepsilon^\mu_{\vec{k}\lambda} \to \varepsilon^\mu_{\vec{k}\lambda} + c_{\vec{k}\lambda} k^\mu$. Here, we will proceed within the framework of classical field theory.

To see that the theory is invariant under $A^\mu \to A^\mu + \partial^\mu \Lambda$, it suffices to show that the action S is invariant under the same transformation. The change in S due to $A^\mu \to A^\mu + \partial^\mu \Lambda$ is

$$\begin{aligned} \Delta S &= \Delta \int d^4x \mathcal{L}_{\rm int} = -e \int d^4x \overbrace{(\partial^\mu \Lambda) j_\mu}^{\text{apply partial integration}} \\ &= e \int d^4x \Lambda \underbrace{\partial^\mu j_\mu}_{0} \\ &= 0\,. \end{aligned} \tag{6.28}$$

In this proof, we have used the conservation of current which is correct only when the fields that form the current follow equations of motion. In deriving the equations of motion of the whole system, one can proceed one field at a time, each time restricting the field to follow the equation of motion before moving to the next field. Thus, we can assume that the current is conserved in the evaluation of action above (as long as the current itself does not contain A^μ, and as long as the derivation of the conserved current is not affected by $A^\mu \to A^\mu + \partial^\mu \Lambda$). Let's take a concrete example of quantum electrodynamics.

In the case of electron photon system, the Lagrangian is the sum of the electron part, the massless spin-1 part, and the interaction part[2] $\mathcal{L}_{\rm int} = -eA^\mu j_\mu$ ($e < 0$) with $j^\mu = \bar\psi \gamma^\mu \psi$:

$$\boxed{\mathcal{L} = \underbrace{\bar\psi(i\partial\!\!\!/ - m)\psi}_{\mathcal{L}_e} \underbrace{-\frac{1}{4} F_{\mu\nu}F^{\mu\nu}}_{\mathcal{L}_\gamma} \underbrace{-eA^\mu(\bar\psi\gamma_\mu\psi)}_{\mathcal{L}_{\rm int}}}\,. \tag{6.29}$$

We first vary the electron field to derive the conserved current. Instead of obtaining the equation of motion for ψ to prove the conservation of current, the action principle can be used to prove the current conservation directly as follows. Suppose $\psi(x)$ is a solution, then any small variation should lead to $\delta S = 0$ due to the action principle. Let us take the variation to be a phase rotation that depends on space-time position:

$$\psi(x) \to e^{i\theta(x)}\psi(x)\,, \quad \bar\psi(x) \to e^{-i\theta(x)}\bar\psi(x)\,. \tag{6.30}$$

Under this change, $\mathcal{L}_\gamma$ is not touched and $\mathcal{L}_{\rm int}$ stays the same since the phase factor cancels out. The electron term $\mathcal{L}_e$, however, changes due to the derivative ∂_μ operating on the space-time dependence of $\theta(x)$:

$$\mathcal{L}_e = \bar\psi(i\partial\!\!\!/ - m)\psi = i\bar\psi\gamma^\mu\partial_\mu\psi - m\bar\psi\psi$$

[2]That the sign of $\mathcal{L}_{\rm int}$ is correct can be seen heuristically as follows: The time component of $\mathcal{L}_{\rm int}$ is $-ej^0A^0$ where j^0 is the electron density and A^0 is the electrostatic potential. This has properly the form $-$(potential energy) as such term should have in Lagrangian.

$$
\begin{aligned}
\to \quad & i(e^{-i\theta}\bar\psi)\gamma^\mu \underbrace{\partial_\mu(e^{i\theta}\psi)}_{e^{i\theta}\partial_\mu\psi + i(\partial_\mu\theta)e^{i\theta}\psi} - m\bar\psi\psi \\
= \; & i\bar\psi\gamma^\mu\partial_\mu\psi - (\bar\psi\gamma^\mu\psi)\partial_\mu\theta - m\bar\psi\psi \\
= \; & \mathcal{L}_e + \delta\mathcal{L}_e \,, \qquad (6.31)
\end{aligned}
$$

with

$$
\delta\mathcal{L}_e = -(\bar\psi\gamma^\mu\psi)\partial_\mu\theta \,. \qquad (6.32)
$$

Then the variation in action is

$$
\begin{aligned}
& -(\bar\psi\gamma^\mu\psi)\partial_\mu\theta \to \text{partial integration} \\
\delta S \; = \; & \int d^4x \, \overbrace{\delta\mathcal{L}_{\rm int}} \\
= \; & \int d^4x \, [\partial_\mu(\bar\psi\gamma^\mu\psi)]\,\theta \,. \qquad (6.33)
\end{aligned}
$$

which should vanish for small and arbitrary $\theta(x)$ (actually, δS given above is valid even for finite $\theta(x)$, but it is besides the point); thus, we have a conserved current

$$
\partial_\mu j^\mu = 0 \,, \quad j^\mu = \bar\psi\gamma^\mu\psi \,, \qquad (6.34)
$$

and one can proceed, as we have done in (6.29), to prove that the theory is invariant under the transformation $A^\mu \to A^\mu + \partial^\mu\Lambda$ based on the fact that the photon field is coupled to the conserved current in the QED Lagrangian.

Note that the conserved current (6.34) is the same as the one obtained by the Noether's theorem based on the invariance of the Lagrangian density under the *global* phase transformation (namely, the phase does not depend on space-time)

$$
\psi \to e^{i\theta}\psi \,, \quad \bar\psi \to e^{-i\theta}\bar\psi \qquad (\theta : \text{ real constant}) \,. \qquad (6.35)
$$

The Lagrangian density, however, is *not* invariant under the *local* phase transformation $\psi \to e^{i\theta(x)}\psi$ (and accordingly for $\bar\psi$) because of the extra term $\delta\mathcal{L}_{\rm int}$ generated by ∂_μ operating on $\theta(x)$ as we have seen in (6.31). Nor is the Lagrangian invariant under $A^\mu(x) \to A^\mu(x) + \partial^\mu\Lambda(x)$ (the action is). There is, however, a curious aspect of the form of the QED Lagrangian which we will demonstrate below; namely, it is invariant if we transform ψ and A^μ *simultaneously* and set $\theta(x) = -e\Lambda(x)$:

$$
\boxed{\psi(x) \to e^{-ie\Lambda(x)}\psi(x) \,, \quad A^\mu(x) \to A^\mu(x) + \partial^\mu\Lambda(x) \,.} \qquad (6.36)
$$

First, the tensor $F^{\mu\nu}$ does not change:

$$
\begin{aligned}
F^{\mu\nu} \; & \equiv \; \partial^\nu A^\mu - \partial^\mu A^\nu \\
\to \quad & \partial^\nu(A^\mu + \partial^\mu\Lambda) - \partial^\mu(A^\nu + \partial^\nu\Lambda) \\
= \; & \partial^\nu A^\mu - \partial^\mu A^\nu + \underbrace{\partial^\nu\partial^\mu\Lambda - \partial^\mu\partial^\nu\Lambda}_{0} = F^{\mu\nu} \,. \qquad (6.37)
\end{aligned}
$$

Thus, the photon part is invariant under (6.36):

$$\delta\mathcal{L}_\gamma = 0\,. \tag{6.38}$$

The variation of the interaction part is

$$\begin{aligned}\delta\mathcal{L}_{\rm int} &= -e(A^\mu + \partial^\mu\Lambda)\underbrace{(e^{ie\Lambda}\bar\psi)\gamma_\mu(e^{-ie\Lambda}\psi)}_{\bar\psi\gamma_\mu\psi} - (-eA^\mu\bar\psi\gamma_\mu\psi)\\ &= -e(\partial^\mu\Lambda)\bar\psi\gamma_\mu\psi\end{aligned} \tag{6.39}$$

Together with $\delta\mathcal{L}_e = e(\partial^\mu\Lambda)\bar\psi\gamma_\mu\psi$ which is obtained by setting $\theta = -e\Lambda$ in (6.32), the total change in $\mathcal{L}$ indeed vanishes:

$$\begin{aligned}\delta\mathcal{L} &= \cancel{\delta\mathcal{L}_e} + \cancel{\delta\mathcal{L}_\gamma} + \delta\mathcal{L}_{\rm int}\\ &= e(\partial^\mu\Lambda)\bar\psi\gamma_\mu\psi - e(\partial^\mu\Lambda)\bar\psi\gamma_\mu\psi = 0\,.\end{aligned} \tag{6.40}$$

Thus, the QED Lagrangian is invariant under the simultaneous space-time dependent transformations of ψ and A^μ given by (6.36). Such symmetry of Lagrangian is called a *local gauge symmetry* or a gauge symmetry of second kind (the first kind being the symmetry under the global phase transformation of ψ). Sometimes it is simply called a *gauge symmetry* and the transformation (6.36) a *gauge transformation*. Note that the invariance holds for finite $\Lambda(x)$ - the transformation does not have to be infinitesimal.

The gauge symmetry of the QED Lagrangian is put in a more transparent form by defining the covariant derivative D_μ by

$$D_\mu \stackrel{\rm def}{\equiv} \partial_\mu + ieA_\mu(x) \tag{6.41}$$

which allows us to write the Lagrangian (6.29) as

$$\begin{aligned}\mathcal{L} &= \bar\psi(i\partial\!\!\!/ - eA\!\!\!/ - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\\ &= \bar\psi(iD\!\!\!/ - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\end{aligned} \tag{6.42}$$

This is of course the minimal substitution we have encountered before. A critical point is that, under the gauge transformation (6.36), the phase factor $e^{-ie\Lambda}$ comes straight out of $D^\mu\psi$:

$$\begin{aligned}D_\mu\psi &\equiv (\partial_\mu + ieA_\mu)\psi\\ &\to\quad [\partial_\mu + ie(A_\mu + \partial_\mu\Lambda)]e^{-ie\Lambda}\psi\\ &= e^{-ie\Lambda}\partial_\mu\psi + (-ie\cancel{\partial_\mu\Lambda})e^{-ie\Lambda}\psi + ie(A_\mu + \cancel{\partial_\mu\Lambda})e^{-ie\Lambda}\psi\\ &= e^{-ie\Lambda}(\partial_\mu + ieA_\mu)\psi\\ &= e^{-ie\Lambda}D_\mu\psi\,.\end{aligned} \tag{6.43}$$

Then, the invariance of the 'kinetic' term $\bar{\psi} i \not{D} \psi$ becomes trivial:

$$\bar{\psi} i\gamma^\mu D_\mu \psi \to (e^{ie\Lambda}\bar{\psi}) i\gamma^\mu (e^{-ie\Lambda} D_\mu \psi) = \bar{\psi} i\gamma^\mu D_\mu \psi \,. \tag{6.44}$$

Together with $m\bar{\psi}\psi \to m\bar{\psi}\psi$ and $F^{\mu\nu} \to F^{\mu\nu}$, we see that the Lagrangian (6.42) is invariant under the gauge transformation.

Even though we have started out by requiring the Lorentz condition $\partial_\mu A^\mu = 0$ to be satisfied, the Lagrangian (6.29) is invariant under the gauge transformation even if $\partial^2 \Lambda \neq 0$ which will break the Lorentz condition; in fact, under $A^\mu \to A^\mu + \partial^\mu \Lambda$ we have

$$\partial_\mu A^\mu \to \partial_\mu A^\mu + \underbrace{\partial^2 \Lambda}_{\neq 0 \text{ in general}} \,. \tag{6.45}$$

When we introduced $\Lambda(x)$ in (6.25), it satisfied $\partial^2 \Lambda = 0$ since it was a linear combination of $e^{\pm ip\cdot x}$ with $p^0 = |\vec{p}|$. However, when we proved that a theory with conserved current is invariant under $A_\mu \to A_\mu + \partial_\mu \Lambda$ in (6.29), $\Lambda(x)$ was no longer required to satisfy $\partial^2 \Lambda = 0$. Nor was it required to show the invariance of $\mathcal{L}$ under the gauge transformation (6.36). Using the flexibility granted by the gauge invariance, one can often simplify calculations by imposing some conditions - called 'fixing the gauge'. Often-used 'gauges' are

$$\begin{aligned} \partial_\mu A^\mu = 0 \;&:\; \text{Lorentz gauge} \\ \vec{\nabla}\cdot\vec{A} = 0 \;&:\; \text{Coulomb (or radiation) gauge} \\ A^0 = 0 \;&:\; \text{temporal gauge} \\ A^3 = 0 \;&:\; \text{axial gauge} \,. \end{aligned} \tag{6.46}$$

Note that, except for Lorentz gauge, they break Lorentz invariance. Each of these conditions does not by themselves exhaust all the degrees of freedom of the gauge transformation. For example, after the Lorentz gauge condition (6.4) was applied, we had three degrees of freedom for free massless spin-1 field as given by the three polarization vectors (6.8). Then, one more degree of freedom can be eliminated by imposing the Coulomb gauge condition $\vec{\nabla}\cdot\vec{A} = 0$ on top of the Lorentz condition, which will remove the third polarization leaving the two transverse polarizations. The resulting field satisfies

$$A^0 = 0 \,, \qquad \vec{\nabla}\cdot\vec{A} = 0 \,. \tag{6.47}$$

This set of conditions, which applies to a free massless spin-1 field, is sometimes called as the Coulomb (or radiation) gauge.

Let us reflect on the flow of the logic we followed. First, we saw that, in the massless limit, the helicity-0 component of a spin-1 particle became unphysical. In order for such unphysical state to have no observable effects, we saw that the corresponding classical theory should be invariant under the transformation $A^\mu \to A^\mu + \partial^\mu \Lambda$ where

Λ is an arbitrary real function. This was accomplished by coupling A^μ to a conserved current which then made the action invariant under $A^\mu \to A^\mu + \partial^\mu \Lambda$, but not the Lagrangian density. We then observed that the Lagrangian density itself was invariant under the local gauge transformation (6.36) where ψ and A^μ are simultaneously transformed space-time dependently.

Now, we could have 'obtained' the QED Lagrangian by following the above reasoning backward, namely, by starting from the free electron Lagrangian $\bar{\psi}(i\partial\!\!\!/ - m)\psi$ and then requiring that the Lagrangian be invariant under the local phase transformation $\psi \to e^{i\theta(x)}$, which leads to an introduction of a spin-1 field that couples to ψ through the covariant derivative $D_\mu = \partial_\mu + ieA_\mu$ and simultaneously transforms to cancel the extra term generated by the space-time dependence of $\theta(x)$. The gauge transformation (6.36) forms a group. Since the phase transformation $e^{i\theta}$ is generated by a 1 by 1 unitary matrix (namely, just 1), it is called $U(1)$ gauge group. When a Lagrangian is invariant under certain global transformations that form a Lie group, one can in general make the Lagrangian invariant under the corresponding *local* transformation by introducing a spin-1 field corresponding to each generator of the Lie group. Such introduction of interactions by requirement of local gauge symmetry is called the *gauge principle*. We will later apply it to $SU(2)$ group (for example, exchange of electron and neutrino), which constitutes a crucial part in constructing the standard model of elementary particles.

Gauge invariance requires that the spin-1 particle be massless. This can be seen by noting that the mass term transforms as

$$\begin{aligned} m^2 A^\mu A_\mu &\to m^2 (A^\mu + \partial^\mu \Lambda)(A_\mu + \partial_\mu \Lambda) \\ &= m^2 \Big(A^\mu A_\mu + \underbrace{(\partial^\mu \Lambda) A_\mu + A_\mu \partial^\mu \Lambda + \partial^\mu \Lambda \partial_\mu \Lambda}_{\neq 0} \Big) \end{aligned} \tag{6.48}$$

The conservation of current, however, is not affected even if photon is massive as can be easily seen by noting that the mass term $m^2 A^\mu A_\mu$ would not affect the derivation of conserved current (6.34). As will see, this allows us to quantize the system by first giving the photon a mass and then taking the massless limit.

6.2 Quantization of quantum electrodynamics

In the previous section, we saw that a theory of massless photon should have a feature that transition matrix elements it produces are invariant when one adds to the polarization vector of a photon an arbitrary term proportional to the 4-momentum of the photon: $\epsilon_{\vec{k}\lambda} \to \epsilon_{\vec{k}\lambda} + ck^\mu$. Then we went on to construct a Lagrangian that would satisfy such requirement in the framework of classical field theory. In order to see that the resulting theory really produces matrix elements that are invariant under such transformation, however, we need to actually quantize the QED Lagrangian (6.29).

In quantizing the QED Lagrangian, one has to quantize a free massless spin-1 particle (namely, photon), and to do so, one has to somehow deal with the two extra degrees of freedom that arises from the fact that the field A^μ has four components while the physical photon has only two degrees of freedom corresponding to helicities ± 1. There are more than one ways to accomplish this, and, luckily, they all give same matrix elements in the end.

The most straightforward is to eliminate all unphysical degrees of freedom and then proceed to apply the method of canonical quantization. A convenient, though not Lorentz-invariant, way to remove the unwanted degrees of freedom is to impose the Coulomb gauge condition

$$\vec{\nabla} \cdot \vec{A}(x) = 0 \,. \tag{6.49}$$

As there is no time derivative of A^0 appearing in the QED Lagrangian (6.29), A^0 does not have corresponding canonical momentum $\partial \mathcal{L}/\partial \dot{A}^0$. Thus, one degree of freedom is already removed. Out of the three polarizations of (6.8), the third violates the Coulomb gauge condition above which reads

$$\vec{p} \cdot \vec{A}(x) = 0$$

for a plane wave. On the other hand, $\epsilon_{\vec{p}1}$ and $\epsilon_{\vec{p}2}$ satisfy it. The expansion of the space part of A^μ is then obtained by taking only $\lambda = 1, 2$ in (6.1)

$$\vec{A}(x) = \sum_{\vec{p},\lambda=1,2} \hat{e}_{\vec{p}\lambda} [a_{\vec{p}\lambda} e_{\vec{p}}(x) + a^\dagger_{\vec{p}\lambda} e^*_{\vec{p}}(x)] \,. \tag{6.50}$$

The time dependence above is for the free field or that in the interaction picture. The quantization condition can be imposed by introducing the standard commutation relations among a's and $a^\dagger$'s:

$$\begin{aligned} [a_{\vec{p},\lambda}, a^\dagger_{\vec{p}',\lambda'}] &= \delta_{\vec{p}\vec{p}'} \delta_{\lambda\lambda'} \\ [a_{\vec{p}\lambda}, a_{\vec{p}'\lambda'}] &= [a^\dagger_{\vec{p}\lambda}, a^\dagger_{\vec{p}'\lambda'}] = 0 \end{aligned} \tag{6.51}$$

One obvious problem is that such quantization is not explicitly Lorentz-invariant since applying a Lorentz transformation to A^μ can lead to the unphysical third polarization $\varepsilon_{\vec{p}3}$; namely, Coulomb gauge condition in one frame is not the same as that in another frame. However, when transition matrix elements are obtained in the end, thay turn out to be Lorentz-invariant. This method is called the Coulomb-gauge quantization.

The second method is to treat all four components of A^μ as independent in order to preserve Lorentz invariance of the procedure (the Gupta-Bleuler method). The polarization vectors are naturally taken to be

$$\begin{aligned} \epsilon_{\vec{p}0} &= (1,0,0,0) \\ \epsilon_{\vec{p}1} &= (0,1,0,0) \\ \epsilon_{\vec{p}2} &= (0,0,1,0) \\ \epsilon_{\vec{p}3} &= (0,0,0,1) \end{aligned} \tag{6.52}$$

in the frame where $p^\mu = (p^0, 0, 0, p^0)$. The photon Lagrangian $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ is incompatible with this scheme since, as mentioned above, it does not contain $\dot{A}^0$ and thus there is no canonical conjugate to A^0. The fix is to add an extra Lorentz-invariant term to the Lagrangian

$$\mathcal{L}_\gamma = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2\alpha}(\partial_\mu A^\mu)^2 \tag{6.53}$$

where α is an arbitrary constant, and impose 'Lorentz condition' as

$$\langle\Psi|\partial_\mu A^\mu|\Psi\rangle = 0 \tag{6.54}$$

for all physical states $|\Psi\rangle$. The Lorentz condition $\partial_\mu A^\mu = 0$ cannot be applied directly to the field since that would revert the Lagrangian (6.53) back to the original form where there is no canonical conjugate to A^0. Then, the commutation relations treating all four components as independent

$$[A_\mu(t,\vec{x}), \pi^\nu(t,\vec{x}')] = i{g_\mu}^\nu\delta^3(\vec{x}-\vec{x}') \tag{6.55}$$

establish the quantized theory. In this scheme, the unphysical 0-th and 3-rd components also propagate as particles (the 'ghosts') with the 0-th component carrying negative probability. However, the two unphysical components always cancel each other in evaluating observable effects. This is reminiscent of the fact that the unphysical polarization $\varepsilon^\mu_{\vec{p}3} = (1,0,0,1)$ in (6.12) did not carry any energy or momentum due essentially to cancellation of the 0-th and 3-rd components.

The method adopted here is to start from the QED Lagrangian with a finite photon mass

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{M^2}{2}A^\mu A_\mu - eA^\mu(\bar{\psi}\gamma_\mu\psi) \tag{6.56}$$

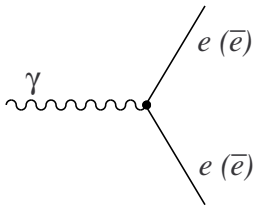
where M is the photon mass that will be set to zero in the end. This Lagrangian is clearly invariant under the global phase rotation $\psi \to e^{i\theta}$, and the Noether theorem gives the conserved current

$$\partial_\mu j^\mu = 0\,, \quad j^\mu = \bar{\psi}\gamma^\mu\psi\,. \tag{6.57}$$

The same conserved current can be obtained by the action principle for the local phase rotation as in (6.34).

Then, quantization of such system is straightforward. In fact, we already know exactly the Feynman rules for external legs and propagators for spin-1/2 and massive spin-1 particles. The vertex factor for electron-photon coupling can be read off the

form of the interaction term $-eA^\mu(\bar\psi\gamma_\mu\psi)$; namely, i times the factor apart from the fields themselves:



$$\text{vertex factor} = -ie\gamma_\mu\,. \tag{6.58}$$

In the massless limit, nothing subtle occurs with the external photon rule; namely, $\epsilon^\mu_{\vec k\lambda}$ for an initial-state photon and $\epsilon^{\mu*}_{\vec k\lambda}$ for a final-state photon. One simply does not have helicity-0 photons in the initial or final states.

The conservation of current still holds even when $j^\mu(x)$ is an operator (in the interaction picture) as shown below. The fields $\psi(x)$ and $\bar\psi(x)$ in the interaction picture satisfy the free-field equations of motion, namely the Dirac equations, as we have seen in (4.357):

$$i\gamma^\mu\partial_\mu\psi = m\psi\,,\quad -i(\partial_\mu\bar\psi)\gamma^\mu = m\bar\psi\,,\quad (\psi,\,\bar\psi\text{: operators}). \tag{6.59}$$

Then, $\partial_\mu j^\mu$ becomes

$$\partial_\mu j^\mu = \partial_\mu(\bar\psi\gamma^\mu\psi) = \underbrace{(\partial_\mu\bar\psi)\gamma^\mu}_{im\bar\psi}\psi + \bar\psi\underbrace{\gamma^\mu\partial_\mu\psi}_{-im\psi} = 0 \quad (j^\mu\text{: operator})\,; \tag{6.60}$$

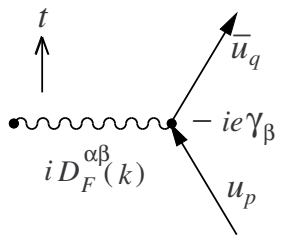
namely, the operator current j^μ is conserved.

The photon propagator needs some care in taking the massless limit, and this is where the conservation of current comes into play. When we set the mass to zero in the propagator of a spin-1 particle (5.327)

$$iD_F^{\alpha\beta}(k) = i\frac{-g^{\alpha\beta} + \frac{k^\alpha k^\beta}{M^2}}{k^2 - M^2 + i\epsilon}\,. \tag{6.61}$$

the term $k^\alpha k^\beta/M^2$ diverges. In general, however, a photon propagator is attached to a fermion current. As an example, let's assume it is attached to an initial-state electron $e^-(p)$ and a final-state electron $e^-(q)$. By the conservation of 4-momentum, the momentum flowing in the propagator is $k = q - p$. There actually is a sign ambiguity of k depending on which direction the arrow is put, but it has no consequence since $D_F^{\alpha\beta}(k)$ is symmetric with respect to the sign of k. Ignoring whatever attaches to the

other side of the propagator, the matrix element is



$$\propto iD_F^{\alpha\beta}(k)\bar{u}_q\gamma_\beta u_p \qquad k = q - p \tag{6.62}$$

where we have suppressed the spin indices for simplicity. Using the Dirac equations $\not{p}u_p = mu_p$ and $\not{q}u_q = mu_q$, we have

$$\begin{aligned} k^\beta(\bar{u}_q\gamma_\beta u_p) &= \bar{u}_q\not{k}u_p = \bar{u}_q(\not{q} - \not{p})u_p \\ &= \underbrace{\bar{u}_q\not{q}}_{m\,\bar{u}_q}\, u_p - \bar{u}_q\, \underbrace{\not{p}u_p}_{m\,u_p} = 0\,. \end{aligned} \tag{6.63}$$

Thus, when the photon propagator is attached to two external electrons, the term $k^\alpha k^\beta/M^2$ strictly vanishes *even before* we take the limit $M \to 0$.

The term $k^\alpha k^\beta/M^2$ of the photon propagator vanishes regardless of whether any of the two external lines are a fermion or a antifermion, or initial-state or final-state as long as the diagram is legitimate (i.e. the charge and the 4-momenta are conserved at the vertex). This can be checked explicitly as above for each case, but it can also be directly derived from the conservation of current as follows. In terms of fields, $\partial_\mu j^\mu = 0$ is

$$0 = \partial_\mu(\bar{\psi}\gamma^\mu\psi) = (\partial_\mu\bar{\psi})\gamma^\mu\psi + \bar{\psi}\gamma^\mu(\partial_\mu\psi)\,. \tag{6.64}$$

Replacing the fields with the momentum expansions

$$\begin{aligned} \psi &= \sum_p (a_p f_p + b_p^\dagger g_p)\,, & \partial_\mu\psi &= \sum_p (-ip_\mu)(a_p f_p - b_p^\dagger g_p)\,, \\ \bar{\psi} &= \sum_q (a_q^\dagger \bar{f}_q + b_q \bar{g}_q)\,, & \partial_\mu\bar{\psi} &= \sum_q (iq_\mu)(a_q^\dagger \bar{f}_q - b_q \bar{g}_q)\,, \end{aligned} \tag{6.65}$$

where we have again suppressed the spin indexes as well as sums over them for simplicity, we have

$$\begin{aligned} 0 &= \sum_q (iq_\mu)(a_q^\dagger \bar{f}_q - b_q\bar{g}_q)\gamma^\mu \sum_p (a_p f_p + b_p^\dagger g_p) \\ &\quad + \sum_q (a_q^\dagger \bar{f}_q + b_q\bar{g}_q)\gamma^\mu \sum_p (-ip_\mu)(a_p f_p - b_p^\dagger g_p) \\ &= i\sum_{p,q} \Big[(q_\mu - p_\mu)(\bar{f}_q\gamma^\mu f_p)\, a_q^\dagger a_p + (q_\mu + p_\mu)(\bar{f}_q\gamma^\mu g_p)\, a_q^\dagger b_p^\dagger + \\ &\qquad\qquad (-q_\mu - p_\mu)(\bar{g}_q\gamma^\mu f_p)\, b_q a_p + (-q_\mu + p_\mu)(\bar{g}_q\gamma^\mu g_p)\, b_q b_p^\dagger \Big]\,. \end{aligned} \tag{6.66}$$

Since the products of creation and annihilation operators in the four terms are independent, each coefficient should vanish. Using $f_p = u_p e^{-ipx}/\sqrt{2p^0 V}$ and $g_p = v_p e^{ipx}/\sqrt{2p^0 V}$, they can be summarized as

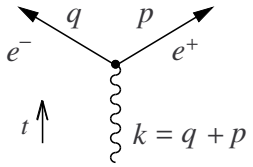
$$k_\mu(\bar{w}_q \gamma^\mu w'_p) = 0 \qquad (w, w' = u \text{ or } v)\,, \tag{6.67}$$

with

$$k = (\pm q) - (\pm p) \tag{6.68}$$

where the signs on $\pm q$ and $\pm p$ are taken to be $+(-)$ if the corresponding spinor is a $u(v)$ spinor. Tracing the above proof backward, one sees that the relation (6.67) is equivalent to $\partial_\mu j^\mu = 0$. In fact, (6.67) can be regarded as the conservation of current expressed in the momentum space.

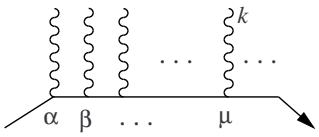
For any of the sign combinations in (6.68), the 4-momentum k is seen to be always the 4-momentum of the photon attached to the fermion current. The conservation of current (6.67) then states that, as long as 4-momentum is properly conserved at the vertex, the photon 4-momentum times the current vanishes in general. If we take both spinors to be u spinors, for example, we have $(q_\mu - p_\mu)\bar{u}_q \gamma^\mu u_p = 0$ recovering (6.63). If both fermions are in the final state, it represents a creation of e^+e^- pair. In this case we have



$$(q_\mu + p_\mu)\bar{u}_q \gamma^\mu v_p = 0\,. \tag{6.69}$$

Note that (6.67) holds regardless of whether the photon is external or a propagator.

What happens if one or more of the fermion line are not external, namely fermion propagators? In such cases, the above argument unfortunately does not hold. We will, however, later extend the above proof to the case where the fermion current contains a series of fermion propagators which may or may not form a loop. Namely, if we denote the fermion current by $J_{\alpha,\beta\ldots}$, where the Lorentz indices are the ones that contract with those of the photon lines, then we have



$$k^\mu J_{\alpha,\beta\ldots\mu\ldots} = 0 \tag{6.70}$$

where k^μ is the 4-momentum of the photon attached at the vertex labeled by the Lorentz index μ. It should be stated here that the identity holds when the diagrams

of the same order are summed (or more precisely, the sum of diagrams where the photon with momentum k^μ is the first photon along the fermion line, second along the fermion line, and so on). This is called the Ward identity which is one form of the more general Ward-Takahashi identities which is a consequence of the conservation of current.

Thus, we can in general ignore the $k^\alpha k^\beta/M^2$ term in the spin-1 propagator for the electron photon system. Then, setting $M = 0$ in the denominator, the photon propagator in massless limit is

$$\boxed{iD_F^{\alpha\beta}(k) = i\frac{-g^{\alpha\beta}}{k^2 + i\epsilon}\,.} \quad \text{(photon propagator)} \tag{6.71}$$

The identity (6.70) also applies when the photon is external. The matrix element that includes an external photon can in general be written as

$$\mathcal{M} = \epsilon^\mu_{\vec{k}\lambda} J_{\alpha,\beta\ldots\mu\ldots} X \equiv \epsilon^\mu_{\vec{k}\lambda} I_\mu \tag{6.72}$$

where $\epsilon^\mu_{\vec{p}\lambda}$ is the photon polarization vector, $J_{\alpha,\beta\ldots\mu\ldots}$ represents the fermion current to which the photon is attached to, and X represents whatever is the rest of $\mathcal{M}$. Then, the identity (6.70) indicates that if we replace the photon polarization vector $\epsilon^\mu_{\vec{p}\lambda}$ by its 4-momentum k^μ, then the matrix element vanishes:

$$\mathcal{M}(\epsilon^\mu_{\vec{k}\lambda} \to k^\mu) = k^\mu (J_{\alpha,\beta\ldots\mu\ldots} X) = 0\,, \tag{6.73}$$

thus verifying that the condition for the self-consistency (6.17) is indeed satisfied for the quantized theory.

Apart from being crucial for the self-consistency, the relation (6.73) greatly simplifies the spin sum for an external photon. Squaring (6.72) and summing over the photon polarization while assuming that photon is massive, we obtain

$$\begin{aligned} \sum_{\lambda=1,2,3} |\mathcal{M}|^2 &= I_\mu I^*_\nu \sum_{\lambda=1,2,3} \epsilon^\mu_{\vec{k}\lambda} \epsilon^{\nu *}_{\vec{k}\lambda} = I_\mu I^*_\nu \left(-g^{\mu\nu} + \frac{k^\mu k^\nu}{M^2}\right) \\ &= I_\mu I^*_\nu (-g^{\mu\nu})\,, \end{aligned} \tag{6.74}$$

where we have used $k^\mu I_\mu = 0$. Thus, the photon spin sum rule is effectively

$$\sum_\lambda \epsilon^\mu_{\vec{k}\lambda} \epsilon^{\nu *}_{\vec{k}\lambda} \to -g^{\mu\nu}\,. \tag{6.75}$$

This is true for a massive photon and should hold as we take the limit $M \to 0$. In the massless case, however, we know that there are only two physical polarizations $\lambda = 1, 2$ over which the sum is taken. Thus, if the spin sum formula above, which is the sum over three polarizations, is to be correct in the massless case where the

sum is over two polarizations, the matrix element for the $\lambda = 3$ component (namely, the helicity-0 component) should smoothly vanish as we take the limit $M \to 0$. It is easy to see that this is indeed the case. The helicity-0 polarization vector $\epsilon^\mu_{\vec{k}3}$ given by (6.2) can be written as a sum of a term proportional to k^μ and another term with zero space part as (note that we have $M \neq 0$ here and thus $k^0 \neq |\vec{k}|$)

$$\epsilon^\mu_{\vec{k}3} = (\eta, \gamma\hat{k}) = \left(\frac{|\vec{k}|}{M}, \frac{k^0}{M}\,\hat{k}\right) = \frac{k^0}{|\vec{k}|}\frac{k^\mu}{M} - \left(\frac{M}{|\vec{k}|}, \vec{0}\right) . \tag{6.76}$$

Since $k^\mu I_\mu = 0$, the helicity-0 matrix element becomes

$$\mathcal{M} = \epsilon^\mu_{\vec{k}3}\, I_\mu = -\frac{M}{|\vec{k}|} I_0 \tag{6.77}$$

which will vanish when we take the limit $M \to 0$ while keeping $\vec{k}$ fixed.

Some cautionary comments are in order in using the formulae (6.71) and (6.75). First, it works only when diagrams of same order, or at least some relevant subset of them, are added up since such addition is required for the Ward identity to hold. Also, the spin sum rule (6.75) in particular should be used at a stage of calculation when whatever the polarization vector is multiplied to, denoted as L_μ, satisfies $k^\mu L_\mu = 0$. Namely, if one modifies the expression using some relation that is satisfied by $\epsilon_{\vec{k}\lambda}$ but not by k^μ, such as $\epsilon^2_{\vec{k}\lambda} = -1$ which is not satisfied by k (namely, $k^2 = 0$ instead), *before* applying the spin sum rule, then the rule will no longer work. Incidentally, the property that the matrix element should vanish when a photon polarization vector is replaced by the corresponding photon momentum can be used as a powerful check of calculation. The same cautions as above should be employed when applying such technique.

6.3 $e^+e^- \to \mu^+\mu^-$

As an example of interaction involving photon, let's take the muon pair creation by e^+e^- annihilation. This mode serves as a model for other important channels such as $e^+e^- \to u\bar{u}$, $d\bar{d}$, or $\tau^+\tau^-$, which can be obtained by simply using the appropriate charge and mass for the final-state fermion. The Lagrangian is essentially the QED Lagrangian (6.29) with muon terms added:

$$\mathcal{L} = \mathcal{L}_e + \mathcal{L}_\mu + \mathcal{L}_\gamma + \mathcal{L}_{\rm int} \tag{6.78}$$

with

$$\mathcal{L}_e = \bar{\psi}_e(i\partial\!\!\!/ - m_e)\psi_e\,, \quad \mathcal{L}_\mu = \bar{\psi}_\mu(i\partial\!\!\!/ - m_\mu)\psi_\mu\,, \tag{6.79}$$

and

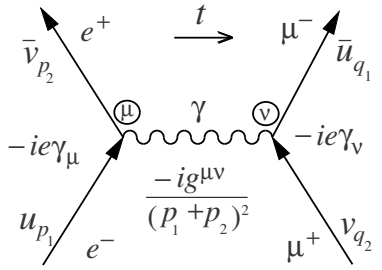
$$\mathcal{L}_{\rm int} = -eA_\alpha(j_e^\alpha + j_\mu^\alpha)\,, \quad j_e^\alpha = \bar\psi_e\gamma^\alpha\psi_e\,, \quad j_\mu^\alpha = \bar\psi_\mu\gamma^\alpha\psi_\mu\,, \tag{6.80}$$

where ψ_e and ψ_μ are the electron field and the muon field, respectively. The photon term $\mathcal{L}_\gamma$ is the same as before. This Lagrangian is a simple extension of the QED lagrangian (6.29) and invariant under a gauge transformation of the type (6.36).

Exercise 6.1 *Verify that the Lagrangian (6.78) is invariant under the following gauge transformation:*

$$\begin{cases} \psi_e(x) \to e^{-ie\Lambda(x)}\psi_e(x) \\ \psi_\mu(x) \to e^{-ie\Lambda(x)}\psi_\mu(x) \end{cases}, \quad \text{and} \quad A_\mu(x) \to A_\mu(x) + \partial_\mu\Lambda(x)\,. \tag{6.81}$$

The matrix element can be derived in exactly the same way as $\nu_\mu e^- \to \mu^-\nu_e$ (5.330); the only differences are that the $V-A$ coupling is replaced by a V-coupling and the spin-1 particle is now massless. Or, we could simply apply the Feynman rules to the diagram below: labeling the spinors by corresponding 4-momenta, we have



$$\begin{aligned} \mathcal{M} &= ie^2(\bar v_{p_2}\gamma_\mu u_{p_1})\frac{g^{\mu\nu}}{(p_1+p_2)^2}(\bar u_{q_1}\gamma_\nu v_{q_2}) \\ &= \frac{ie^2}{s}(\bar v_{p_2}\gamma_\mu u_{p_1})(\bar u_{q_1}\gamma^\mu v_{q_2}) \end{aligned} \tag{6.82}$$

where $s \equiv (p_1+p_2)^2$ is the invariant C.M. mass squared, and we have ignored $+i\varepsilon$ in the denominator of the photon propagator since there is no singularity at $\varepsilon \to 0$.

In the following, we ignore the electron mass, but not the muon mass:

$$m_e = 0\,, \quad m_\mu \neq 0\,. \tag{6.83}$$

We assume that all fermions are unpolarized and sum over all spins and divide by the number of initial-state spin configurations (which is four) to obtain the spin-averaged matrix element squared:

$$\begin{aligned} \overline{|\mathcal{M}|^2} &= \frac{1}{4}\sum_{\rm spins}|\mathcal{M}|^2 \\ &= \frac{e^4}{4s^2}\sum_{\rm spins}[(\bar v_{p_2}\gamma_\mu u_{p_1})(\bar u_{q_1}\gamma^\mu v_{q_2})][(\bar v_{p_2}\gamma_\nu u_{p_1})(\bar u_{q_1}\gamma^\nu v_{q_2})]^* \\ &= \frac{e^4}{4s^2}\sum_{\rm spins}(\bar v_{p_2}\gamma_\mu u_{p_1})(\bar u_{p_1}\gamma_\nu v_{p_2})\sum_{\rm spins}(\bar u_{q_1}\gamma^\mu v_{q_2})(\bar v_{q_2}\gamma^\nu u_{q_1}) \qquad (6.84) \\ &= \frac{e^4}{4s^2}{\rm Tr}\not p_2\gamma_\mu\not p_1\gamma_\nu\,{\rm Tr}(\not q_1 + m_\mu)\gamma^\mu(\not q_2 - m_\mu)\gamma^\nu\,. \qquad (6.85) \end{aligned}$$

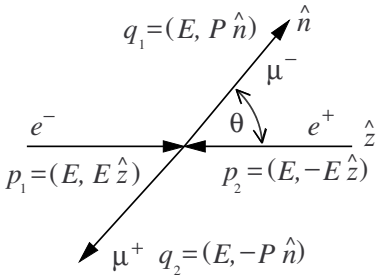
Using trace theorems, we have

$$\begin{aligned} \mathrm{Tr}\not{p}_2\gamma_\mu\not{p}_1\gamma_\nu &= 4(p_{2\mu}p_{1\nu} - p_1\cdot p_2\, g_{\mu\nu} + p_{1\mu}p_{2\nu})\,, \\ \mathrm{Tr}(\not{q}_1 + m_\mu)\gamma^\mu(\not{q}_2 - m_\mu)\gamma^\nu &= 4(q^{1\mu}q^{2\nu} - q_1\cdot q_2\, g^{\mu\nu} + q^{2\mu}q^{1\nu}) - 4m_\mu^2 g^{\mu\nu}\,. \end{aligned} \tag{6.86}$$

Taking the product of the two traces and contracting μ and ν, we obtain (note $g^{\mu\nu}g_{\mu\nu} = 4$)

$$\overline{|\mathcal{M}|^2} = \frac{8e^4}{s^2}[p_2\cdot q_1\; p_1\cdot q_2 + p_2\cdot q_2\; p_1\cdot q_1 + m_\mu^2\, p_1\cdot p_2]\,. \tag{6.87}$$

Let's evaluate this in the C.M. system whose kinematic configuration as well as the definitions of parameters are shown in the figure below. The 4-momentum products are then written as



$$\begin{aligned} p_1\cdot p_2 &= 2E^2\,, \\ p_1\cdot q_1 &= p_2\cdot q_2 = E^2 - EP\cos\theta \\ &\qquad = E^2(1-\beta\cos\theta)\,, \\ p_1\cdot q_2 &= p_2\cdot q_1 = E^2 + EP\cos\theta \\ &\qquad = E^2(1+\beta\cos\theta)\,. \end{aligned} \tag{6.88}$$

$(\beta \equiv P/E:$ the velocity of $\mu)$

Together with $e^2 = 4\pi\alpha$ (α: the fine structure constant) and $s = (2E)^2$, $\overline{|\mathcal{M}|^2}$ is now

$$\begin{aligned} \overline{|\mathcal{M}|^2} &= \frac{8(4\pi\alpha)^2}{s^2}\left[E^4(1+\beta\cos\theta)^2 + E^4(1-\beta\cos\theta)^2 + 2E^2m_\mu^2\right] \\ &= (4\pi\alpha)^2(2-\beta^2\sin^2\theta)\,. \end{aligned} \tag{6.89}$$

Using the cross section formula (5.287) with $s \equiv M^2$, we then obtain

$$\frac{d\sigma}{d\Omega} = \frac{\overline{|\mathcal{M}|^2}}{(8\pi)^2 s}\frac{|\vec{p}_f|}{|\vec{p}_i|} = \frac{(4\pi\alpha)^2(2-\beta^2\sin^2\theta)}{(8\pi)^2 s}\underbrace{\frac{P}{E}}_{\beta}$$

$$\rightarrow \quad \boxed{\frac{d\sigma(e^+e^-\to\mu^+\mu^-)}{d\Omega} = \frac{\alpha^2}{4s}\beta\,(2-\beta^2\sin^2\theta)} \quad (m_e = 0)\,. \tag{6.90}$$

We see that the angular distribution is uniform just above the threshold of $\mu^+\mu^-$ production where $\beta \sim 0$, and approaches $1+\cos^2\theta$ at high-energy ($\beta \sim 1$).

Integrating over $d\Omega = 2\pi\, d\cos\theta$, the total cross section is

$$\boxed{\sigma(e^+e^- \to \mu^+\mu^-) = \begin{cases} \dfrac{2\pi\alpha^2}{s}\beta\left(1 - \dfrac{\beta^2}{3}\right) & (m_e = 0) \\ \dfrac{4\pi\alpha^2}{3s} & (m_e = m_\mu = 0)\,, \end{cases}} \tag{6.91}$$

where in the second expression we have also set $m_\mu = 0$ (namely, the high-energy limit).

6.4 Spin correlations in $e^+e^- \to \mu^+\mu^-$

In the massless limit, a $V-A$ current creates left-handed (helicity $-$) fermion or right-handed (helicity $+$) antifermion, as we have seen in the decay $\mu^- \to \nu_\mu e^- \bar{\nu}_e$ (p234). The scattering $e^+e^- \to \mu^+\mu^-$ occurs through vector currents, and there is no specific helicity associated with fermions or antifermions. As we will see now, however, there are spin correlations between the spins of e^+ and e^- and between those of μ^+ and μ^- If a high-energy electron is left-handed, then the corresponding spinor u satisfies

$$P_L u = u \qquad (e^- \text{ left-handed}). \tag{6.92}$$

Similarly, if a high-energy positron is left-handed, its spinor v satisfies

$$P_R v = v \qquad (e^+ \text{ left-handed}). \tag{6.93}$$

Then, the e^+e^- vector current of the initial state vanishes if both particles are left-handed:

$$\bar{v}\gamma^\mu u = \overline{P_R v}\gamma^\mu P_L u = \bar{v} P_L \gamma^\mu P_L u = \bar{v}\gamma^\mu \underbrace{P_R P_L}_{0} u = 0\,. \tag{6.94}$$

Thus, the e^+e^- pair cannot annihilate through a virtual photon if both are left-handed. By exchanging P_L and P_R in the above, we see that the process is also forbidden if both are right-handed. Similarly, in the high-energy limit, the $\mu^+\mu^-$ pair cannot be created from a virtual photon if both are left-handed or both are right-handed.

Similar selection rules apply when the vector current represents a particle in initial state scattering to a particle in final state. When a right-handed electron in initial state scatters to a left-handed electron in final state, the spinors satisfies

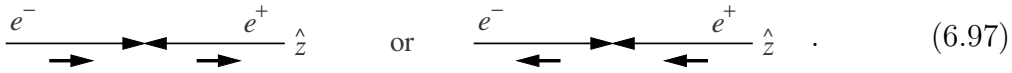
$$P_R u_i = u_i\,, \quad P_L u_f = u_f\,, \tag{6.95}$$

where i and f indicates initial state and final state, respectively. Then, the vector current again vanishes:

$$\bar{u}_f \gamma^\mu u_i = \overline{P_L u_f}\gamma^\mu P_R u_i = 0\,. \tag{6.96}$$

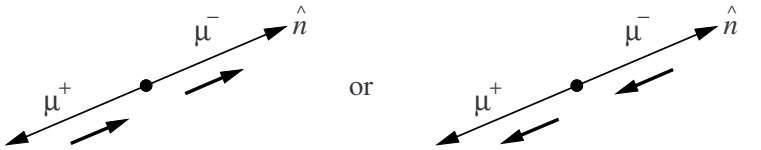
By exchanging P_R and P_L in the above, and also applying similar logic to an antifermion, we see that a high-energy fermion or an antifermion cannot flip helicity in a scattering caused by a vector current.

One can use these spin correlations to obtain the angular distribution of $e^+e^- \to \mu^+\mu^-$ withtout actually calculating the Feynman diagrams. In the C.M. frame of $e^+e^- \to \mu^+\mu^-$, let us take the spin quantization axis for e^+ and e^- to be the $\hat{z}$ direction (the e^- direction). As we have seen above, the interaction proceeds only when the spin of e^+ and that of e^- are aligned:



(6.97)

This can be interpreted as the virtual photon having polarization $|1,1\rangle_z$ or $|1,-1\rangle_z$ where the subscript z indicates that the quantization axis for these states is $\hat{z}$. Then, the creation of $\mu^+\mu^-$ can be considered as decays of this virtual photon to a $\mu^+\mu^-$ pair. On the other hand, the allowed spin configurations for the $\mu^+\mu^-$ pair is



(6.98)

where the spin quantization axis is taken to be $\hat{n}$ (the direction of μ^-). Then, the virtual photon coupling to the $\mu^+\mu^-$ pair should have spin state $|1,1\rangle_n$ or $|1,-1\rangle_n$ where the subscript n indicates that the quantization axis is $\hat{n}$. The angular distribution is then given by the inner product of the initial and final spin states of the virtual photon (four combinations):

$$\frac{d\sigma}{d\Omega} \propto \sum_{m,m'=\pm 1} |{}_n\langle 1,m|1,m'\rangle_z|^2 = \sum_{m,m'=\pm 1} |d^1_{m,m'}(\theta)|^2 \tag{6.99}$$

where the inner products of angular momentum eigenstates with different quantization axes are given by the d-functions:

$$d^1_{1,1}(\theta) = d^1_{-1,-1}(\theta) = \frac{1+\cos\theta}{2}\,, \quad d^1_{1,-1}(\theta) = d^1_{-1,1}(\theta) = \frac{1-\cos\theta}{2}\,. \tag{6.100}$$

Thus, we obtain the angular distribution without actually calculating the matrix element in spinor space:

$$\frac{d\sigma}{d\Omega} \propto 1+\cos^2\theta\,. \tag{6.101}$$

The above selection rules applies at high-energy limit. When the masses are not negligible compared to energies involved, the selection rules are not exact, but still work approximately to the extent the masses are small. Then, exactly how often is the $\mu^+\mu^-$ pair created with both left-handed or both right-handed when the muon mass is finite? To answer the question, we will use the trick with the spin projection operator. Namely, we will insert a spin projection operator in front of each muon spinor:

$$u_{q_1} \to \Sigma_1 u_{q_1}\ (\mu^-)\,, \quad v_{q_2} \to \Sigma_2 u_{q_2}\ (\mu^+)\,, \tag{6.102}$$

with

$$\Sigma_i = \frac{1+\gamma_5 \not{s}_i}{2}\,, \quad (i=1,2)\,, \tag{6.103}$$

then sum over the spins of muons. This way, we obtain the rate for the spin states specified by the spin projection operators while allowing the use of trace techniques. Let's denote the spin component along the μ^- direction $\hat{n}$ (in unit of 1/2) as δ_i $(i=1,2)$; for example, if both particles are right-handed, then we have $\delta_1 = 1$ and $\delta_2 = -1$. The polarization 4-vectors s_1 and s_2 are obtained by boosting the values in the particle rest frame

$$s_i^\mu = (0, \delta_i \hat{n}) = \delta_i (0, \hat{n})\,, \quad (i=1,2) \tag{6.104}$$

to the C.M. frame of the interaction; namely, the boost is in the $\hat{n}$ direction for 1 (μ^-) and in the $-\hat{n}$ direction for 2 (μ^+):

$$s_1^\mu = \delta_1(\eta, \gamma\hat{n})\,, \quad s_2^\mu = \delta_2(-\eta, \gamma\hat{n})\,, \tag{6.105}$$

where $\gamma \equiv E/m$ and $\eta \equiv P/m$. Substituting (6.102) in (6.82) and using $\overline{\Sigma}_i = \Sigma_i$, the matrix element is

$$\begin{aligned} \mathcal{M} &= \frac{ie^2}{s}(\bar{v}_{p_2}\gamma_\mu u_{p_1})(\overline{\Sigma_1 u_{q_1}}\gamma^\mu \Sigma_2 v_{q_2}) \\ &= \frac{ie^2}{s}(\bar{v}_{p_2}\gamma_\mu u_{p_1})(\bar{u}_{q_1}\Sigma_1\gamma^\mu\Sigma_2 v_{q_2})\,. \end{aligned} \tag{6.106}$$

Now we square this matrix element, sum over all spins and divide by four for the number of initial-state spin configurations:

$$\begin{aligned} \overline{|\mathcal{M}|^2} &= \frac{e^4}{4s^2}\mathrm{Tr}\not{p}_2\gamma_\mu\not{p}_1\gamma_\nu \ \cdot\ \mathrm{Tr}(\not{q}_1+m)\Sigma_1\gamma^\mu\Sigma_2(\not{q}_2-m)\Sigma_2\gamma^\nu\Sigma_1 \\ &= \frac{e^4}{4s^2}\mathrm{Tr}\not{p}_2\gamma_\mu\not{p}_1\gamma_\nu \ \cdot\ \mathrm{Tr}(\not{q}_1+m)\Sigma_1\gamma^\mu(\not{q}_2-m)\Sigma_2\gamma^\nu\,, \end{aligned} \tag{6.107}$$

where

$$m \equiv m_\mu\,, \tag{6.108}$$

and we have used the fact that $\not{q}_i$ commutes with Σ_i for same i (3.283) and $\Sigma_i^2 = \Sigma_i$. The first trace is the same as before and given in (6.87). The second trace is straightforward if cumbersome. Using the trace theorems, $s_i \cdot q_i = 0$, and dropping the terms with $\epsilon^{\mu\nu\alpha\beta}$ (since it is multiplied to the first trace which is symmetric under $\mu \leftrightarrow \nu$), we obtain

$$\begin{aligned}\mathrm{Tr}(\not{q}_1 + m)\Sigma_1\gamma^\mu(\not{q}_2 - m)\Sigma_2\gamma^\nu &= q^{1\mu}q^{2\nu} - q_1\cdot q_2\, g^{\mu\nu} + q^{2\mu}q^{1\nu} - m_\mu^2 g^{\mu\nu} \\ &-(q_1\cdot q_2 + m^2)(s^{1\mu}s^{2\nu} - s_1\cdot s_2\, g^{\mu\nu} + s^{2\mu}s^{1\nu}) - s_1\cdot q_2\, s_2\cdot q_1\, g^{\mu\nu} \\ &+\Big[(s_1\cdot q_2\, s_2^\mu q_1^\nu + s_2\cdot q_1\, s_1^\mu q_2^\nu - s_1\cdot s_2\, q_2^\mu q_1^\nu) + (\mu\leftrightarrow\nu)\Big]\,. \end{aligned} \tag{6.109}$$

Executing the product of the two traces and using (6.88) together with the following explicit evaluations

$$\begin{aligned} q_2\cdot s_1 &= -q_1\cdot s_2 = 2\frac{E^2}{m}\beta\delta_i\,, \quad (i : \text{subscript of } s) \\ p_2\cdot s_1 &= -p_1\cdot s_2 = \frac{E^2}{m}(\beta + \cos\theta)\delta_i\,, \\ p_1\cdot s_1 &= -p_2\cdot s_2 = \frac{E^2}{m}(\beta - \cos\theta)\delta_i\,, \end{aligned} \tag{6.110}$$

the spin-averaged $|\mathcal{M}|^2$ is found to be

$$\overline{|\mathcal{M}|^2} = \frac{e^4}{s}\Big[(E^2+m^2) + P^2\cos^2\theta + \delta_1\delta_2\Big((E^2+m^2)\cos^2\theta + P^2\Big)\Big]\,. \tag{6.111}$$

Using the formula (5.287), the scattering cross section becomes

$$\frac{d\sigma}{d\Omega} = \begin{cases} \dfrac{\alpha^2}{8s}\,\beta(1+\cos^2\theta) & (\delta_1\delta_2 = 1) \\ \dfrac{\alpha^2}{8s}\,\beta(1-\beta^2)\sin^2\theta & (\delta_1\delta_2 = -1) \end{cases}\,, \tag{6.112}$$

where $\delta_1\delta_2 = 1$ corresponds to a favored spin configuration of $\mu^+\mu^-$ pair (one is right-handed and the other is left-handed), and $\delta_1\delta_2 = -1$ corresponds to a suppressed configuration (both left-handed or both right-handed).

Let us now appreciate the physical meaning of this result. First, the cross section for $\delta_1\delta_2 = -1$ vanishes for $\beta \to 1$ as expected - the selection rule is exact in the massless limit. Note that the favored spin configuration is seen to have the same angular distribution, $1+\cos^2\theta$, as the massless case. This is because the spin angular momentum along the $\hat{n}$ direction is already specified to be ± 1, thus the same argument as in (6.99) still holds. The angular distribution for the suppressed case can also be derived similarly. In this case, the muon spins add up to zero along $\hat{n}$ and thus the

virtual photon that couples to the final state should have the zero component along $\hat{n}$. Thus, the angular distribution should be given by

$$\frac{d\sigma}{d\Omega} \propto \sum_{m=\pm 1} |_n\langle 1,0|1,m\rangle_z|^2 = \sum_{m=\pm 1} |d^1_{0,m}(\theta)|^2 = \sin^2\theta\,, \tag{6.113}$$

where we have used $d^1_{0,1}(\theta) = -d^1_{0,-1}(\theta) = \sin\theta/\sqrt{2}$. We see that the rate for suppressed spin configuration is maximum at $\theta = \pi/2$ and becomes zero for $\theta = 0$ or π. This can be understood as follows: when the $\mu^+\mu^-$ pair is emitted along the beam direction, the z component of the angular momentum of the virtual photon, $m = \pm 1$, has to be carried by that of the muon pair. Since the orbital angular momentum cannot have component along the line of the pair, the sum of the spin z components of the pair has to be ± 1 which means that the spins have to be aligned along the z direction. Integrating (6.112) over the angle and taking the ratio, we obtain

$$\frac{\sigma(\delta_1\delta_2 = -1)}{\sigma(\delta_1\delta_2 = 1)} = \frac{1-\beta^2}{2} = \frac{m^2}{2E^2}\,. \tag{6.114}$$

The factor m^2/E^2 is the typical suppression factor that occurs when a fermion spin is opposite to what is allowed in the massless limit.

6.5 Scalar-photon interaction

In order to introduce the interaction of photon with charged scalar (denoted $H^\pm$), we apply the gauge principle to the free charged scalar Lagrangian (4.228)

$$\mathcal{L}_H = (\partial_\mu\phi)^\dagger\partial^\mu\phi - m^2\phi^\dagger\phi\,, \tag{6.115}$$

which is clearly invariant under the phase transformation $\phi(x) \to e^{i\theta}\phi(x)$ where θ is a real constant. In order to make the Lagrangian invariant under phase transformation that depends on space time, we introduce a massless spin-1 field A_μ and replace ∂_μ with the covariant derivative $D_\mu = \partial_\mu + ieA_\mu$ as before. The resulting Lagrangian

$$\mathcal{L} = (D_\mu\phi)^\dagger D^\mu\phi - m^2\phi^\dagger\phi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}\,, \tag{6.116}$$

is invariant under the simultaneous transformation of $\phi(x)$ and $A_\mu(x)$ given by

$$\phi(x) \to e^{-ie\Lambda(x)}\phi(x)\,, \quad A_\mu(x) \to A_\mu(x) + \partial_\mu\Lambda(x)\,.$$

This can be seen by noting that the covariant derivative of the field $D_\mu\phi$ transforms by a simple phase rotation:

$$D_\mu\phi \to e^{-ie\Lambda(x)}D_\mu\phi\,,$$

which is derived in the same way as (6.43). Note that we used the replacement $\partial_\mu\psi^\dagger \to (D_\mu\psi)^\dagger$ rather than $\partial_\mu\psi^\dagger \to D_\mu(\psi^\dagger)$ in order to make the Lagrangian real (hermitian).

The scaler-photon interaction is contained in the kinetic term:

$$\begin{aligned}
&(D_\mu\phi)^\dagger D^\mu\phi - m^2\phi^\dagger\phi \\
&\quad = (\partial_\mu - ieA_\mu)\phi^\dagger(\partial^\mu + ieA^\mu)\phi - m^2\phi^\dagger\phi \\
&\quad = \underbrace{(\partial_\mu\phi)^\dagger\partial^\mu\phi - m^2\phi^\dagger\phi}_{\mathcal{L}_H} + \underbrace{(-ieA_\mu)\phi^\dagger\partial^\mu\phi + (\partial_\mu\phi)^\dagger ieA^\mu\phi}_{-eA^\mu i[\phi^\dagger(\partial_\mu\phi) - (\partial_\mu\phi)^\dagger\phi]} + e^2A_\mu A^\mu\phi^\dagger\phi \\
&\quad = \mathcal{L}_H + \mathcal{L}_{\rm int}\,,
\end{aligned} \tag{6.117}$$

where

$$\mathcal{L}_{\rm int} = -eA^\mu j_\mu + e^2A_\mu A^\mu\phi^\dagger\phi\,, \tag{6.118}$$

with

$$j_\mu \equiv i[\phi^\dagger(\partial_\mu\phi) - (\partial_\mu\phi)^\dagger\phi] \equiv i\phi^\dagger\overset{\leftrightarrow}{\partial}_\mu\phi\,. \tag{6.119}$$

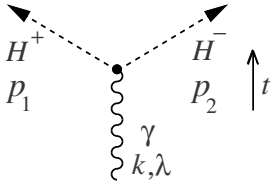
This is the same conserved current we obtained for the free Klein-Gordon field in (2.44). In the interaction picture, the operator field $\phi(x)$ satisfies the free Klein-Gordon equation; and thus the same derivation that led to the conservation of (2.44) shows that the operator current j_μ above in the interaction picture satisfies

$$\partial_\mu j^\mu = 0\,.$$

The proper canonical quantization involves defining the conjugate field π which has an extra term arising from $\partial_0\phi$ found in $\mathcal{L}_{\rm int}$, forming the Hamiltonian, identifying the interaction Hamiltonian and introducing the standard commutation relation between ϕ and π. As a result, the interaction Hamiltonian $\mathcal{H}_{\rm int}$ is not simply $-\mathcal{L}_{\rm int}$. In the interaction picture, the resulting interaction Hamiltonian can be written as

$$\mathcal{H}_{\rm int} = ieA^\mu(\phi^\dagger\overset{\leftrightarrow}{\partial}_\mu\phi) + e^2\phi^\dagger\phi\vec{A}^2 \tag{6.120}$$

which differ from $-\mathcal{L}_{\rm int}$ by $e^2\phi^\dagger\phi A^{0\,2}$. Let us ignore the second term for now, which is second order in e, and find the vertex rule for the first term. First, consider the 'decay' of a virtual photon $\gamma \to H^+H^-$. The virtual photon is actually a propagator, but one may assume that the photon is massive and on-shell for this exercise - the result does not depend on this assumption. The initial and final states are then



$$\begin{aligned}
|i\rangle &= a^\dagger_{k\lambda}|0\rangle\,, \\
|f\rangle &= a^\dagger_{p_1}b^\dagger_{p_2}|0\rangle\,.
\end{aligned} \tag{6.121}$$

The S matrix element is first-order in the Dyson series (5.76):

$$\begin{aligned} S_{fi} &= -i\int d^4x \langle f|\mathcal{H}_{\rm int}(x)|i\rangle \\ &= -i\int d^4x \langle 0|b_{p_2}a_{p_1}\Big[ie(\underbrace{\phi^\dagger \overleftrightarrow{\partial}_\mu \phi)A^\mu}_{1=\sum_n |n\rangle\langle n|}\Big]a^\dagger_{k\lambda}|0\rangle \\ &= e\int d^4x \langle 0|b_{p_2}a_{p_1}(\phi^\dagger \overleftrightarrow{\partial}_\mu \phi)|0\rangle\langle 0|A^\mu a^\dagger_{k\lambda}|0\rangle\,, \end{aligned} \tag{6.122}$$

where in the second line we have inserted identity and used the fact that there are only photon operators to its right and only scalar operators to its left, thus only $|0\rangle\langle 0|$ survives. In the second expectation value, only $a_{k\lambda}\epsilon^\mu_{k\lambda}e_k(x)$ in A^μ survives:

$$\langle 0|A^\mu a^\dagger_{k\lambda}|0\rangle = \epsilon^\mu_{k\lambda}e_k(x)\,. \tag{6.123}$$

In the first vacuum expectation value, only the term with $a^\dagger_{p_1}$ and $b^\dagger_{p_2}$ survives:

$$\begin{aligned} \langle 0|b_{p_2}a_{p_1}(\phi^\dagger \overleftrightarrow{\partial}_\mu \phi)|0\rangle &= \langle 0|b_{p_2}a_{p_1}\sum_p(\boxed{a^\dagger_p e^*_p} + b_p e_p)\overleftrightarrow{\partial}_\mu \sum_q (a_q e_q + \boxed{b^\dagger_q e^*_q})|0\rangle \\ &= \langle 0|b_{p_2}a_{p_1}(a^\dagger_{p_1}e^*_{p_1}\overleftrightarrow{\partial}_\mu b^\dagger_{p_2}e^*_{p_2})|0\rangle = e^*_{p_1}(x)\overleftrightarrow{\partial}_\mu e^*_{p_2}(x) \\ &= -i(p_1-p_2)_\mu\, e^*_{p_1}(x)e^*_{p_2}(x)\,. \end{aligned} \tag{6.124}$$

We then identify the Lorentz-invariant matrix element $\mathcal{M}$ in S_{fi}:

$$S_{fi} = \underbrace{\int d^4x e^*_{p_1}(x)e^*_{p_2}(x)e_k(x)}_{\text{4-momentum conservation}}\ \underbrace{\Big[-ie(p_1-p_2)_\mu\epsilon^\mu_{k\lambda}\Big]}_{\mathcal{M}} \tag{6.125}$$

The factor $\epsilon^\mu_{k\lambda}$ in $\mathcal{M}$ is the photon external leg factor we have seen before and the vertex factor of photon scalar coupling is then identified as $-ie(p_1-p_2)_\mu$.

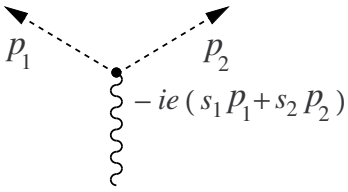
What signs should be given to the 4-momenta in general cases? We note that final state particles always should be matched with creation operators $a^\dagger$ or $b^\dagger$ (the particle has to be 'created') and initial state particles with annihilation operators (the particle has to be 'destroyed'). In momentum expansion of fields, creation operators are always attached to $e^*_p(x)$ and annihilation operators to $e_p(x)$. Thus, a final state particle with momentum p will be associated with $e^*_p(x)$ and an initial-state particle with momentum p with $e_p(x)$. Thus, ∂_μ will pull down the factor ip_μ for a final-state particle and $-ip_\mu$ for an initial-state particle. Also, there is an extra minus sign if the desired operator is found in $\phi^\dagger$ (namely, $a^\dagger$ or b), since it sits to the left of the

operator $\overset{\leftrightarrow}{\partial}_\mu$ in $(\phi^\dagger \overset{\leftrightarrow}{\partial}_\mu \phi)$. The signs are thus

$-ie(s_1p_1 + s_2p_2)$	$\phi^\dagger$	$(a^\dagger$	$b)$	$\overset{\leftrightarrow}{\partial}_\mu$	ϕ	$(a$	$b^\dagger)$
to the right(+) or left(−) of $\overset{\leftrightarrow}{\partial}$		−	−			+	+
initial(+) or final(−) state		−	+			+	−
over-all sign (s_i)		+	−			+	−

; (6.126)

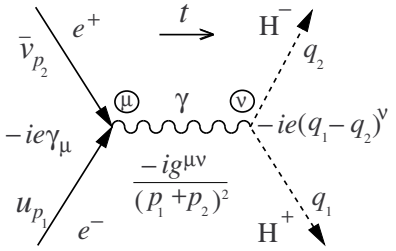
namely, the vertex factor is $-ie(s_1p_1 + s_2p_2)$ where sign s_i is plus if it corresponds to a particle and minus if antiparticle regardless of whether it appears in the initial state or in the final state. This applies even when one of the legs are a propagator as long as the momentum flowing in the propagator is determined by particle or antiparticle in the propagator to be moving forward in time; namely, when one draws a line as a particle propagating forward in time or an antiparticle propagating forward in time, the 4-momentum of the propagator has opposite sign which is canceled by the stated rule giving the same vertex factor in the end. Thus, the vertex rule of scalar-photon coupling is



s_i : + if particle, − if antiparticle. (6.127)

The photon can be dealt with in the same way as in the photon-electron system. We can assign a finite mass to the photon, obtain the spin-1 propagator as before, and take the limit of massless photon. Here, we need to worry if the $k^\mu k^\nu/M^2$ term in the propagator strictly vanishes - otherwise, it will diverge when the massless limit is taken.

We can apply these rules to the reaction $e^+e^- \to H^+H^-$ which occurs through a virtual photon as in the case of $e^+e^- \to \mu^+\mu^-$. The matrix element is then



$$\begin{aligned} \mathcal{M} &= ie^2(\bar{v}_{p_2}\gamma_\mu u_{p_1})\frac{g^{\mu\nu}}{(p_1+p_2)^2}(q_1-q_2)^\nu \\ &= \frac{ie^2}{s}(\bar{v}_{p_2}\not{k}u_{p_1}) \\ (s \equiv (p_1+p_2)^2\,, &\quad k \equiv q_1 - q_2) \end{aligned} \tag{6.128}$$

where the direction of arrows indicates the actual flow of 4-momentum; namely, those of e^+e^- flow into the interaction region and those of H^+H^- flows out of the interaction region. Square it and spin averaging over the initial-state, we obtain (also ignoring the electron mass)

$$\begin{aligned}\overline{|\mathcal{M}|^2} &= \frac{1}{4}\sum_{\text{spin}}|\mathcal{M}|^2 = \frac{e^4}{4s^2}\text{Tr}\not{p}_2\not{k}\not{p}_1\not{k} \\ &= \frac{e^4}{s^2}(2\,p_2\cdot k\,p_1\cdot k - p_2\cdot p_1\,k^2) \end{aligned} \tag{6.129}$$

In the C.M. frame, the kinematics is the same as in the case of $e^+e^- \to \mu^+\mu^-$ (6.88) where (μ^-, μ^+) is replaced by (H^+, H^-). Then, we have

$$\begin{aligned} k &= q_1 - q_2 = (0, 2P\hat{n})\,, \\ k^2 &= -4P^2\,, \\ p_2\cdot k &= -p_1\cdot k = 2EP\cos\theta\,, \\ p_1\cdot p_2 &= 2E^2\,. \end{aligned} \tag{6.130}$$

Using these expressions, $\overline{|\mathcal{M}|^2}$ becomes

$$\overline{|\mathcal{M}|^2} = \frac{e^4}{2}\beta^2\sin^2\theta\,, \tag{6.131}$$

which give the differential cross section

$$\frac{d\sigma}{d\Omega} = \frac{\overline{|\mathcal{M}|^2}}{(8\pi)^2 s}\frac{P}{E} = \frac{\alpha^2}{8s}\beta^3\sin^2\theta\,. \tag{6.132}$$

Integrating this over $d\Omega = 2\pi d\cos\theta$, we get the total cross section

$$\sigma(e^+e^- \to H^+H^-) = \frac{\pi\alpha^2}{3s}\beta^3\,. \tag{6.133}$$

The angular distribution can be understood in terms of the spin state of the virtual photon which is $|1, \pm 1\rangle_z$ by the same argument as in the case of $e^+e^- \to \mu^+\mu^-$. Since scalar particle is spin zero, the angular momentum of the virtual photon has to be carried by the orbital angular momentum of the H^+H^- pair. Thus, the angular wave function of the H^+H^- pair should be $Y^1_{\pm 1}(\theta, \phi)$ which should add up incoherently since the two cases correspond to different initial sates:

$$\frac{d\sigma}{d\Omega} \propto |Y^1_1(\theta,\phi)|^2 + |Y^1_{-1}(\theta,\phi)| \propto \sin^2\theta\,. \tag{6.134}$$

Note also the threshold behavior is $\sigma \propto \beta^3$; this is typical for a P-wave threshold production (namely, orbital angular momentum = 1). This can be compared to the

fermion pair production which behaved as $\sigma \propto \beta$ near the threshold. At well above threshold, the cross section for $e^+e^- \to H^+H^-$ is 1/4 of the muon pair case. Thus, the size of total cross section, the threshold behavior, as well as the angular distribution can be (and have been) used to determine the spin of the particle-antiparticle pairs being produced in e^+e^- annihilation.

6.6 Wick's theorem

In order to obtain the complete set of Feynman rules of QED, we will now introduce a powerful theorem that allows one to expand the time ordered products in the Dyson series into terms that contain propagators. Using $h(t) = \int d^3x \mathcal{H}_{\rm int}(x)$, the Dyson series (5.74) can be written as

$$S_{fi} = \sum_{n=1}^{\infty} \frac{(-i)^n}{n\,!} \int d^4x_1 \cdots d^4x_n \langle f|T\Big(\mathcal{H}_{\rm int}(x_1)\cdots\mathcal{H}_{\rm int}(x_n)\Big)|i\rangle\,. \tag{6.135}$$

where we have assumed $|i\rangle \neq |f\rangle$. In evaluating $\langle f|T(\mathcal{H}_{\rm int}(x_1)\cdots\mathcal{H}_{\rm int}(x_n))|i\rangle$, we note that terms that survive should have annihilation operators that exactly match the initial state particles and creation operators that exactly match the final-state particles. If there are extra operators in a given term, which would happen for higher-order terms, then they should be matched within themselves in order for the term not to vanish. We have seen an example in the $\nu_\mu e \to \mu\nu_e$ scattering where extra operators were the creation and annihilation operators of W, resulting in the W propagator. We will now see how each order in the expansion above becomes a set of diagrams with Feynman propagators properly inserted.

First, we define the Wick contraction of two fields $\phi_1(x_1)$ and $\phi_2(x_2)$ by the difference between their time-ordered product and the normal product:

$$\overbrace{\phi_1(x_1)\phi_2(x_2)} \stackrel{\rm def}{\equiv} T(\phi_1(x_1)\phi_2(x_2)) - :\phi_1(x_1)\phi_2(x_2):\,, \tag{6.136}$$

where ϕ_i may be a scalar field, a component of a fermion or vector field, and can be separated into an annihilation part and creation part as in (4.269) or (4.358):

$$\phi_i(x_i) = \phi_i^a(x_i) + \phi_i^c(x_i)\,, \tag{6.137}$$

where the superscript 'a' ('c') stands for the annihilation part (the creation part). The fields in (6.136) may be $\phi^{a,c}(x)$, or for that matter any operators that are functions of x and constructed out of creation and annihilation operators. To simplify expressions, we will use the shorthand

$$\phi_i \stackrel{\rm def}{\equiv} \phi_i(x_i) \quad \phi_i^{a,c} \stackrel{\rm def}{\equiv} \phi_i^{a,c}(x_i) \tag{6.138}$$

Recall that the time-ordered product is defined by [see (5.75)]

$$T(\phi_1\phi_2) = \begin{cases} \phi_1\phi_2 & (x_1^0 \geq x_2^0) \\ s\,\phi_2\phi_1 & (x_1^0 < x_2^0) \end{cases} , \tag{6.139}$$

and the normal-ordered product is given by reordering the fields such that ϕ_i^c is always to the left of ϕ_i^a:

$$\begin{aligned} :\phi_1\phi_2: &= :(\phi_1^a + \phi_1^c)(\phi_2^a + \phi_2^c): \\ &= :\phi_1^a\phi_2^a + \underline{\phi_1^a\phi_2^c} + \phi_1^c\phi_2^a + \phi_1^c\phi_2^c: \\ &= \phi_1^a\phi_2^a + \underline{s\,\phi_2^c\phi_1^a} + \phi_1^c\phi_2^a + \phi_1^c\phi_2^c\,, \end{aligned} \tag{6.140}$$

where s in (6.139) and (6.140) is -1 if ϕ_1 and ϕ_2 are both fermion fields and $+1$ otherwise (namely, if at least one of them is a boson field). Then, if ϕ_1 and ϕ_2 commute (or anticommute if both are fermion) then $T(\phi_1\phi_2)$ and $:\phi_1\phi_2:$ are both equal to the original product; namely, the Wick contraction is zero:

$$\overbrace{\phi_1\phi_2} = 0\,, \quad \text{if } \phi_1 \text{ and } \phi_2 \text{ (anti)commute}\,. \tag{6.141}$$

In the above and hereafter, when we write '(anti)commute', 'anticommute' applies only when both fields in question are fermion. When the two fields are of same type, they may not (anti)commute. In such cases, the effect of reordering comes from the (anti)commutation relations

$$a_{\vec{p}\sigma}a^\dagger_{\vec{p}'\sigma'} - s\,a^\dagger_{\vec{p}'\sigma'}a_{\vec{p}\sigma} = \delta_{\vec{p}\vec{p}'}\delta_{\sigma\sigma'}\,, \tag{6.142}$$

where σ is a spin index if any, and the time-ordered product and the normal-ordered product differ from the original product by a mere c-number. This makes the Wick contraction a c-number, and sandwiching the defining equation (6.136) between the vacuum gives

$$\overbrace{\phi_1\phi_2} = \langle 0|T(\phi_1\phi_2)|0\rangle\,, \tag{6.143}$$

where we have used the fact that the vacuum expectation value of a normal-ordered product is zero by definition (4.202). Thus, the Feynman propagators are nothing but Wick contractions; namely,

$$\begin{aligned} \overbrace{\phi(x)\phi^\dagger}(y) = i\Delta_F(x-y)\,, \qquad & \overbrace{\psi_n(x)\bar{\psi}_m}(y) = iS_{Fnm}(x-y)\,, \\ \overbrace{A^\mu(x)A^\nu}(y) \;&=\; iD_F{}^{\mu\nu}(x-y)\,. \end{aligned} \tag{6.144}$$

Now define a contraction within a normal product by

$$: \phi_1 \cdots \overbrace{\phi_i \cdots \phi_j} \cdots \phi_n : \stackrel{\text{def}}{\equiv} \underbrace{s^{1\cdots n}_{ij1\cdots\not{i}\cdots\not{j}\cdots n}}_{d_{ij}} \; \overbrace{\phi_i \phi_j} \; : \phi_1 \cdots \not{\phi}_i \, \not{\phi}_j \cdots \phi_n : , \tag{6.145}$$

where $ij1\cdots\not{i}\cdots\not{j}\cdots n$ indicates that i and j are missing from the ascending series $1\cdots n$, and

$$\begin{cases} s^{i_1\cdots i_n}_{j_1\cdots j_n} \stackrel{\text{def}}{\equiv} (-1)^l & \text{l: number of fermion flips needed to reorder } \phi_{i_1}\cdots\phi_{i_n} \text{ to } \phi_{j_1}\cdots\phi_{j_n}. \\ d_{ij} \stackrel{\text{def}}{\equiv} (-1)^k & \text{k: number of fermion flips to move } \phi_i \text{ and } \phi_j \text{ next to each other without changing their relative order.} \end{cases} \tag{6.146}$$

These two sign factors are equivalent since if two fields are of different type then the whole expression is zero due to (6.141), and once two fields of same type are put next to each other, they can be moved together without changing sign. Multiple contractions are defined similarly:

$$\begin{aligned} &: \phi_1 \cdots \overbrace{\phi_i \cdots \overbrace{\phi_k \cdots \phi_j} \cdots \phi_l} \cdots \phi_n : \\ &\qquad \stackrel{\text{def}}{\equiv} s^{1\cdots n}_{ijkl\,1\cdots\not{i}\cdots\not{k}\cdots\not{j}\cdots\not{l}\cdots n} \; \overbrace{\phi_i \phi_j} \overbrace{\phi_k \phi_l} \; : \phi_1 \cdots \not{\phi}_i \, \not{\phi}_k \not{\phi}_j \, \not{\phi}_l \cdots \phi_n : , \end{aligned} \tag{6.147}$$

Note that the order of lower indices of s is the same as the order of fields that follow including the contracted ones. The final ordering of a time-ordered product is uniquely determined by the time arguments of the fields. Thus, it does not matter in what order the fields are placed in the time-ordered product - up to a sign. Suppose $x^0_{k_1} > \cdots > x^0_{k_n}$, then by the definition of time-ordered product,

$$\begin{aligned} T(\phi_1 \cdots \phi_n) &= s^{1\cdots n}_{k_1\cdots k_n} \, \phi_{k_1} \cdots \phi_{k_n} \\ T(\phi_{i_1} \cdots \phi_{i_n}) &= s^{i_1\cdots i_n}_{k_1\cdots k_n} \, \phi_{k_1} \cdots \phi_{k_n} \, . \end{aligned} \tag{6.148}$$

Thus,

$$T(\phi_{i_1} \cdots \phi_{i_n}) = \underbrace{s^{1\cdots n}_{k_1\cdots k_n} s^{i_1\cdots i_n}_{k_1\cdots k_n}}_{s^{1\cdots n}_{i_1\cdots i_n}} T(\phi_1 \cdots \phi_n) \tag{6.149}$$

According to the definition (6.139), this does not hold if any of the times are identical; for example, if $x^0_k = x^0_l$, we have [3]

$$T(\phi_k \phi_l) = \phi_k \phi_l \, , \quad T(\phi_l \phi_k) = \phi_l \phi_k \neq s \phi_k \phi_l \text{ in general.} \tag{6.150}$$

[3] If we define the time-ordered product for fields with equal time to be $T(\phi_1(t)\phi_2(t)) = \phi_1(t)\phi_2(t) + s\phi_2(t)\phi_1(t)$, then $T(\phi_1(t_1)\phi_2(t_2)) = sT(\phi_2(t_2)\phi_1(t_1))$ holds also for $t_1 = t_2$. This will, however, lead to some complications later on, and we will stick to the definition (6.139).

However, in the following proof, we will not have to exchange the order if the times are equal.

Similarly, the result of a normal ordering does not change up to a sign when one starts from a different ordering of the same fields. The sign is again given by the same factor as in the case of time ordering:

$$: \phi_{i_1} \cdots \phi_{i_n} : = s^{1\cdots n}_{i_1 \cdots i_n} : \phi_1 \cdots \phi_n : \ . \tag{6.151}$$

Applying these to $n = 2$, we see that the Wick contraction also changes sign if two fermion fields are switched:

$$\overbracket{\phi_2 \phi_1} = T(\phi_2 \phi_1) - : \phi_2 \phi_1 : = s \, [T(\phi_1 \phi_2) - : \phi_1 \phi_2 : \,] = s \, \overbracket{\phi_1 \phi_2} \, , \tag{6.152}$$

which works for $x_1^0 \neq x_2^0$.

Now, Wick's theorem states that

$$T(\phi_1 \cdots \phi_n) = apc(\phi_1 \cdots \phi_n) \, , \qquad \text{(Wick's theorem)} \tag{6.153}$$

where $apc(\phi_1 \cdots \phi_n)$ stands for the sum of all possible contractions of $: \phi_1 \cdots \phi_n :$; namely,

$$\begin{aligned} apc(\phi_1 \cdots \phi_n) \ &= \ : \phi_1 \cdots \phi_n : \qquad \text{(0 contraction)} \\ &+ \ : \phi_1 \cdots \overbracket{\phi_i \cdots \phi_j} \cdots \phi_n : + \ \cdots \qquad \text{(1 contraction)} \\ &+ \ : \phi_1 \cdots \overbracket{\phi_i \cdots \overbracket{\phi_k \cdots \phi_j} \cdots \phi_l} \cdots \phi_n : + \ \cdots \qquad \text{(2 contractions)} \\ &+ \ \cdots \, , \end{aligned} \tag{6.154}$$

where for example the 2-contraction terms are for all possible ways to pick two pairs (i, j) and (k, l) out of $(1 \cdots n)$.

The proof of Wick's theorem is a bit cumbersome, in particular to keep track of signs arising from fermion fields. We will prove it without waving hands and I believe it would be useful to follow the next few pages; if you choose to skip it, however, just make sure that you know the exact meaning of Wick's theorem and are able to use it correctly.

For n=2, it trivially holds since it is nothing but the definition of the Wick contraction (6.136). In proving this theorem, let's use a further-simplified notation:

$$i \overset{\text{def}}{\equiv} \phi_i \, , \quad i^{a,c} \overset{\text{def}}{\equiv} \phi_i^{a,c} \, . \tag{6.155}$$

We will prove this in three steps: (a) without losing generality, we can assume that $x_1^0 \geq \cdots \geq x_n^0$, (b) the following relation holds:

$$: 1 \cdots (n-1) : n = : 1 \cdots n : + \sum_{k=1}^{n-1} : 1 \cdots \overbracket{k \cdots n} : \, , \tag{6.156}$$

(c) if Wick's theorem holds for $n-1$, then it holds for n.

(a) Suppose the fields $1\cdots n$ are not time-ordered. When they are reordered such that $x^0_{i_1}\geq\cdots\geq x^0_{i_n}$, the time-ordered product on the left-hand side of the theorem picks up the factor $s^{1\cdots n}_{i_1\cdots i_n}$ according to (6.149). If any of the times are equal, then the relative order is not changed. For a given term in $apc(\phi_{i_1}\cdots\phi_{i_n})$, there is one term in $apc(\phi_1\cdots\phi_n)$ that has exactly the same fields contracted, say k and l ($k<l$). They are by definition

$$:1\cdots\overbrace{k\cdots l}\cdots n:=s^{1\cdots n}_{kl\,1\cdots\not{k}\cdots\not{l}\cdots n}\,\overbrace{kl}\;:1\cdots\not{k}\cdots\not{l}\cdots n: \tag{6.157}$$

and

$$:i_1\cdots\overbrace{l\cdots k}\cdots i_n:=s^{i_1\cdots i_n}_{lk\,i_1\cdots\not{l}\cdots\not{k}\cdots i_n}\,\overbrace{lk}\;:i_1\cdots\not{l}\cdots\not{k}\cdots i_n:, \tag{6.158}$$

where we have assumed that k appear to the left of l in $(i_1\cdots i_n)$. Due to (6.151) and (6.152), we can switch k and l and reorder $(i_1\cdots\not{l}\cdots\not{k}\cdots i_n)$ in (6.158) as long as we do it consistently in the lower indices of s and for the fields. Thus,

$$\begin{aligned}:i_1\cdots\overbrace{l\cdots k}\cdots i_n:\;&=\underbrace{s^{i_1\cdots i_n}_{kl\,1\cdots\not{k}\cdots\not{l}\cdots n}}_{s^{1\cdots n}_{i_1\cdots i_n}s^{1\cdots n}_{kl\,1\cdots\not{k}\cdots\not{l}\cdots n}}\overbrace{kl}\;:1\cdots\not{k}\cdots\not{l}\cdots n:\\&=s^{1\cdots n}_{i_1\cdots i_n}:1\cdots\overbrace{k\cdots l}\cdots n:\,.\end{aligned} \tag{6.159}$$

Similarly, all other terms in $apc(\phi_{i_1}\cdots\phi_{i_n})$ pick up the same relative sign with respect to the corresponding terms in $apc(\phi_1\cdots\phi_n)$. Namely, when one reorders the fields inside a normal product with contractions, it simply picks up the sign $s^{1\cdots n}_{i_1\cdots i_n}$ as long as the same fields are contracted before and after the reordering. Thus,

$$apc(\phi_{i_1}\cdots\phi_{i_n})=s^{1\cdots n}_{i_1\cdots i_n}\,apc(\phi_1\cdots\phi_n)\,. \tag{6.160}$$

Together with (6.149), we see that if Wick's theorem holds for $x^0_1\geq\cdots\geq x^0_n$, then it holds for any ordering of fields.

(b) We now assume $x^0_1\geq\cdots\geq x^0_n$. By induction, we will first prove

$$:1\cdots(n-1):n^c=:1\cdots n^c:+\sum_{k=1}^{n-1}:1\cdots\overbrace{k\cdots n}^{c}:\,. \tag{6.161}$$

For $n=2$, it trivially holds by the definition of the Wick contraction:

$$\begin{aligned}&\overbrace{T(12^c)\text{ since }x^0_1\geq x^0_2}\\&\overbrace{12^c}\;=:12^c:+\overbrace{12}^{c}\quad\to\quad\overbrace{12}^{c}=T(12^c)-:12^c:\,.\end{aligned} \tag{6.162}$$

Note that a creation field at the left-hand-most inside a normal ordering or an annihilation field at the right-hand most comes straight out of the normal ordering sign:

$$\overleftarrow{:}\boxed{1^c}2\cdots n:=1^c:2\cdots n:,\quad :1\cdots(n-1)\boxed{n^a}\overrightarrow{:}=:1\cdots(n-1):n^a \tag{6.163}$$

and together with (6.151),

$$:\boxed{1^a}\overrightarrow{2\cdots n}:=s^{1\cdots n}_{2\cdots n\,1}:2\cdots n\boxed{1^a}\overrightarrow{:}=s^{1\cdots n}_{2\cdots n\,1}:2\cdots n:1^a\,. \tag{6.164}$$

Thus, we have

$$:1\cdots n:=\overleftarrow{:(\boxed{1^c}+\boxed{1^a})\,2\cdots n:}=1^c:2\cdots n:+s^{1\cdots n}_{2\cdots n\,1}:2\cdots n:1^a\,. \tag{6.165}$$

Similar relation holds even when there are contractions. Using (6.159),

$$\begin{aligned}
:1\cdots\overbrace{k\cdots l}\cdots n:&=:(1^c+1^a)\cdots\overbrace{k\cdots l}\cdots n:\\
&=:1^c2\cdots\overbrace{k\cdots l}\cdots n:+\underbrace{:1^a2\cdots\overbrace{k\cdots l}\cdots n:}_{s^{1\cdots n}_{2\cdots n\,1}:2\cdots\overbrace{k\cdots l}\cdots n1^a:}\\
&=1^c:2\cdots\overbrace{k\cdots l}\cdots n:+s^{1\cdots n}_{2\cdots n\,1}:2\cdots\overbrace{k\cdots l}\cdots n:1^a\,,
\end{aligned} \tag{6.166}$$

and similarly for the cases with more than one contractions. Also, since $x^0_1\geq x^0_n$, we have

$$\begin{aligned}
\overbrace{1n} &= T(1\,n)-:1\,n:=(1^a+1^c)(n^a+n^c)-:(1^a+1^c)(n^a+n^c):\\
&=1^an^c-s^{1\,n}_{n\,1}\,n^c1^a\,.
\end{aligned} \tag{6.167}$$

Then, using (6.165),

$$\begin{aligned}
&:1\cdots(n-1):n^c\\
&\quad=\Big[1^c:2\cdots(n-1):+s^{1\cdots(n-1)}_{2\cdots(n-1)\,1}:2\cdots(n-1):1^a\Big]n^c\\
&\quad=1^c:2\cdots(n-1):n^c\;+s^{1\cdots(n-1)}_{2\cdots(n-1)\,1}:2\cdots(n-1):\underbrace{1^an^c}_{(6.167)\to\,s^{1\,n}_{n\,1}\,n^c1^a+\overbrace{1n}}\\
&\quad=1^c\underbrace{:2\cdots(n-1):n^c}_{(*)}+\underbrace{s^{1\cdots(n-1)}_{2\cdots(n-1)\,1}s^{1\,n}_{n\,1}}_{s^{1\cdots n}_{2\cdots n\,1}}\underbrace{:2\cdots(n-1):n^c}_{(*)}1^a\\
&\qquad+\underbrace{s^{1\cdots(n-1)}_{2\cdots(n-1)\,1}\overbrace{1n}:2\cdots(n-1):}_{:\overbrace{1\cdots n}:},
\end{aligned} \tag{6.168}$$

where in the last step we have used $s^{1\cdots(n-1)}_{2\cdots(n-1)\,1} = d_{1n}$ which corresponds to moving 1 next to n.

Now assuming that (6.161) holds for $n-1$, we apply it to $(*)$ above:

$$
\begin{aligned}
:1\cdots(n-1):n^c \;&=\; :\overbrace{1\cdots n}: \\
&\quad + 1^c:2\cdots n^c: + \sum_{k=2}^{n-1} 1^c:2\cdots\overbrace{k\cdots n}^c: \\
&\quad + s^{1\cdots n}_{2\cdots n\,1}:2\cdots n^c:1^a + \sum_{k=2}^{n-1} s^{1\cdots n}_{2\cdots n\,1}:2\cdots\overbrace{k\cdots n}^c:1^a \\
&= :\overbrace{1\cdots n}: + \underbrace{1^c:2\cdots n^c: + s^{1\cdots n}_{2\cdots n\,1}:2\cdots n^c:1^a}_{:12\cdots n^c:\ \text{by (6.165)}} \\
&\quad + \sum_{k=2}^{n-1}\Big(\underbrace{1^c:2\cdots\overbrace{k\cdots n}^c: + s^{1\cdots n}_{2\cdots n\,1}:2\cdots\overbrace{k\cdots n}^c:1^a}_{:12\cdots\overbrace{k\cdots n}^c:\ \text{by (6.166)}}\Big) \\
&\quad :12\cdots n^c: + \sum_{k=1}^{n-1} :12\cdots\overbrace{k\cdots n}^c: \qquad (6.169)
\end{aligned}
$$

Thus, (6.161) holds for n and it is proven by induction. Adding

$$
:1\cdots(n-1):n^a = :1\cdots(n-1)n^a: \qquad (6.170)
$$

to (6.161), and noting

$$
\begin{aligned}
\overbrace{kn} &= \underbrace{\overbrace{kn}^a}_{T(kn^a)-\,:kn^a:\,=0} + \overbrace{kn}^c = \overbrace{kn}^c\,, \qquad (6.171)
\end{aligned}
$$

we establish (6.156).

(c) Since the Wick's theorem holds for $n=2$, we again use induction and assume it holds for $n-1$; then, since $x^0_1 \geq \cdots \geq x^0_n$,

$$
T(1\cdots n) = T(1\cdots(n-1))\,n = apc(1\cdots(n-1))\,n\,. \qquad (6.172)
$$

Now, we write out $apc(1\cdots(n-1))$ and use (6.156). The first term $:1\cdots(n-1):n$ gives $1\cdots n:$ and all possible contractions involving n. Then the term

$$
:1\cdots\overbrace{i\cdots j}\cdots(n-1):n \qquad (6.173)
$$

adds $: 1 \cdots \overbrace{i \cdots j} \cdots n :$ and all possible double contractions involving n, and similarly for higher-order contractions. Thus, the net result is that we have all possible contractions of $(1 \cdots n)$ and this proves the Wick's theorem.

The Wick's theorem holds for any fields for which time ordering and normal ordering can be defined, and they may be annihilation or creation parts of fermion fields, ψ_i^a, $\bar{\psi}_i^c$ etc. (note that $\bar{\psi}^c$ contains $a^\dagger_{\vec{p}s}$, not $b^{(\dagger)}_{\vec{p}s}$). When a group of fields in the time-ordered product have equal time and already normal-ordered, which can happen when they are part of an interaction Hamiltonian with a given x, then the only difference is that contractions within the group do not occur. This can be seen as follows: Suppose 1 and 2 are normal-ordered and $x_1^0 = x_2^0$, then

$$\begin{aligned} T(:1\,2:X) &= T(:(1^a+1^c)(2^a+2^c):X) \\ &= T((1^a2^a+1^c2^a+s\,2^c1^a+1^c2^c)X) \\ &= T(1^a2^aX)+T(1^c2^aX)+s\,T(2^c1^aX)+T(1^c2^cX) \end{aligned} \tag{6.174}$$

Applying the Wick's theorem to each of the four time-ordered products, the terms with only 1 and 2 contracted are

$$\underbrace{1^{\overbrace{a}}2^{\overbrace{a}}}_{0}:X:+1^{\overbrace{c}}2^{\overbrace{a}}:X:+s\,2^{\overbrace{c}}1^{\overbrace{a}}:X:+\underbrace{1^{\overbrace{c}}2^{\overbrace{c}}}_{0}:X:=(1^{\overbrace{c}}2^{\overbrace{a}}+s\,2^{\overbrace{c}}1^{\overbrace{a}}):X:, \tag{6.175}$$

where we have used the fact that the contraction of two fields that (anti)commute vanish. Since $x_1^0 = x_2^0$ we have

$$1^{\overbrace{c}}2^{\overbrace{a}} = \underbrace{T(1^c2^a)}_{1^c2^a} - \underbrace{:1^c2^a:}_{1^c2^a} = 0\,, \tag{6.176}$$

and $2^{\overbrace{c}}1^{\overbrace{a}}$ similarly vanishes. Thus, we can write

$$:\overbrace{\phi_1(t)\phi_2}(t):=0\,. \tag{6.177}$$

What happens if we ignore the normal ordering of the interaction term and proceed with applying Wick's theorem? As we will see later, we will simply obtain a constant 'infinity' which can be removed by the renormalization process.

6.7 Feynman rules of quantum electrodynamics

Let us apply the Wick's theorem to simple cases of quantum electrodynamics to see how it works. For the scattering $e^+e^- \to \mu^+\mu^-$, the interaction Lagrangian is given by (6.80):

$$\mathcal{H}_{\rm int}(x) = e:A_\alpha(x)(j_e^\alpha(x)+j_\mu^\alpha(x)):, \tag{6.178}$$

where we have explicitly written the normal ordering. Since the process should annihilate e^+e^- and create $\mu^+\mu^-$ it cannot be a first-order process. The terms in $\mathcal{H}_{\rm int}(x)\mathcal{H}_{\rm int}(y)$ that can do the job are

$$(a)\ e^2 A_\alpha(x) j_e^\alpha(x) A_\beta(y) j_\mu^\beta(y) \quad \text{and} \quad (b)\ e^2 A_\alpha(x) j_\mu^\alpha(x) A_\beta(y) j_e^\beta(y)\,. \tag{6.179}$$

The S_{fi} for the first term above, denoted as $S_{fi}^{(a)}$, is then

$$S_{fi}^{(a)} = \frac{(-ie)^2}{2}\int d^4x d^4y \langle \mu^+\mu^- | T\Big((:A_\alpha\, j_e^\alpha:)_x (:A_\beta\, j_\mu^\beta:)_y\Big)|e^-e^+\rangle\,, \tag{6.180}$$

where the subscript x and y indicates the space-time arguments:

$$F_x \equiv F(x)\,, \quad \text{(F: any operator)}\,. \tag{6.181}$$

Now we apply the Wick's theorem to the time-ordered product. Among all possible contractions, the only terms that survive are the one with the photon fields contracted since there are no photons in the initial or final states. There is one such term which, by the definition (6.145), is

$$\overbrace{A_{\alpha x} A_{\beta y}} : j_{ex}^\alpha j_{\mu y}^\beta : = + i D_{F\,\alpha\beta}(x-y) : j_{ex}^\alpha j_{\mu y}^\beta :\ . \tag{6.182}$$

Thus, we have

$$S_{fi}^{(a)} = \frac{(-ie)^2}{2}\int d^4x d^4y\, i D_{F\,\alpha\beta}(x-y) \langle \mu^+\mu^- | : j_{ex}^\alpha j_{\mu y}^\beta : |e^-e^+\rangle\,, \tag{6.183}$$

The S_{fi} due to the second term of (6.179) can be obtained by simply exchanging e and μ for the currents:

$$S_{fi}^{(b)} = \frac{(-ie)^2}{2}\int d^4x d^4y\, i \underbrace{D_{F\,\alpha\beta}(x-y)}_{D_{F\,\beta\alpha}(y-x)} \langle \mu^+\mu^- | \underbrace{: j_{\mu x}^\alpha j_{ey}^\beta :}_{: j_{ey}^\beta j_{\mu x}^\alpha :} |e^-e^+\rangle\,, \tag{6.184}$$

where we have used the fact that $D_F^{\alpha\beta}(x-y)$ is symmetric under $\alpha \leftrightarrow \beta$ or $x \leftrightarrow y$ and that j_{ey}^β commutes with $j_{\mu x}^\beta$ (the ordering does not matter anyway since they are normal-ordered). Thus, we see that the integrands of (6.183) and (6.184) are related by $x \leftrightarrow y$ and thus give the same result when integrated over x and y, and this cancels the factor $1/2$ in front of the integral. The evaluation of the remaining matrix element proceeds as usual; namely, matching operators are moved next to the initial or final states and annihilated. Using a notation similar to the Wick contraction to indicate a matching of operators (it is *not* Wick contraction), we have

$$\overbrace{\langle \mu^+ \overbrace{\mu^- | : (\bar\psi_\mu \gamma^\beta \psi_\mu)_y (\overbrace{\bar\psi_e \gamma^\alpha \psi_e)_x : |e^-}^{\bar g_e \quad f_e} e^+\rangle}^{\bar f_\mu \quad g_\mu}} = (\bar f_{\mu^-}\gamma^\beta g_{\mu^+})_y (\bar g_{e^+}\gamma^\alpha f_{e^-})_x\,. \tag{6.185}$$

In this particular example, there is no minus sign arising from the shuffling. Using the Fourier transform of $iD_{F\,\alpha\beta}(x-y)$ (5.327) and executing the integrals over x, y and the momentum variable in $iD_{F\,\alpha\beta}(x-y)$, we obtain

$$\begin{aligned} S_{fi} &= 2S_{fi}^{(a)} = (-ie)^2 \int d^4x d^4y\, iD_{F\,\alpha\beta}(x-y)(\bar{g}_{e^+}\gamma^\alpha f_{e^-})_x(\bar{f}_{\mu^-}\gamma^\beta g_{\mu^+})_y \\ &= \frac{(2\pi)^4\delta^4(p_{\mu^+}+p_{\mu^-}-p_{e^+}-p_{e^-})}{\sqrt{2p^0_{\mu^+}V 2p^0_{\mu^-}V}\sqrt{2p^0_{e^+}V 2p^0_{e^-}V}} \\ &\qquad \times \underbrace{(-ie)^2 iD_{F\,\alpha\beta}(p_{e^+}+p_{e^-})(\bar{v}_{e^+}\gamma^\alpha u_{e^-})(\bar{u}_{\mu^-}\gamma^\beta v_{\mu^+})}_{\mathcal{M}}, \end{aligned} \tag{6.186}$$

where $\mathcal{M}$ is identified by the definition (5.109). It reproduces the matrix element obtained earlier in (6.82).

Let us look at a case where there will be multiple diagrams where we have to be careful about the relative sign among them. Consider the scattering

$$e^-e^- \to e^-e^- \tag{6.187}$$

called the Møller scattering. This is also caused by the QED interaction $\mathcal{H}_{\rm int} = eA^\mu\bar{\psi}\gamma_\mu\psi$ at the second order where the photon fields are contracted. No electron fields should be contracted at this order since we need all four fields of $(\bar{\psi}\gamma_\mu\psi)_x(\bar{\psi}\gamma_\mu\psi)_y$ to annihilate two electrons and create two electrons. Thus,

$$\begin{aligned} S_{fi} &= \frac{(-i)^2}{2}\int d^4x d^4y\, \langle e_3^- e_4^-|T(\mathcal{H}_{\rm int}(x)\mathcal{H}_{\rm int}(y))|e_2^- e_1^-\rangle \\ &= \frac{(-ie)^2}{2}\int d^4x d^4y\, \overbrace{A^\mu_x A^\nu_y}\, \langle e_3^- e_4^-| : (\bar{\psi}\gamma_\mu\psi)_x(\bar{\psi}\gamma_\nu\psi)_y : |e_2^- e_1^-\rangle \end{aligned} \tag{6.188}$$

Now, the final-state electrons can be created by $\bar{\psi}_x$ or $\bar{\psi}_y$, and the initial-state electrons can be annihilated by ψ_x or ψ_y. Thus, there are four ways to match the fields with the initial and final states. Among the four cases, two of them are related the other two by the exchange $x \leftrightarrow y$. Thus, we resolve it by requiring that e_4^- of the final state be always matched with $\bar{\psi}_x$ and drop the factor 1/2 for S_{fi}. This will require that e_3^- be matched with $\bar{\psi}_y$. There are still two ways to match the operators corresponding to the exchange of the initial-state electrons:

$$\begin{aligned} &\langle e_3^- e_4^-| : (\bar{\psi}\gamma_\mu\psi)_x(\bar{\psi}\gamma_\nu\psi)_y : |e_2^- e_1^-\rangle \\ &= \overbrace{\langle e_3^- e_4^-| : (\bar{\psi}\gamma_\mu\underbrace{\psi})_x(\bar{\psi}\gamma_\nu\underbrace{\psi})_y : |e_2^- e_1^-\rangle}^{\bar{f}_4\ f_2\ \bar{f}_3\ f_1} + \overbrace{\langle e_3^- e_4^-| : (\bar{\psi}\gamma_\mu\underbrace{\psi})_x(\bar{\psi}\gamma_\nu\underbrace{\psi})_y : |e_2^- e_1^-\rangle}^{\bar{f}_4\ f_1\ \bar{f}_3\ f_2} \\ &\quad + \quad (x \leftrightarrow y) \\ &= (\bar{f}_4\gamma_\mu f_2)_x(\bar{f}_3\gamma_\nu f_1)_y - (\bar{f}_4\gamma_\mu f_1)_x(\bar{f}_3\gamma_\nu f_2)_y \quad + \quad (x \leftrightarrow y) \end{aligned} \tag{6.189}$$

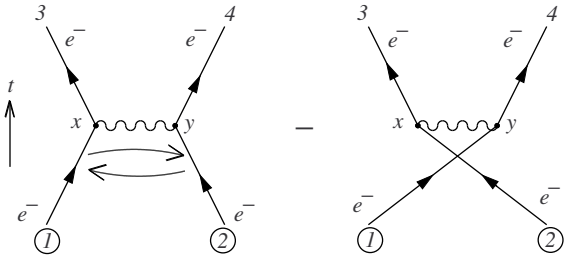


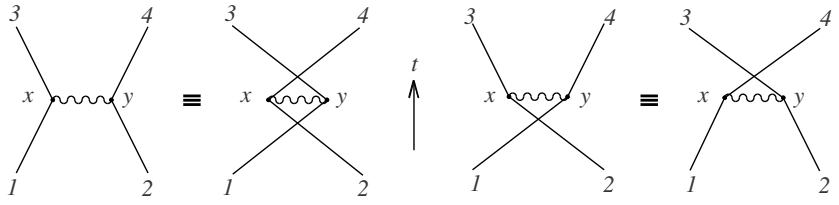
Figure 6.2: Two diagrams contributing to Møller scattering $e^-e^- \to e^-e^-$. They are related by the exchange of the legs for the initial-state electrons which results in the relative minus sign.

Here, we are being careful about the ordering of the particles written inside the initial and final states so that they reflect the ordering of the creation and annihilation operators:

$$|e_2^- e_1^-\rangle \stackrel{\text{def}}{\equiv} a_2^\dagger a_1^\dagger |0\rangle\,, \quad \langle e_3^- e_4^-| \stackrel{\text{def}}{\equiv} \langle 0|a_3 a_4\,, \tag{6.190}$$

then the relative minus sign above between the two amplitudes arises due to the signs that occur when we move the matching operators in the fields next to the creation and annihilation operators of the initial and final states. One sees that it is effectively due to the exchange of the role of the underlined fields ψ_x and ψ_y in (6.189), which is graphically shown in Figure 6.2.

The two diagrams we have obtained are all the topologically independent diagrams that have two vertices. Note that what counts in distinguishing diagrams is which external legs connect to which vertices topologically and it does not matter where one draws the vertices; for example,



. (6.191)

It corresponds to the exchange $x \leftrightarrow y$ which is already taken care of by eliminating the $1/n!$ factor in the Dyson series. Putting all pieces together, the Lorentz-invariant matrix element is

$$\begin{aligned} \mathcal{M} &= (-ie)^2 i D_F^{\alpha\beta}(p_1 - p_3)(\bar{u}_4\gamma_\beta u_2)(\bar{u}_3\gamma_\alpha u_1) \\ &- (-ie)^2 i D_F^{\alpha\beta}(p_2 - p_3)(\bar{u}_4\gamma_\beta u_1)(\bar{u}_3\gamma_\alpha u_2)\,. \end{aligned} \tag{6.192}$$

Note that the second terms is obtained from the first by relabeling $1 \leftrightarrow 2$.

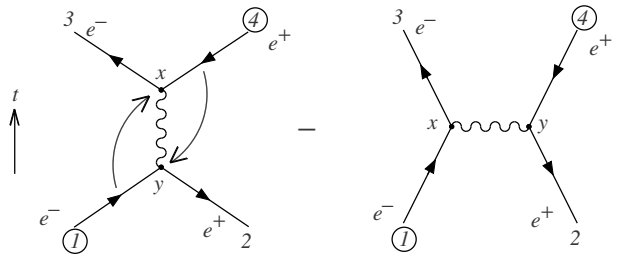


Figure 6.3: Two diagrams contributing to Bhabha scattering $e^+e^- \to e^+e^-$. If one interchanges the lines for e^+ in the final state and e^- in the initial state in the first diagram, one obtains the second.

Similar minus sign occurs when two diagrams are related by exchange of an initial-state fermion with a final-state antifermion, or exchange of an initial-state antifermion with a final-state fermion. To demonstrate it, let us consider the scattering

$$e^+e^- \to e^+e^- \tag{6.193}$$

which is called the Bhabha scattering. The corresponding S_{fi} can be obtained by simply changing the initial and final states In (6.188):

$$S_{fi} = \frac{(-ie)^2}{2} \int d^4x d^4y \, \overbrace{A^\mu_x A^\nu_y} \, \langle e^+_4 e^-_3 | : (\bar\psi\gamma_\mu\psi)_x (\bar\psi\gamma_\nu\psi)_y : | e^-_1 e^+_2 \rangle \tag{6.194}$$

with

$$\begin{aligned}
&\langle e^+_4 e^-_3 | : (\bar\psi\gamma_\mu\psi)_x (\bar\psi\gamma_\nu\psi)_y : | e^-_1 e^+_2 \rangle \\
&\quad = \langle \overbrace{e^+_4 e^-_3 | : (\bar\psi\gamma_\mu \underline{\psi})_x (\bar\psi\gamma_\nu \underline{\psi})_y : | e^-_1 e^+_2}^{\bar f_3 \quad g_4 \quad \bar g_2 \quad f_1} \rangle + \langle \overbrace{e^+_4 e^-_3 | : (\bar\psi\gamma_\mu \underline{\psi})_x (\bar\psi\gamma_\nu \underline{\psi})_y : | e^-_1 e^+_2}^{\bar f_3 \quad f_1 \quad \bar g_2 \quad g_4} \rangle \\
&\qquad + \quad (x \leftrightarrow y) \\
&\quad = (\bar f_3\gamma_\mu g_4)_x (\bar g_2\gamma_\nu f_1)_y - (\bar f_3\gamma_\mu f_1)_x (\bar g_2\gamma_\nu g_4)_y \quad + \quad (x \leftrightarrow y)
\end{aligned} \tag{6.195}$$

where the redundancy due to $x \leftrightarrow y$ was resolved by requiring that e^-_3 be always matched with a field at x. The relative minus sign arises in the process of moving the matching operators next to the initial and final states and annihilating them one by one. As in the case of Møller scattering, the role of the underlined fields are interchanged between the two terms; this time, however, the electron in the initial state and the positron in the final state are exchanged as shown in Figure 6.3. The Lorentz-invariant matrix element is then given by

$$\begin{aligned}
\mathcal{M} \;&=\; (-ie)^2 i D_F^{\alpha\beta}(p_3+p_4)(\bar u_3\gamma_\beta v_4)(\bar v_2\gamma_\alpha u_1) \\
&-\; (-ie)^2 i D_F^{\alpha\beta}(p_3-p_1)(\bar u_3\gamma_\beta u_1)(\bar v_2\gamma_\alpha v_4)\,.
\end{aligned} \tag{6.196}$$

This can be obtained by simply writing down all topologically independent diagrams with two vertices and assigning a relative minus sign between the resulting two diagrams.

Note that this sign rule arose because a given field (ψ or $\bar{\psi}$) could be matched with either of two external legs. If two diagrams are related by exchange of a fermion and an antifermion in the initial state (or in the final state), then there is no single field that can be matched with both of the two external legs since it would require that a single ψ or $\bar{\psi}$ field should contain both a and b operators (or both $a^\dagger$ and $b^\dagger$ operators if the two legs are in the final state). Thus, the sign rule does not apply in such cases. Similarly, it does not apply if two diagrams are related by an exchange of a fermion in the initial state and a fermion in the final state since a single field cannot have both a and $a^\dagger$ operators, nor when they are related by an exchange of an antifermion in the initial state with an antifermion in the final state since a single field cannot have both b and $b^\dagger$ operators.

Next, we will study the rules realted to fermion loops. Let's consider the second-order photon to photon transition $\gamma_i \to \gamma_f$. In general, the initial and final state photons can be off-shell; we will, however, treat them as on-shell since it still demonstrates essential points. Since there are no fermions in the initial or final states, the fermion fields must be contracted. If we ignore the normal ordering within $\mathcal{H}_{\rm int}$, the S matrix element is

$$\begin{aligned} S_{fi} &= \frac{(-i)^2}{2}\int d^4x d^4y\, \langle\gamma_f|T(\mathcal{H}_{\rm int}(x)\mathcal{H}_{\rm int}(y))|\gamma_i\rangle \\ &= \frac{(-ie)^2}{2}\int d^4x d^4y \Big[\langle\gamma_f| : A^\mu_x A^\nu_y \overbrace{(\bar\psi\gamma_\mu\psi)_x(\bar\psi\gamma_\nu\psi)_y} : |\gamma_i\rangle \quad (a) \\ &\qquad + \langle\gamma_f| : A^\mu_x A^\nu_y (\overbrace{\bar\psi\gamma_\mu\psi})_x(\overbrace{\bar\psi\gamma_\nu\psi})_y : |\gamma_i\rangle \quad (b) \\ &\qquad + \langle\gamma_f| : \overbrace{A^\mu_x A^\nu_y}(\overbrace{\bar\psi\gamma_\mu\psi})_x(\overbrace{\bar\psi\gamma_\nu\psi})_y : |\gamma_i\rangle \quad (c) \\ &\qquad + \langle\gamma_f| : \overbrace{A^\mu_x A^\nu_y}(\overbrace{\bar\psi\gamma_\mu\psi})_x(\overbrace{\bar\psi\gamma_\nu\psi})_y : |\gamma_i\rangle\Big] \quad (d)\,. \end{aligned} \tag{6.197}$$

Let us examine the term (a) which is called the vacuum polarization or the photon self energy. By the definition of contraction within normal product (6.145) and the relation (6.144),

$$\begin{aligned} : A^\mu_x A^\nu_y \overbrace{(\bar\psi\gamma_\mu\psi)_x(\bar\psi\gamma_\nu\psi)_y} : &= : A^\mu_x A^\nu_y \overbrace{\bar\psi_{xi}\psi_{xj}\bar\psi_{yk}\psi_{yl}} : \gamma_{\mu ij}\gamma_{\nu kl} \\ &= \underbrace{s^{ijkl}_{iljk}}_{+} \underbrace{\bar\psi_{xi}\psi_{yl}}\ \overbrace{\psi_{xj}\bar\psi_{yk}} : A^\mu_x A^\nu_y : \gamma_{\mu ij}\gamma_{\nu kl} \\ &\qquad -\overbrace{\psi_{yl}\bar\psi_{xi}} \text{ by (6.152)} \\ &= -iS_{Fli}(y-x)\,\gamma_{\mu ij}\, iS_{Fjk}(x-y)\gamma_{\nu kl} : A^\mu_x A^\nu_y : \end{aligned}$$

$$= -\mathrm{Tr}\Big[iS_F(y-x)\gamma_\mu iS_F(x-y)\gamma_\nu\Big] : A^\mu_x A^\nu_y : \tag{6.198}$$

The photon matrix element has two terms

$$\begin{aligned}\langle\gamma_f| : A^\mu_x A^\nu_y : |\gamma_i\rangle &= \langle\gamma_f| :\overbrace{A^\mu_x}\overbrace{A^\nu_y : |\gamma_i\rangle} + \langle\gamma_f| :\overbrace{A^\mu_x A^\nu_y : |\gamma_i\rangle} \\ &= h^{\mu *}_f(x) h^\nu_i(y) + h^{\nu *}_f(y) h^\mu_i(x)\,,\end{aligned} \tag{6.199}$$

where $h^\mu_{\vec{p}\lambda}(x) = \epsilon^\mu_{\vec{p}\lambda} e_{\vec{p}}(x)$. Then, the term (a) is now

$$\begin{aligned}(a) = -\frac{(-ie)^2}{2}\int d^4x d^4y \Big[\ &\mathrm{Tr}\, iS_F(y-x)\not{h}^*_f(x)\, iS_F(x-y)\not{h}_i(y) \\ &+\ \mathrm{Tr}\, iS_F(y-x)\not{h}_i(x)\, iS_F(x-y)\not{h}^*_f(y)\ \Big]\,.\end{aligned} \tag{6.200}$$

The two traces are related by $x \leftrightarrow y$ and thus give same result when integrated over x and y, canceling out the factor $1/2$ in front. Thus, we can again require that the final state be matched with $A^\mu(x)$ and drop the factor $1/2$. Using the Fourier transform of the propagator (5.351)

$$\begin{aligned}iS_F(y-x) &= \int \frac{d^4p}{(2\pi)^4} iS_F(p) e^{-ip\cdot(y-x)} \\ iS_F(x-y) &= \int \frac{d^4p'}{(2\pi)^4} iS_F(p') e^{-ip'\cdot(x-y)}\end{aligned}\,, \tag{6.201}$$

and integrating over x and y, one obtains two delta functions corresponding to 4-momentum conservation at each vertex:

$$\int d^4x d^4y \underbrace{e^{-ip\cdot(y-x)}}_{S_F(y-x)} \underbrace{e^{-ip'\cdot(x-y)}}_{S_F(x-y)} \underbrace{e^{ik_f\cdot x}}_{h^{\mu *}_f(x)} \underbrace{e^{-ik_i\cdot y}}_{h^\nu_i(y)} = (2\pi)^4\delta^4(p+k_f-p')(2\pi)^4\delta^4(p'-p-k_i)\,, \tag{6.202}$$

where k_i and k_f are the initial and final 4-momenta, respectively. Upon integrating over p', p' acquires value $p+k_i = p+k_f$, and we are left with one delta function $(2\pi)^4\delta^4(k_i-k_f)$ and the integration over p. Thus,

$$(a) = \underbrace{-(-ie)^2\int \frac{d^4p}{(2\pi)^4}\mathrm{Tr}\Big[iS_F(p)\not{\epsilon}^*_f iS_F(p+k_i)\not{\epsilon}_i\Big]}_{\mathcal{M}} \frac{(2\pi)^4\delta^4(k_f-k_i)}{\sqrt{2k^0_f V}\sqrt{2k^0_i V}}\,. \tag{6.203}$$

This is graphically shown in Figure 6.4 (a). Since the 4-momentum should be conserved at each vertex, the initial and final 4-momenta should be equal as indicated by the delta function above. Since the propagators form a loop, the momenta of the propagators are not uniquely determined by the external momenta, as the result

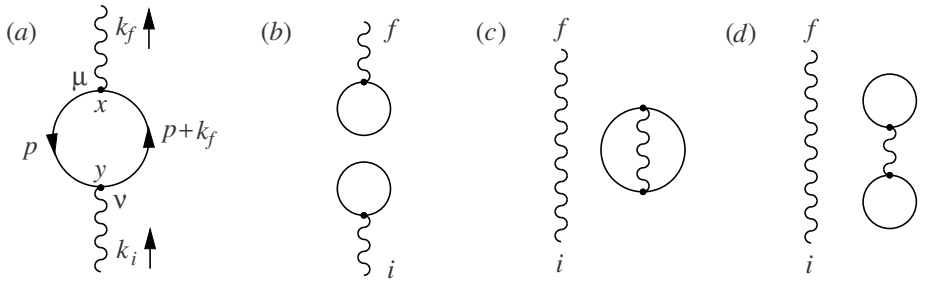


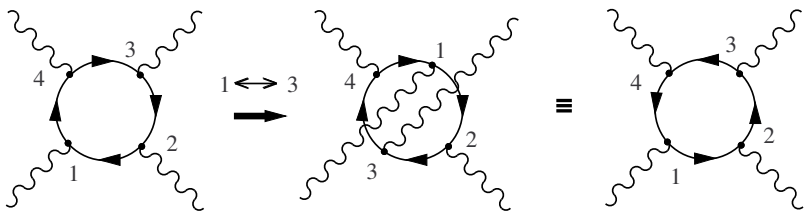
Figure 6.4: Second-order diagrams for the photon to photon transition. (a) vacuum polarization, (b) tadpoles, and (c) and (d) with vacuum bubbles.

there is an integration over the momentum flowing in the loop. Note how the trace is formed: draw arrows on the loop indicating the fermion number flow (the direction happens to be irrelevant in this case) and write down 4-momenta such that they are conserved at each vertex, and then starting somewhere on the loop one forms the trace by picking up the elements moving against the arrow on the loop.

There is an minus sign that arose when one of the propagators was formed by flipping the order of fields in order to match the definition of propagator in (6.198). This is a general feature for a fermion loop regardless of how many photons are attached. For example, when there are four external photons the process is 4-th order, and possible contractions are

$$: A_1^\alpha A_2^\beta A_3^\mu A_4^\nu \overbrace{(\bar\psi\gamma_\alpha\psi)_1(\bar\psi\gamma_\beta\psi)_2(\bar\psi\gamma_\mu\psi)_3(\bar\psi\gamma_\nu\psi)_4}^{} : \tag{6.204}$$

where subscript i indicates that the corresponding term is a function of x_i. In this example, the contraction of ψ_4 and $\bar\psi_1$ needs to be flipped to make it the propagator $-iS_F(x_4 - x_1)$, picking up a minus sign in the process. Incidentally, the sense of the loop does matter if there are more than two external photons for a fermion loop. This is in contrast to the case with two external photons, and it is because the change of the sense of the loop is equivalent to topologically exchanging two diagonally opposing photons:



. (6.205)

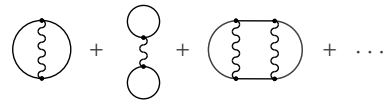
For a fixed set of external photon momenta, the two diagrams will have different momenta flowing in the photon propagators and results in different matrix elements.

The term (b) of (6.197) may be graphically represented in Figure 6.4 (b) and called 'tadpoles'. There is also a diagram where the photon fields are also contracted which is shown as Figure 6.4 (d). As we have seen in (6.177), such terms do not occur when $\mathcal{H}_{\rm int}$ is normal-ordered. Sometimes, however, it is more convenient not to use normal ordering in $\mathcal{H}_{\rm int}$, as in the case of the systematic study of divergences. and then in general the effect of tadpoles will be removed later when the theory is renormalized. In QED, however, tadpoles vanish due to the Furry's theorem which we will discuss later.

The third term (c) contains a part which is not connected to any external lines and forms a multiplicative factor for the amplitude:

$$(c) = \left[\frac{(-i)^2}{2} \int d^4x d^4y \, : \overbrace{A^\mu_x A^\nu_y} (\overbrace{\bar\psi\gamma_\mu\psi)_x (\bar\psi\gamma_\nu\psi)_y} : \right] \langle \gamma_f | \gamma_i \rangle \, . \tag{6.206}$$

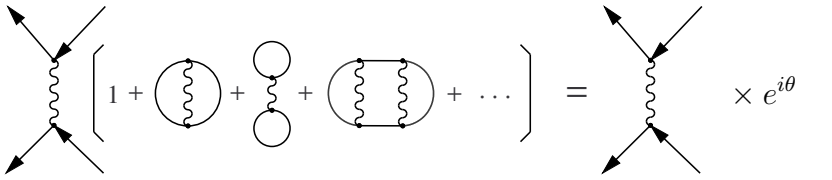
Generally, disconnected parts of a diagram form simple multiplicative factors, which is also true for the diagram (b) and (d). Diagrams that are not connected to any external lines are fully contracted and are called vacuum bubbles, and appear when one evaluates the vacuum to vacuum transition amplitude:

$\langle 0|S|0\rangle \;=\;$  (6.207)

Since the vacuum is expected to stay as vacuum; namely, if there is no particles at all at $t = -\infty$ then there will be no particles at $t = +\infty$, when all is properly renormalized the vacuum to vacuum transition amplitude should be unity up to a phase factor:

$$\langle 0|S|0\rangle = e^{i\theta} \, . \tag{6.208}$$

This amplitude should then multiply to any process in question. For example, $e^+e^- \to \mu^+\mu^-$ have vacuum bubble corrections that factor out as a phase factor:

 (6.209)

This phase factor is common to any diagrams, and thus the vacuum bubbles can in general be ignored.

In addition, we note that disconnected diagrams are usually not relevant for actual evaluations of decay rates and scattering cross sections. If an incoming line continues to the final state without being attached to any other lines, then it is not decaying nor scattering. Even if the incoming line does interact, disconnected parts are usually better treated separately.

To summarize, the Feynman rules in the momentum space for quantum electrodynamics is as follows: the Lorentz-invariant matrix elements at n-th order are obtained as follows:

1. Draw all topologically distinct diagrams with n vertices which connect initial-state particles to final state. No disconnected diagrams or vacuum bubbles need to be included.

2. Impose 4-momentum conservation at each vertex.

3. For each vertex, give a vertex factor $-ie\gamma^\mu$ where μ is the Lorentz index uniquely assigned to the vertex.

4. External lines have factors given by

	initial state	final state
fermion	$u_{\vec{p}\vec{s}}$	$\bar{u}_{\vec{p}\vec{s}}$
antifermion	$\bar{v}_{\vec{p}\vec{s}}$	$v_{\vec{p}\vec{s}}$
photon	$\epsilon^\mu_{\vec{p}\lambda}$	$\epsilon^{\mu *}_{\vec{p}\lambda}$

(6.210)

   The Lorentz index of $\epsilon^\mu_{\vec{p}\lambda}$ is the one assigned to the vertex the photon attaches to.

5. Internal lines (propagators) are associated with

	propagator	symbol
fermion	$\dfrac{i(\not{p}+m)}{p^2-m^2+i\epsilon}$	$iS_F(p)$
photon	$\dfrac{-ig^{\mu\nu}}{p^2-m^2+i\epsilon}$	$iD_F^{\mu\nu}(p)$

(6.211)

   The Lorentz indices of $iD_F^{\mu\nu}(p)$ are the ones assigned to the two vertices they attach to. If not part of a loop, the sign of p for the fermion propagator is given by drawing the fermion propagating forward in time.

6. For a fermion loop, assign an extra minus sign for the amplitude, then integrate over the momentum flowing in the loop: $\int d^4p/(2\pi)^4$.

7. Relative minus sign if two diagrams are related by exchange of two fermion lines. Namely, exchanges of two fermions or two antifermions in the initial or final state, or a fermion in the initial state and an antifermion in the final state, or an antifermion in the initial state and a fermion in the final state.

Exercise 6.2 *Photon-photon scattering.*
Consider the photon-photon scattering

$$\gamma_1 + \gamma_2 \to \gamma_3 + \gamma_4 \,.$$

Start from the 4-th order S matrix element

$$S_{fi} = \frac{(-i)^4}{4!} \int d^4x_1 \cdots d^4x_4 \langle \gamma_3\gamma_4 | T(\mathcal{H}_{\rm int}(x_1) \cdots \mathcal{H}_{\rm int}(x_4)) | \gamma_1\gamma_2 \rangle$$

with

$$\mathcal{H}_{\rm int}(x) = eA^\mu(x)\bar\psi(x)\gamma_\mu\psi(x) \,,$$

and apply the Wick's theorem to systematically enumerate all the distinct connected diagrams it generates. Does the factor $1/4!$ *canceled by the redundancy due to the exchange of* $x_1 \cdots x_4$ *? How many distinct diagrams are there? Then draw the diagrams.*

6.8 Pair annihilation $e^+e^- \to \gamma\gamma$

Let us consider the annihilation of e^+e^- pair into two photons

$$e^-(p_1, s_1) + e^+(p_2, s_2) \to \gamma(k_a, \epsilon_a) + \gamma(k_b, \epsilon_b) \,. \tag{6.212}$$

There are two topologically distinct graphs as shown in Figure 6.5. Since the particles in the initial state are a fermion and an antifermion, there is no relative minus sign associated with their exchange. Then following the Feynman rules of QED, the Lorentz-invariant matrix element is

$$\mathcal{M} = -ie^2 \Big(\underbrace{\bar v_2 \not\epsilon_b^* \frac{\not p_1 - \not k_a + m}{(p_1 - k_a)^2 - m^2} \not\epsilon_a^* u_1}_{A} + \underbrace{\bar v_2 \not\epsilon_a^* \frac{\not p_1 - \not k_b + m}{(p_1 - k_b)^2 - m^2} \not\epsilon_b^* u_1}_{B} \Big) \,. \tag{6.213}$$

Note that B is obtained from A by the exchange of labels $a \leftrightarrow b$. If we were not interested in the polarization of the photons, we would square this and use the photon spin sum rule $\sum \epsilon^{*\mu}\epsilon^\nu = -g^{\mu\nu}$. Here, however, we will proceed without summing over

the photon spins so that we can obtain the correlation of photon polarizations. We will evaluate the rate in the e^- rest frame [namely, $p_1 = (m, \vec{0})$] and use the Coulomb gauge defined in that frame and the linear basis :

$$\begin{aligned} \epsilon_a^{\mu *} = \epsilon_a^\mu = (0, \hat{e}_a)\,, \quad \epsilon_b^{\mu *} = \epsilon_b^\mu = (0, \hat{e}_b)\,, \\ k_a \cdot \epsilon_a = 0\,, \quad k_b \cdot \epsilon_b = 0\,. \end{aligned} \tag{6.214}$$

This allows us to simplify the numerators of A and B substantially. First, we note that

$$p_1 = (m, \vec{0}) \quad \rightarrow \quad p_1 \cdot \epsilon_a = 0\,, \quad p_1 \cdot \epsilon_b = 0\,, \tag{6.215}$$

which means that $\not{p}_1$ anticommutes with $\not{\epsilon}_{a,b}$:

$$\not{p}_1 \not{\epsilon}_a + \not{\epsilon}_a \not{p}_1 = 2 p_1 \cdot \epsilon_a = 0\,, \quad \not{p}_1 \not{\epsilon}_b + \not{\epsilon}_b \not{p}_1 = 2 p_1 \cdot \epsilon_b = 0\,. \tag{6.216}$$

This and the Dirac equation $(\not{p}_1 - m) u_1 = 0$ gives

$$\underbrace{\not{p}_1 \not{\epsilon}_a}_{-\not{\epsilon}_a \not{p}_1} u_1 = -\not{\epsilon}_a \underbrace{\not{p}_1 u_1}_{m u_1} = -m \not{\epsilon}_a u_1 \tag{6.217}$$

Then, the numerator of A simplifies as

$$(\underbrace{\not{p}_1}_{\rightarrow -m} - \not{k}_a + m) \not{\epsilon}_a u_1 = -\not{k}_a \not{\epsilon}_a u_1\,. \tag{6.218}$$

Using

$$(p_1 - k_a)^2 - m^2 = \underbrace{p_1^2}_{m^2} - 2 p_1 \cdot k_a + \underbrace{k_a^2}_{0} - m^2 = -2 p_1 \cdot k_a\,. \tag{6.219}$$

in the denominator, and repeating the same procedure for B, we have

$$A = \frac{\bar{v}_2 \not{\epsilon}_b \not{k}_a \not{\epsilon}_a u_1}{2\, p_1 \cdot k_a}\,, \quad B = (a \leftrightarrow b)\,. \tag{6.220}$$

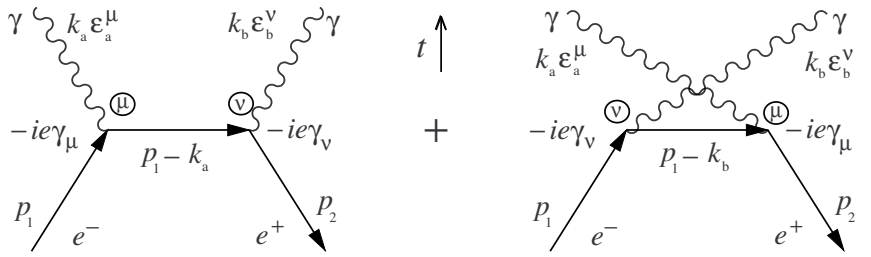


Figure 6.5: Two diagrams for the pair annihilation $e^+e^- \rightarrow \gamma\gamma$.

In evaluating

$$|\mathcal{M}|^2 = e^4(|A|^2 + |B|^2 + A^*B + AB^*)\,, \tag{6.221}$$

we can obtain $|B|^2$ from $|A|^2$ by $a \leftrightarrow b$, and AB^* from A^*B by simply taking the complex conjugate. Summing over the e^+e^- spins and dividing by the number of initial-state spin configurations (which is four), the spin-averaged $|A|^2$ becomes

$$\begin{aligned}\overline{|A|^2} &= \frac{1}{4}\sum_{s_1,s_2}|A|^2 \\ &= \frac{1}{4}\frac{1}{4(p_1\cdot k_a)^2}\mathrm{Tr}(\not p_1 + m)\not\epsilon_a\not k_a\not\epsilon_b(\not p_2 - m)\not\epsilon_b\not k_a\not\epsilon_a \\ &= \frac{1}{16(p_1\cdot k_a)^2}(\mathrm{Tr}\not p_1\not\epsilon_a\not k_a\not\epsilon_b\not p_2\not\epsilon_b\not k_a\not\epsilon_a - m^2\mathrm{Tr}\not\epsilon_a\not k_a\not\epsilon_b\not\epsilon_b\not k_a\not\epsilon_a)\,. \end{aligned}\tag{6.222}$$

We now have to use variety of tricks to reduce the number of γ matrices. Note that

$$\not k_a\not k_a = k_a^2 = 0\,, \quad \not\epsilon_a\not\epsilon_a = \epsilon_a^2 = -1\,, \quad \text{and } (a \leftrightarrow b) \tag{6.223}$$

Then we immediately see that the second trace vanishes:

$$\mathrm{Tr}\not\epsilon_a\not k_a\underbrace{\not\epsilon_b\not\epsilon_b}_{-1}\not k_a\not\epsilon_a = -\mathrm{Tr}\not\epsilon_a\underbrace{\not k_a\not k_a}_{0}\not\epsilon_a = 0\,. \tag{6.224}$$

Similarly, the strategy in calculating the first trace is to move two $\not k_a$'s or two $\not\epsilon_a$'s next to each other and then use (6.223):

$$\begin{aligned}\mathrm{Tr}\underbrace{\not p_1\not\epsilon_a}_{-\not\epsilon_a\not p_1}\not k_a\not\epsilon_b\not p_2\not\epsilon_b\not k_a\not\epsilon_a &= -\mathrm{Tr}\overbrace{\not\epsilon_a\not p_1\not k_a\not\epsilon_b\not p_2\not\epsilon_b\not k_a\not\epsilon_a}^{-1} \\ &= \mathrm{Tr}\underbrace{\not p_1\not k_a}_{2p_1\cdot k_a - \not k_a\not p_1}\not\epsilon_b\not p_2\not\epsilon_b\not k_a = 2p_1\cdot k_a\mathrm{Tr}\not\epsilon_b\not p_2\not\epsilon_b\not k_a - \mathrm{Tr}\overbrace{\not k_a\not p_1\not\epsilon_b\not p_2\not\epsilon_b\not k_a}^{0}\,. \end{aligned}\tag{6.225}$$

Then, the trace theorem for $\mathrm{Tr}\not\epsilon_b\not p_2\not\epsilon_b\not k_a$ finishes the job:

$$\overline{|A|^2} = \frac{2p_1\cdot k_a}{16(p_1\cdot k_a)^2}\mathrm{Tr}\not\epsilon_b\not p_2\not\epsilon_b\not k_a = \frac{1}{2\,p_1\cdot k_a}(2\,\epsilon_b\cdot p_2\,\epsilon_b\cdot k_a - \underbrace{\epsilon_b^2}_{-1}\,p_2\cdot k_a)\,. \tag{6.226}$$

Let's use p_1 instead of p_2 wherever possible since the space part of p_1 is zero which will simplify expressions later. To do so, we can use

$$p_1 + p_2 = k_a + k_b \to \begin{cases} (p_1 - k_b)^2 = (p_2 - k_a)^2 & \to p_1\cdot k_b = p_2\cdot k_a \\ \epsilon_b\cdot(p_1+p_2) = \epsilon_b\cdot(k_a+k_b) & \to \epsilon_b\cdot p_2 = \epsilon_b\cdot k_a \end{cases}\,, \tag{6.227}$$

where we have used $\epsilon_b \cdot p_1 = 0$ (6.215) and $\epsilon_b \cdot k_b = 0$ (Coulomb gauge). Then $|A|^2$ can now be written as

$$\overline{|A|^2} = \frac{1}{2\,p_1 \cdot k_a}[2(k_a \cdot \epsilon_b)^2 + p_1 \cdot k_b]\,. \tag{6.228}$$

Taking $a \leftrightarrow b$, we have

$$\overline{|B|^2} = \frac{1}{2\,p_1 \cdot k_b}[2(k_b \cdot \epsilon_a)^2 + p_1 \cdot k_a]\,. \tag{6.229}$$

The evaluation of the spin-averaged A^*B and AB^* is left as an exercise:

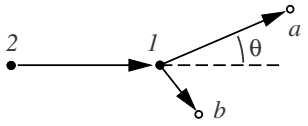
$$\overline{A^*B} + \overline{AB^*} = -\Big[2(\epsilon_a \cdot \epsilon_b)^2 - 1 + \frac{(k_a \cdot \epsilon_b)^2}{p_1 \cdot k_a} + \frac{(k_b \cdot \epsilon_a)^2}{p_1 \cdot k_b}\Big]\,. \tag{6.230}$$

Putting all pieces together, we obtain

$$\overline{|\mathcal{M}|^2} = e^4\overline{|A+B|^2} = \frac{e^4}{2}\Big[\frac{p_1 \cdot k_b}{p_1 \cdot k_a} + \frac{p_1 \cdot k_a}{p_1 \cdot k_b} - 4(\epsilon_a \cdot \epsilon_b)^2 + 2\Big]\,. \tag{6.231}$$

Using the cross section formula (5.291)

$$\frac{d\sigma}{d\Omega_a} = \frac{|\vec{p}_a||\mathcal{M}|^2}{64\pi^2 m|\vec{p}_2|\Big[m + E_2(1 - \frac{\beta_2}{\beta_a}\cos\theta)\Big]} \tag{6.232}$$



with

$$\beta_a = 1\,, \quad |\vec{p}_a| = k_a^0\,, \quad p_1 \cdot k_{a,b} = m\,k_{a,b}^0\,, \quad \epsilon_a \cdot \epsilon_b = -\hat{e}_a \cdot \hat{e}_b, \tag{6.233}$$

and $e^2 = 4\pi\alpha$, we finally obtain the cross section for $e^+e^- \to \gamma\gamma$:

$$\frac{d\sigma}{d\Omega_a} = \frac{k_a^0\alpha^2}{8m|\vec{p}_2|\,[m + E_2(1 - \beta_2\cos\theta)]}\Big[\frac{k_b^0}{k_a^0} + \frac{k_a^0}{k_b^0} - 4(\hat{e}_a \cdot \hat{e}_b)^2 + 2\Big]\,, \tag{6.234}$$

which was first obtained in 1930 by Dirac.

The physical meaning of the polarization correlation can best be appreciated in the low velocity limit; namely, when the e^+e^- pair annihilates at rest. Then we have $\beta_2 \sim 0$ and

$$k_a^0 \sim k_b^0 \sim m\,, \quad |\vec{p}_2| \sim m\beta_2\,, \quad E_2 \sim m\,. \tag{6.235}$$

and the cross section becomes

$$\frac{d\sigma}{d\Omega_a} = \frac{\alpha^2}{4m^2\beta_2}\,[1 - (\hat{e}_a \cdot \hat{e}_b)^2] \tag{6.236}$$

Apparently, if $\hat{e}_a$ and $\hat{e}_b$ are parallel, then the cross section is zero. Classically, the photon wave function is

$$A^\mu = \epsilon^\mu e_{\vec{k}}(x) + c.c. \tag{6.237}$$

and the corresponding electric field is given by (with $A^0 = 0$)

$$\vec{E} \equiv -\partial_0 \vec{A} - \vec{\nabla} A^0 = -\partial_0 \vec{A} = ik^0\, \hat{e}\, e_{\vec{k}}(x) + c.c.\,; \tag{6.238}$$

where $\epsilon^\mu = (0, \hat{e})$. Namely, the electric field oscillates along $\hat{e}$. Thus, the electric fields of the two photons created by an e^+e^- annihilation at rest are perpendicular to each other:

. (6.239)

Later, we will learn that such correlation indicates that the annihilating state has negative parity quantum number.

In order to calculate the total cross section, we have to sum over the photon polarizations. Noting that $\hat{e}_{\vec{k}1}$, $\hat{e}_{\vec{k}2}$, and $\hat{k}$ form a complete orthonormal basis, we have

$$\begin{aligned} &(\hat{e}_{\vec{k}1})_i(\hat{e}_{\vec{k}1})_j + (\hat{e}_{\vec{k}2})_i(\hat{e}_{\vec{k}2})_j + \hat{k}_i\hat{k}_j = \delta_{ij} \\ \to \quad &\sum_{\lambda=1,2} (\hat{e}_{\vec{k}\lambda})_i(\hat{e}_{\vec{k}\lambda})_j = \delta_{ij} - \hat{k}_i\hat{k}_j\,. \end{aligned} \tag{6.240}$$

Then, for general directions $\vec{k}$ and $\vec{k}'$,

$$\begin{aligned} \sum_{\lambda\lambda'} (\hat{e}_{\vec{k}\lambda}\cdot\hat{e}_{\vec{k}'\lambda'})^2 &= \sum_{\lambda\lambda'}\sum_{ij} \overbrace{(\hat{e}_{\vec{k}\lambda})_i(\hat{e}_{\vec{k}'\lambda'})_i(\hat{e}_{\vec{k}\lambda})_j(\hat{e}_{\vec{k}'\lambda'})_j} \\ &= \sum_{ij} (\delta_{ij} - \hat{k}_i\hat{k}_j)(\delta_{ij} - \hat{k}'_i\hat{k}'_j) \\ &= \sum_{ij} \Big(\underbrace{\delta_{ij}\delta_{ij}}_{3} - \underbrace{\hat{k}^2}_{1} - \underbrace{\hat{k}'^2}_{1} + (\vec{k}\cdot\vec{k}')^2 \Big) \\ &= 1 + (\vec{k}\cdot\vec{k}')^2\,. \end{aligned} \tag{6.241}$$

For our case at hand, we have $\hat{k}_a \cdot \hat{k}_b = -1$ which leads to $\sum(\hat{e}_a\cdot\hat{e}_b)^2 = 2$. Thus, the cross section summed over photon spin is

$$\frac{d\sigma}{d\Omega_a} = \frac{\alpha^2}{4m^2\beta_2} \underbrace{\sum_{\lambda_a\lambda_b} [1 - (\hat{e}_a\cdot\hat{e}_b)^2)]}_{4-2} = \frac{\alpha^2}{2m^2\beta_2}\,, \tag{6.242}$$

which has an uniform angular distribution. This is not surprising since the annihilation occurs at rest and we have summed over all spins. When integrating over angle, we note that the exchange of $\vec{k}_a$ and $\vec{k}_b$ results in exactly the same final state which should not be counted twice. Thus, we integrate over 4π steradian and divide by two to obtain

$$\sigma_{\rm tot}(e^+e^- \to \gamma\gamma) = \frac{\pi\alpha^2}{m^2\beta_2} \qquad \text{(low energy limit)}\,, \tag{6.243}$$

or

$$\sigma_{\rm tot}(e^+e^- \to \gamma\gamma) = \frac{\pi r_e^2}{\beta_2} \qquad \text{(low energy limit)}\,, \tag{6.244}$$

where $r_e \equiv \alpha/m$ is the classical electron radius.

Three distances in QED
There are three distance scales in the electron photon system, and they are related by the fine structure constant $\alpha \sim 1/137$:

$$\begin{array}{lll} \text{classical electron radius} & r_e = \dfrac{\alpha}{m} & \sim 2.8179\ \text{fm}\,(= 10^{-13}\text{cm}) \\ \text{electron compton wave length} & \lambda\!\!\!^{-}_e = \dfrac{1}{m} & \sim 386.1\ \text{fm} \\ \text{Bohr radius} & a_\infty = \dfrac{1}{m\alpha} & \sim 0.5292\ \text{Å}\,(= 10^{-8}\text{cm}) \end{array} \tag{6.245}$$

Since the radius of proton is about 0.5 fm, the classical electron radius r_e is about five times larger than the radius of proton. ∎

6.9 The Ward identity

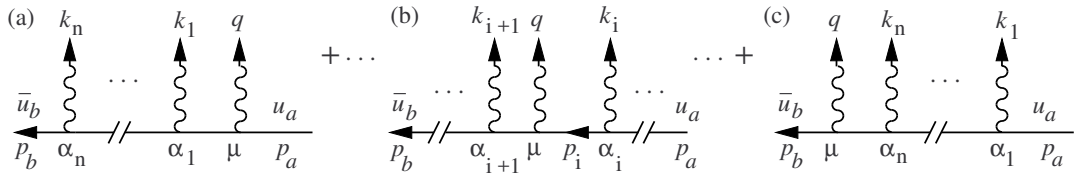
We are now well equipped to prove the Ward identity (6.70). Let us first restate the claim. When an external photon is involved in a process, the Lorentz-invariant matrix element $\mathcal{M}$ is linear in the polarization vector ϵ^μ of the photon. Then, it vanishes if one replaces ϵ^μ with the 4-momentum of the photon q^μ:

$$\mathcal{M} = \epsilon^\mu I_\mu \quad \to \quad q^\mu I_\mu = 0\,. \tag{6.246}$$

As we will see below, the photon does not have to be on-shell. When a photon is attached to a fermion line (loop or not), then, regardless of the on-shellness of the photon, the current associated with the fermion line vanishes when the gamma matrix at the vertex is contracted with the 4-momentum of the photon. This latter aspect was needed for the term $q^\mu q^\nu/M^2$ of the spin-1 propagator to vanish in (6.61).

The original Ward identity is a relation between the electron propagator and the electron-photon vertex function in the context of perturbation theory (with higher-order corrections), which was generalized by Takahashi using a formulation exact to all orders. It has become customary, however, to include the relation presented here in the 'Ward identities'.

A photon is always attached to a fermion line which may have two ends or may be a loop. The fermion line may be attached to photons (with momenta $k_1 \cdots k_n$ other than the photon of interest (with momentum q). The photons may be on-shell or off-shell. Let us start from the case where the fermion line has ends. We assume that the ends are u spinors, but the proof works also when one or both are v spinors. For given incoming and outgoing momenta of the fermion line (p_a and p_b) and the momenta of photons ($q, k_1 \cdots k_n$), there are $n+1$ ways the photon of interest attaches to the fermion line depending on the relative position of the series $k_1 \cdots k_n$ and q:



(6.247)

The system that these photons attach to is identical for each of these diagrams, thus we can sum the these diagrams to prove the cancellation in terms of the entire matrix element. There are diagrams with $k_1 \cdots k_n$ reordered, but it turns out that the cancellation occurs within a given ordering of $k_1 \cdots k_n$.

As we have seen in (5.358), because of the relation

$$\frac{\not{p} + m}{p^2 - m^2} = (\not{p} - m)^{-1} \,; \tag{6.248}$$

we symbolically write the fermion propagator as

$$iS_F(p) = i\frac{\not{p} + m}{p^2 - m^2 + i\epsilon} \stackrel{\text{def}}{\equiv} \frac{i}{\not{p} - m + i\epsilon} \,. \tag{6.249}$$

For example, the diagram (a) of (6.247) can be written as

$$(a) = \bar{u}_b \gamma_{\alpha_n} \frac{1}{\not{p}_b + \not{k}_n - m} \gamma_{\alpha_{n-1}} \cdots \gamma_{\alpha_1} \frac{1}{\not{p}_a - \not{q} - m} \gamma_\mu u_a \,, \tag{6.250}$$

where we have dropped the i's and $-ie$'s (e: electric charge) since they are common to all diagrams and irrelevant in demonstrating the cancellation. We have also ignored the $+i\epsilon$ terms which do not affect the proof. Our task then is to show that the

net result is zero when we contract γ_μ with q^μ in each diagram and sum the $n+1$ diagrams. We further simplify the notation by writing the inverse propagator as

$$[p] \stackrel{\text{def}}{\equiv} \not{p} - m\,. \tag{6.251}$$

Then, the Dirac equations for u_a and $\bar{u}_b$ are

$$[p_a]u_a = 0\,, \quad \bar{u}_b[p_b] = 0\,, \tag{6.252}$$

and we have for any p^μ and p'^μ

$$[p] - [p'] = (\not{p} - m) - (\not{p}' - m) = \not{p} - \not{p}'\,. \tag{6.253}$$

Then the diagram (a) with γ_μ contracted with q^μ is

$$\begin{aligned}(a):\ \bar{u}_b\gamma_{\alpha_n}\cdots\frac{1}{[p_a-q-k_1]}\gamma_{\alpha_1}\frac{1}{[p_a-q]}\overbrace{\not{q}}^{[p_a]-[p_a-q]\text{ by (6.253)}}u_a \\ = \bar{u}_b\gamma_{\alpha_n}\cdots\frac{1}{[p_a-q-k_1]}\gamma_{\alpha_1}\frac{1}{[p_a-q]}\overbrace{[p_a]u_a}^{0\text{ by (6.252)}} \\ -\bar{u}_b\gamma_{\alpha_n}\cdots\frac{1}{[p_a-q-k_1]}\gamma_{\alpha_1}\overbrace{\frac{1}{[p_a-q]}[p_a-q]}^{1}u_a \\ = -\bar{u}_b\gamma_{\alpha_n}\cdots\frac{1}{[p_a-q-k_1]}\gamma_{\alpha_1}u_a\,.\end{aligned} \tag{6.254}$$

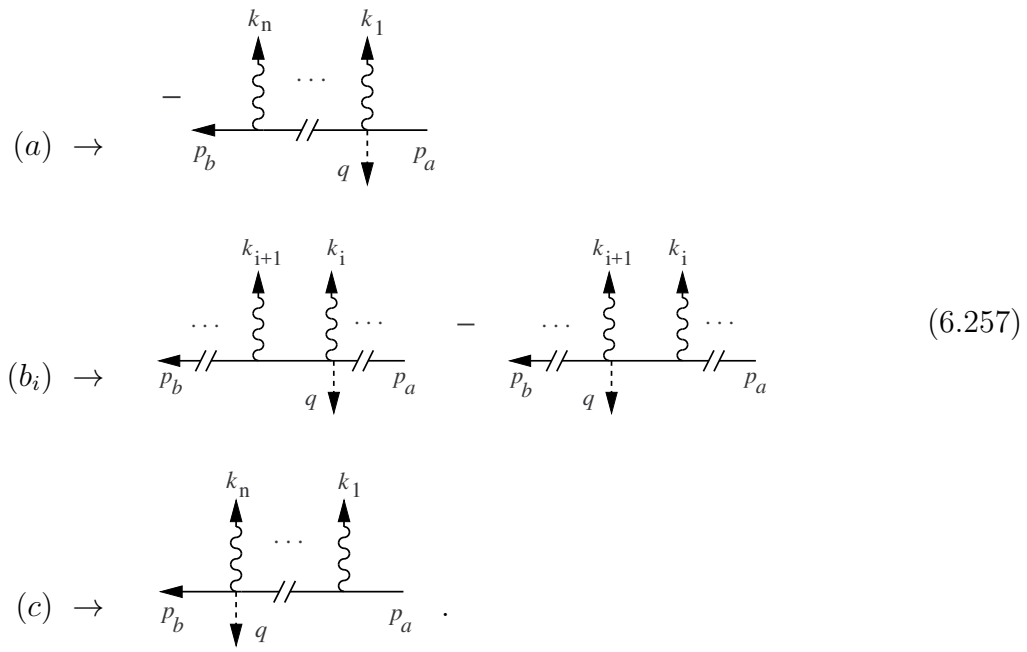
Similarly for (b) and (c), the contraction with q^μ results in

$$(c):\ \bar{u}_b\overbrace{\not{q}}^{[p_b+q]-[\not{p}_b]\quad(\leftarrow\bar{u}_b[p_b]=0)}\frac{1}{[p_b+q]}\gamma_{\alpha_n}\frac{1}{[p_b+q+k_n]}\cdots u_a = \bar{u}_b\gamma_{\alpha_n}\frac{1}{[p_b+q+k_n]}\cdots u_a\,, \tag{6.255}$$

and

$$\begin{aligned}(b_i):\ \bar{u}_b\cdots\frac{1}{[p_i-q-k_{i+1}]}\gamma_{\alpha_{i+1}}\frac{1}{[p_i-q]}\overbrace{\not{q}}^{[p_i]-[p_i-q]}\frac{1}{[p_i]}\gamma_{\alpha_i}\frac{1}{[p_i+k_i]}\cdots u_a \\ = \bar{u}_b\cdots\frac{1}{[p_i-q-k_{i+1}]}\gamma_{\alpha_{i+1}}\frac{1}{[p_i-q]}\gamma_{\alpha_i}\frac{1}{[p_i+k_i]}\cdots u_a \\ -\bar{u}_b\cdots\frac{1}{[p_i-q-k_{i+1}]}\gamma_{\alpha_{i+1}}\frac{1}{[p_i]}\gamma_{\alpha_i}\frac{1}{[p_i+k_i]}\cdots u_a\,,\end{aligned} \tag{6.256}$$

where we have labeled as (b_i) the diagram in which the photon of interest (with momentum q) is attached between the $(i+1)$-th and i-th photons and p_i is the momentum of the propagator just to the left of the i-th photon. In order to keep track of momenta flowing in the propagators, it is convenient to use a graphical method where the momenta are uniquely determined by the momentum conservation at each vertex. We will use an extra arrow with dotted line to indicate that momentum q is flowing out of the vertex and the momentum conservation is imposed on each vertex, and otherwise the normal rules for propagators and spinors are applied. The expression (6.254) can then be represented as



(6.257)

Now it is quite clear that (a) cancels the first term of (b_1), and the second terms of (b_1) cancels the first term of (b_2), and so on; and finally the second term of (b_{n-1}) cancels (c).

If p_a is carried by a positron in the final state, then the starting momentum of the current is $-p_a$ rather than p_a; e.g. the momentum of the propagator just to the left of q in (6.247a) is $-p_a - q$. On the other hand, the positron spinor v_a satisfies

$$[-p_a]v_a = (-\not{p}_a - m)v_a = 0\,, \tag{6.258}$$

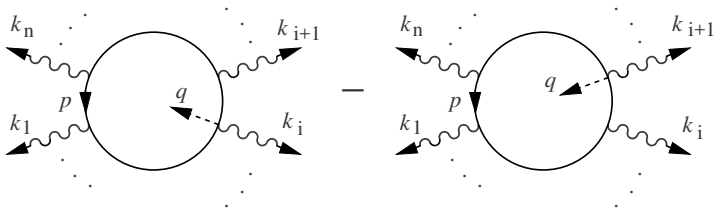
and the proof can be carried out in the same way as before. Similarly, the proof works fine when the spinor at the end of the current is a v spinor.

When the photon in question is attached to a fermion loop, the same reasoning works just fine. This time the photon with momentum q is attached to a fermion loop

with n other photons with momenta $k_1 \cdots k_n$. There are n diagrams depending on where the photon in question is attached with respect to the series of other photons, and these diagrams need to be summed. The diagram with q placed between k_i and k_{i-1} gives (upon contracting q^μ with the γ_μ at the vertex)

$$\begin{aligned}
&\mathrm{Tr}\left(\gamma_{\alpha_n}\cdots\frac{1}{[p_i-q-k_{i+1}]}\gamma_{\alpha_{i+1}}\frac{1}{[p_i-q]}\overbrace{\not{q}\,\frac{1}{[p_i]}}^{[p_i]-[p_i-q]}\gamma_{\alpha_i}\frac{1}{[p_i+k_i]}\cdots\frac{1}{[p]}\right) \qquad (6.259)\\
&\qquad=\mathrm{Tr}\left(\gamma_{\alpha_n}\cdots\frac{1}{[p_i-q-k_{i+1}]}\gamma_{\alpha_{i+1}}\frac{1}{[p_i-q]}\gamma_{\alpha_i}\frac{1}{[p_i+k_i]}\cdots\frac{1}{[p]}\right)\\
&\qquad\quad-\mathrm{Tr}\left(\gamma_{\alpha_n}\cdots\frac{1}{[p_i-q-k_{i+1}]}\gamma_{\alpha_{i+1}}\frac{1}{[p_i]}\gamma_{\alpha_i}\frac{1}{[p_i+k_i]}\cdots\frac{1}{[p]}\right),
\end{aligned}$$

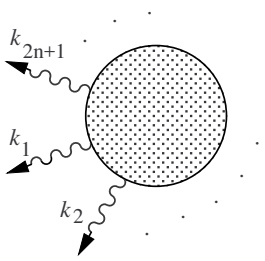
which can be graphically represented as

(i) : 

. (6.260)

Let's label this pair of diagrams as (i). Then, it is seen that the second term of (i) cancels the first term of $(i+1)$ and so on, and the second term of (n), where q is placed between k_1 and k_n, cancels the first term of (1). Thus, the net total again vanishes and the Ward identity (6.246) holds even when the photon is attached to a fermion loop. This completes the proof.

Incidentally, when there are odd number of photons attached to a fermion loop, then there is a cancellation between two diagrams with opposite loop directions (this is for the ordinary matrix element without contracting any photon momentum with the corresponding γ matrix). The photons need not be on-shell for this to hold. It is also easy to see that the amplitude vanishes when odd number of photons and no fermion lines are attached to a 'blob' which represents any QED diagram:



$= 0$ (Furry's theorem) : (6.261)

Since there are no external fermion lines, all fermion lines inside the blob are loops, and since all photon lines that do not come out of the blob have two vertices there is at least one fermion loop with odd number of photons attached. The amplitude of this loop multiplies to the whole amplitude and thus the diagram vanishes. This is called the Furry's theorem and is a consequence of the symmetry of QED under charge conjugation C which in particular prohibits transitions between odd and even numbers of photons. We will come back to this point later. When there is only one photon attached to the blob, it represents a tadpole (and corrections to it) and it vanishes due to the Furry's theorem.

Exercise 6.3 *Furry's theorem.*
(*a*) *Write down the two diagrams for a fermion loop with three external photons with momenta* k_1, k_2, *and* k_3 *flowing out of the loop. These diagrams are related by reversing the direction of the fermion loop. Each vertex has a gamma matrix with a given lorentz index* (*call them* γ_{μ_1}, γ_{μ_2}, *and* γ_{μ_3}). *Write down the corresponding amplitudes including the integral over the loop momentum. You can ignore the* i*'s and* e*'s as well as the* $+i\epsilon$ *terms.*
(*b*) *Show that the two amplitudes cancel.* (*hint: Use* $\mathrm{Tr}\gamma_{i_1}\dots\gamma_{i_n} = \mathrm{Tr}\gamma_{i_n}\dots\gamma_{i_1}$ (*reverse order*), *and that the trace of odd number of* γ*'s is zero. Also, note that the momentum conservation at each vertex demands* $k_1 + k_2 + k_3 = 0$.)
(*c*) *Extend the proof to the case with* $(2n+1)$ *photons attached to a fermion loop with* $n > 2$.

Problems

6.1 *Radiative decay: Higgs → fermion pair + photon.*
The coupling of the neutral Higgs to fermion is given by the interaction Lagrangian

$$c\phi\bar{\psi}\psi$$

where c is a dimensionless real coupling constant, ϕ is the Higgs field, which is neutral and spin-0, and ψ is the fermion field. Let's consider the process where Higgs decays to a fermion pair + photon:

$$H(q) \to f(p_1, s_1) + \bar{f}(p_2, s_2) + \gamma(k, \epsilon)$$

There are two diagrams to be summed to obtain the decay amplitude corresponding to which of f and $\bar{f}$ emits the photon.
(a) Draw the relevant Feynman diagrams and write down the Lorentz-invariant matrix element $\mathcal{M}$. Assume that the fermion charge is e. Note that the momentum used in the fermion propagator is defined to be that of the fermion propagating forward in time and not that of anti-fermion propagating forward in time.
(b) Assume that the fermion mass is small; namely, it is set to zero. Actually, in the standard model, the Higgs coupling to fermion is proportional to the fermion mass, but it does not concern us here. Square the matrix element and sum over the fermion spins and the photon spin. For the photon spin sum, use

$$\sum_\lambda \epsilon_\lambda^{\mu *} \epsilon_\lambda^\nu \to -g^{\mu\nu} \,.$$

Convert it to the differential decay rate using

$$\frac{d\Gamma}{ds_1 ds_2} = \frac{\overline{|\mathcal{M}|^2}}{(2\pi)^3 32 M^3}$$

where M is the Higgs mass, s_1 is the invariant mass squared of the fermion and photon, s_2 is that of the antifermion and photon. Express the result in terms of M, s_1, s_2, $\alpha = e^2/(4\pi)$ and c. You may want to use $\not{p}_1 + \not{k} = \not{q} - \not{p}_2$ (4-momentum conservation) in the numerator of a fermion propagator, then use the Dirac equation to eliminate $\not{p}_2$. Similarly for the other propagator.
(c) This time, we will use the Coulomb gauge in the rest frame of Higgs, namely the polarization vector has no time component and the space part is transverse to $\hat{k}$, and simplify the matrix element as much as possible without summing over the photon spin at the beginning. Thus, you can use relations such as

$$q \cdot \epsilon = 0 \,, \quad k \cdot \epsilon = 0$$

Show that the decay rate is given by (summed over fermion spins but not over photon spin)

$$\frac{d\Gamma}{ds_1 ds_2} = \frac{\alpha c^2}{32\pi^2 M^3}\left(\frac{1}{s_1}+\frac{1}{s_2}\right)^2 [s_1 s_2 - 4M^2(p_1\cdot\epsilon)(p_2\cdot\epsilon)]$$

You can combine the two terms into one using $q\cdot\epsilon = 0$ (i.e. one trace to deal with)
(d) Now use $\sum_\lambda \epsilon^{\mu *}_\lambda \epsilon^\nu_\lambda \to -g^{\mu\nu}$ in $(p_1\cdot\epsilon)(p_2\cdot\epsilon)$ above to 'sum over' the photon spin. You will not get the same answer as in (b), why?
(e) Explicitly sum over the two linear polarizations. You can use

$$\sum_{\lambda=1,2} \hat{e}^i_{\vec{k}\lambda}\hat{e}^j_{\vec{k}\lambda} = \delta_{ij} - \hat{k}_i\hat{k}_j\,,$$

where $\hat{e}_{\vec{k}\lambda}$ is the space part of the 4-vector $\epsilon_{\vec{k}\lambda}$. This time, you should get the same result as in (b).
[*comment: The result shows that the decay rate is divergent as the invariant mass of the photon and one of the fermion pair goes to zero. In fact the total decay rate obtained by integrating over s_1 and s_2 is infinity. Such divergence is known to be canceled by a divergence encountered in the diagram where a photon emitted from fermion is absorbed by anti-fermion which is called the vertex correction. It is an example of a more general theorem called Kinoshita-Lee-Nauenberg, or KLN, theorem.*]

6.2 *$e^+e^- \to \mu^+\mu^-$ with transversely polarized beam.*
When an electron or positron beam goes around a storage ring, the intrinsic magnetic moment tends to get aligned to the magnetic field of the bending magnet which is vertical (in order to circulate the beams in a horizontal plane). Since electron and positron have opposite charges, their spins tend to polarize in opposite direction (vertically). Take the y-exis to be pointing up and the z-zxis to be in the beam direction at the collision point, and assume that electron is polarized in the $+y$ direction and positron in $-y$ direction. Assume that electron is massless but do NOT assume that muon is massless.
(a) Set the momentum/spin of electron and positron to be (p_1, s_1) and (p_2, s_2) respectively. What is the numerical values of s_1 and s_2 in the lab system?
(b) Using the Feynman rules, write down the invariant matrix element $\mathcal{M}$ without summing over the spins. Momenta/spin of final state μ^- and μ^+ are (q_1, s'_1) and (q_2, s'_2) respectively. Consider only the lowest order; namely, via a single photon.
(c) Sum over the spins of final state muons, but not over the spins of initial state particles. Use the relation

$$u_{p,s}\bar{u}_{p,s} = \frac{1+\gamma_5 \not{s}}{2}(\not{p} + m)$$

etc. to convert it to traces. Yes, it becomes traces even though not summing up the intial state spins. This is equivalent to inserting spin projection operators and then summing over spins.

(*d*) *Complete the calculation of the differential cross section* $d\sigma/d\Omega$ *and express it in terms of* α (*fine structure constant*), s (*total c.m. energy squared*), β (*velocity of muon*), θ (*polar angle of muon with respect to the beam axis*), *and* ϕ (*azimuthal angle* - $\phi = 0$ *for the* x *axis*). *The result is almost as simple as the non-polarized case.*
(*e*) *Obtain the differential cross section* $d\sigma/d\Omega$ *for the case when the polarizations of electron and positron are in a same direction; namely, both up or both down. Sketch the azimuthal distribution at* $\theta = \pi/2$ *for the two cases - polarizations parallel and anti-parallel - in the massless limit.*

6.3 *Electronic decay width of* J/Ψ.
(*a*) *Closely follow the calculation of* $e^+e^- \to \mu^+\mu^-$ *to obtain the spin-averaged Lorentz-invariant matrix element squared for*

$$c(p_1)\bar{c}(p_2) \to e^-(q_1)e^+(q_2)\,.$$

This time keep the mass of the charm quark ($m = 1.5\ GeV/c^2$) *as non-zero. You can ignore the electron mass. The charge of c quark is* $+2/3$. *You should get*

$$\overline{|\mathcal{M}|^2} = \frac{32e^4}{9s^2}[p_2\cdot q_1\, p_1\cdot q_2 + p_2\cdot q_2\, p_1\cdot q_1 + m^2 q_1\cdot q_2]\,.$$

(*b*) *Express the differential cross section* $d\sigma/d\Omega$ *in the CM system in terms of the velocity of the charm quark* (β), *the CM energy squared* (s), *the angle between the incomiong charm quark and the outgoing electron* (q), *and the fine structure constatn* $a = e^2/4\pi$.

$$\left(\textit{answer:}\qquad \frac{d\sigma}{d\Omega} = \frac{\alpha^2}{9s\beta}(2-\beta^2\sin^2\theta)\,.\right)$$

(*c*) *The result above can be used to roughly estimate the electronic decay rate* $\Gamma(\Psi \to e^+e^-)$, *where* Ψ *is made of a pair of charm and anti-charm quarks coupled to form spin 1. Imagine, naively, that* Ψ *is a system in which charm quark and anti-charm quark are rotating in opposite direction at non-relativistic speed. You can assume that the charm quark is moving in the cloud of anti-charm quark with uniform density* $\rho = 1/V$ *where* V *is the volume of* Ψ *meson. Assume a ball of radius* 0.25×10^{-13} *cm.* (*or 0.25 fm*). *Estimate the decay in unit of keV.* [*hint: First integrate the cross section over angles assuming* $\beta \ll 1$. *Note that the number of reaction per unit time - i.e. decay rate - is* $\sigma\times\rho\times$*(relative velocity of projectile and target). The result is independent of the velocity* β. *The cross section obtained in* (*b*) *is for the spin averaged case. What happens if the incoming* $c\bar{c}$ *pair is already coupled to spin 1? The experimental value is 5.26*$\pm$*0.37 keV.*]

6.4 *QED scattering* $H^+H^- \to H^+H^-$.

(*a*) *Draw the two lowest-order diagrams relevant for the QED reaction*

$$H^+(p_1) + H^-(p_2) \to H^+(q_1) + H^-(q_2)\,,$$

and write down the Lorentz-invariant matrix element in terms of the 4-momenta. You should have one term inversely proportional to $t \equiv (p_1 - q_1)^2$ (*'t-channel'*) *and another term inversely proportional to* $s = (p_1 + p_2)^2$ (*'s-channel'*).
(*b*) *Evaluate the differential cross section* $d\sigma/d\Omega$ *in the C.M. system and express it in terms of the fine structure constant* α*, the velocity of the particles* β*,* s*, and the angle* θ *between the incoming and outgoing* H^+*. Examine the limit* $\beta \to 0$ *and comment on the angular distribution. Which of the two channels* (s *or* t) *dominates in that limit?*