Exercise 1.3

(a) Raising α in the expression $(M^{\mu\nu})_{\alpha\beta} = g^\mu{}_\alpha\, g^\nu{}_\beta - g^\mu{}_\beta\, g^\nu{}_\alpha$,

$$(M^{\mu\nu})^\alpha{}_\beta = g^{\mu\alpha}\, g^\nu{}_\beta - g^\mu{}_\beta\, g^{\nu\alpha}\,.$$

Then, noting that $g^{0i} = 0 (i = 1, 2, 3)$,

$$\begin{aligned}(K_i K_j)^\alpha{}_\gamma &= (M^{0i})^\alpha{}_\beta (M^{0j})^\beta{}_\gamma \\ &= (g^{0\alpha}\, g^i{}_\beta - g^0{}_\beta\, g^{i\alpha})(g^{0\beta}\, g^j{}_\gamma - g^0{}_\gamma\, g^{j\beta}) \\ &= -g^{i\alpha} g^j{}_\gamma - g^{0\alpha} g^0{}_\gamma g^{ji}\,.\end{aligned}$$

Exchanging i and j,

$$(K_j K_i)^\alpha{}_\gamma = -g^{j\alpha} g^i{}_\gamma - g^{0\alpha} g^0{}_\gamma g^{ij}\,.$$

Then, the commutator $[K_i, K_j]$ is

$$[K_i, K_j]^\alpha{}_\gamma = g^{j\alpha} g^i{}_\gamma - g^{i\alpha} g^j{}_\gamma = (M^{ji})^\alpha{}_\gamma = -(M^{ij})^\alpha{}_\gamma\,,$$

namely,

$$[K_i, K_j] = -L_k \quad (i, j, k : \text{cyclic})\,.$$

(b) Since any nonzero components in the matrixes that appear in $[L_i, L_j] = \epsilon_{ijk} L_k$ have purely space indeces, it suffices to consider the 3×3 space parts only. Using $(L_i)^j{}_k = -\epsilon_{ijk}$,

$$\begin{aligned}(L_i L_j)^l{}_m &= (L_i)^l{}_k (L_j)^k{}_m = \epsilon_{ilk}\epsilon_{jkm} = -\epsilon_{ilk}\epsilon_{jmk} \\ &= -(\delta_{ij}\delta_{lm} - \delta_{im}\delta_{lj})\,,\end{aligned}$$

where we have used the formula $\epsilon_{ilk}\epsilon_{jmk} = \delta_{ij}\delta_{lm} - \delta_{im}\delta_{lj}$. Exchanging i and j,

$$(L_j L_i)^l{}_m = -(\delta_{ji}\delta_{lm} - \delta_{jm}\delta_{li})\,.$$

Then, the commutator becomes,

$$[L_i, L_j]^l{}_m = \delta_{im}\delta_{lj} - \delta_{jm}\delta_{li} = \epsilon_{ijk}\epsilon_{mlk} = \epsilon_{ijk}(L_k)^l{}_m\,,$$

namely,

$$[L_i, L_j] = \epsilon_{ijk} L_k\,.$$

(c) Since $(L_i)^0{}_\mu = (L_i)^\mu{}_0 = 0$ and $(K_i)^j{}_k = 0$,

$$[L_i, K_j]^k{}_l = (L_i)^k{}_\mu (K_j)^\mu{}_l - (K_j)^k{}_\mu (L_i)^\mu{}_l = 0\,.$$

On the other hand,

$$[L_i, K_j]^k{}_0 = \underbrace{(L_i)^k{}_\mu (K_j)^\mu{}_0}_{-\epsilon_{ikl}\delta_{jl}} - (K_j)^k{}_\mu \underbrace{(L_i)^\mu{}_0}_{0} = -\epsilon_{ikj} = \epsilon_{ijl}\delta_{lk} = \epsilon_{ijl}(K_l)^k{}_0\,.$$

Similarly,

$$[L_i, K_j]^0{}_k = \epsilon_{ijl}(K_l)^0{}_k \,.$$

Putting all together, we have

$$[L_i, K_j] = \epsilon_{ijl} K_l \,.$$

Exercise 1.4
(a) Using $P_\parallel = \vec{P}\cdot\vec{n}$ and $\vec{P}_\perp = \vec{P} - P_\parallel \vec{n}$, the boost can be written as

$$\begin{aligned} E' &= \gamma E + \eta P_\parallel = \gamma E + \eta \vec{P}\cdot\vec{n} = \gamma E + \beta\gamma\vec{P}\cdot\frac{\vec{\beta}}{\beta} \\ &= \gamma E + \gamma\vec{P}\cdot\vec{\beta} = \gamma E + \gamma\beta_x P_x + \gamma\beta_y P_y + \gamma\beta_z P_z\,, \end{aligned}$$

and

$$\begin{aligned} \vec{P}' &= P'_\parallel\vec{n} + \vec{P}'_\perp = (\eta E + \gamma P_\parallel)\vec{n} + \vec{P}_\perp \\ &= (\eta E + \gamma\vec{P}\cdot\vec{n})\vec{n} + \vec{P} - (\vec{P}\cdot\vec{n})\vec{n} \\ &= (\eta E + (\gamma-1)\vec{P}\cdot\vec{n})\vec{n} + \vec{P} \\ &= \gamma E\vec{\beta} + \frac{\gamma-1}{\beta^2}(\vec{P}\cdot\vec{\beta})\vec{\beta} + \vec{P}\,, \end{aligned}$$

which writes in terms of components,

$$\begin{aligned} P'_x &= \gamma\beta_x E + \rho\beta_x(\vec{\beta}\cdot\vec{P}) + P_x \\ P'_y &= \gamma\beta_x E + \rho\beta_x(\vec{\beta}\cdot\vec{P}) + P_y \\ P'_z &= \gamma\beta_y E + \rho\beta_y(\vec{\beta}\cdot\vec{P}) + P_z\,, \end{aligned}$$

with $\rho \equiv (\gamma-1)/\beta^2$. Putting them in matrix form,

$$\begin{pmatrix} E' \\ P'_x \\ P'_y \\ P'_z \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta_x & \gamma\beta_y & \gamma\beta_z \\ \gamma\beta_x & 1+\rho\beta_x^2 & \rho\beta_x\beta_y & \rho\beta_x\beta_z \\ \gamma\beta_y & \rho\beta_y\beta_x & 1+\rho\beta_y^2 & \rho\beta_y\beta_z \\ \gamma\beta_z & \rho\beta_z\beta_x & \rho\beta_z\beta_y & 1+\rho\beta_z^2 \end{pmatrix} \begin{pmatrix} E \\ P_x \\ P_y \\ P_z \end{pmatrix}.$$

(b) When $\beta \ll 1$, $\gamma \sim 1 + \beta^2/2$ and thus $\gamma - 1 \sim \beta^2/2$. Namely, $\rho \sim 1/2$ and $\gamma \sim 1$. Then, the Lorentz tranformation obtained above becomes,

$$\begin{pmatrix} 1 & \beta_x & \beta_y & \beta_z \\ \beta_x & 1 & 0 & 0 \\ \beta_y & 0 & 1 & 0 \\ \beta_z & 0 & 0 & 1 \end{pmatrix} = I + \beta_i K_i\,.$$

(c) The particle with mass m at rest in K has 4-momentum $P^\nu = m g^\nu{}_0$. Then, the 4-momentum in K' is given by

$$P'^\mu = \Lambda^\mu{}_\nu P^\nu = m\Lambda^\mu{}_\nu g^\nu{}_0 = m\Lambda^\mu{}_0\,.$$

On the other hand, the same 4-momentum should be

$$P'^\mu = m\,\eta'^\mu_K\,.$$

where η'^{μ}_{K} is the 4-velocity corresponding to β'_K:

$$\eta'^{0}_{K} = \frac{1}{\sqrt{1-\beta'^{2}_{K}}}\,, \quad \eta'^{i}_{K} = \frac{\beta'_{Ki}}{\sqrt{1-\beta'^{2}_{K}}}\,.$$

Thus, we should have

$$\Lambda^{\mu}{}_{0} = \eta'^{\mu}_{K}\,.$$

If we put a mass m at rest in K', the 4-momenta there is $P'^{\nu} = mg^{\nu}{}_{0}$. The 4-momentum in K is given by the inverse transformation of Λ:

$$P^{\mu} = (\Lambda^{-1})^{\mu}{}_{\nu}P'^{\nu} = m(\Lambda^{-1})^{\mu}{}_{0} = m\Lambda_{0}{}^{\mu}\,,$$

where we have used $(\Lambda^{-1})^{\mu}{}_{\nu} = \Lambda_{\nu}{}^{\mu}$. On the other hand, the same 4-momentum can be written in terms of $\vec{\beta}_{K'}$ as

$$P^{\mu} = m\,\eta^{\mu}_{K'}\,,$$

where $\eta^{\mu}_{K'}$ is the 4-velocity corresponding to $\vec{\beta}_{K'}$. Thus, we should have $\Lambda_{0}{}^{\mu} = \eta^{\mu}_{K'}$, or

$$\Lambda^{0}{}_{\mu} = \eta_{K'\mu}\,.$$

Problem 1.3

That a rotation around an axis commute with a boost in that direction: Using the commutations relations between K_i and L_j as well as $\theta_j = c\,\xi_j$,

$$[\xi_i K_i, \theta_j L_j] = \xi_i \theta_j \underbrace{[K_i, L_j]}_{\epsilon_{ijk} K_k} = c\,\xi_i \xi_j \epsilon_{ijk} K_k = 0$$

where we noted that $\xi_i \xi_j$ is symmetric while ϵ_{ijk} is antisymmetric under $i \leftrightarrow j$, thus cancel out when summed over i and j. Then, using the CBH theorem,

$$e^{\xi_i K_i} e^{\theta_j L_j} = e^{\xi_i K_i + \theta_j L_j + \underbrace{\cdots}_{0}}$$

where the terms indicated by dots are zero since their innermost commutator is $[\xi_i K_i, \theta_j L_j]$ which is zero as shown above. Namely, $e^{\xi_i K_i}$ and $e^{\theta_j L_j}$ commute.

That two boosts in the same direction commute: Similarly as above,

$$[\xi_i K_i, \xi'_j K_j] = \xi_i \xi'_j \underbrace{[K_i, K_j]}_{-\epsilon_{ijk} L_k} = -c\,\xi_i \xi_j \epsilon_{ijk} L_k = 0\,.$$

Thus,

$$e^{\xi_i K_i} e^{\xi'_j K_j} = e^{\xi_i K_i + \xi'_j K_j + \underbrace{\cdots}_{0}}\,.$$

Namely, $e^{\xi_i K_i}$ and $e^{\xi'_j K_j}$ commute.