Exercise 3.5

In terms of γ matrices, B_i^b and B_i^r can be written as

$$B_i^b = \frac{1}{2}\gamma^0\gamma^i \qquad B_i^r = \frac{1}{2}\gamma^j\gamma^k \ (ijk : \text{cyclic})$$

We first evaluate $[B_i^b, B_j^b]$ where (i,j,k) are cyclic. Using the anticommutation of γ matrixes and ${\gamma^0}^2 = 1$,

$$\begin{aligned}[B_i^b, B_j^b] &= \frac{1}{4}[\gamma^0\gamma^i, \gamma^0\gamma^j] = \frac{1}{4}(\gamma^0 \underbrace{\gamma^i\gamma^0}_{-\gamma^0\gamma^i}\gamma^j - \gamma^0 \underbrace{\gamma^j\gamma^0}_{-\gamma^0\gamma^j}\gamma^i) \\ &= -\frac{1}{4}(\gamma^i\gamma^j - \gamma^j\gamma^i) = -\frac{1}{2}\gamma^i\gamma^j = -B_k^b\,.\end{aligned}$$

This is consistent with

$$[B_i^b, B_j^b] = -\epsilon_{ijk}B_k^r\,. \qquad (*)$$

where (i,j) is in cyclic order. When (i,j) is in anti-cyclic order, the above equation is correct since LHS and RHS both change sign. Also, it is trivially correct when $i=j$ since both sides are zero. Thus, $(*)$ is correct for all (i,j).

Next we evaluate $[B_i^r, B_j^r]$ where (i,j) are in cyclic oder. Taking (ijk) as cyclic,

$$[B_i^r, B_j^r] = \frac{1}{4}[\gamma^j\gamma^k, \gamma^k\gamma^i] = \frac{1}{4}(\gamma^j \underbrace{\gamma^k\gamma^k}_{-1}\gamma^i - \underbrace{\gamma^k\gamma^i\gamma^j\gamma^k}_{\underbrace{\gamma^k\gamma^k}_{-1}\gamma^i\gamma^j}) = \frac{1}{2}\gamma^i\gamma^j = B_k^r\,,$$

and thus we get

$$[B_i^r, B_j^r] = \epsilon_{ijk}B_k^r$$

as in the discussion below $(*)$.

Again for (ijk) in cyclic order,

$$[B_i^r, B_j^b] = \frac{1}{4}[\gamma^j\gamma^k, \gamma^0\gamma^j] = \frac{1}{4}(\underbrace{\gamma^j\gamma^k\gamma^0\gamma^j}_{\underbrace{\gamma^j\gamma^j}_{-1}\gamma^k\gamma^0} - \gamma^0 \underbrace{\gamma^j\gamma^j}_{-1}\gamma^k) = \frac{1}{2}\gamma^0\gamma^k = B_k^b\,,$$

which leads to

$$[B_i^r, B_j^b] = \epsilon_{ijk}B_k^b$$

as in the discussion below $(*)$.

Exercise 3.6

(a) We trivially have $1^2 = 1$, $\gamma^{\mu 2} = 1$ or -1. For $\sigma^{\mu\nu}$ $(\mu \neq \nu)$,

$$\sigma^{\mu\nu 2} = i\gamma^\mu\gamma^\nu i\gamma^\mu\gamma^\nu = \gamma^{\mu 2}\gamma^{\nu 2} = \pm 1\,.$$

We also have ${\gamma_5}^2 = 1$ (see text). For axial vectors,

$$(\gamma^\mu\gamma_5)^2 = \gamma^\mu \underbrace{\gamma_5\,\gamma^\mu}_{-\gamma^\mu\gamma_5} \gamma_5 = -\gamma^{\mu 2}{\gamma_5}^2 = \pm 1\,.$$

(b) For γ^μ or $\gamma^\mu\gamma_5$, γ_5 does the job:

$$\gamma^\mu\gamma_5 = -\gamma_5\gamma^\mu\,, \qquad (\gamma^\mu\gamma_5)\gamma_5 = -\gamma_5(\gamma_5\gamma^\mu)\,.$$

For γ_5, a γ^μ does the job as shown above. For $\sigma^{\mu\nu}$ $(\mu \neq \nu)$, the γ^μ anticommutes with it:

$$(i\gamma^\mu\gamma^\nu)\gamma^\mu = -\gamma^\mu(i\gamma^\mu\gamma^\nu)\,.$$

Thus, for any Γ_i $(i \neq 1)$, there is at least one Γ_k that anticommutes with it. Now the trace of $\Gamma_k\Gamma_i\Gamma_k$ can be written in two ways as

$$\mathrm{Tr}(\Gamma_k\Gamma_i\Gamma_k) = \begin{cases} \mathrm{Tr}(\Gamma_i\Gamma_k^2) & (\text{by } \mathrm{Tr}AB = \mathrm{Tr}BA) \\ -\mathrm{Tr}(\Gamma_i\Gamma_k^2) & (\text{by } \{\Gamma_i, \Gamma_k\} = 0) \end{cases} \quad \rightarrow \quad \mathrm{Tr}(\Gamma_i\Gamma_k^2) = 0\,.$$

On the other hand, $\Gamma_k^2 = \pm 1$ by (a); thus, we have $\mathrm{Tr}\Gamma_i = 0$ $(i \neq 1)$.

(c) By moving the same γ^μ's to next to each other and using $\gamma^{\mu 2} = \pm 1$, any product of γ^μ's can be uniquely reduced to the form

$$c\gamma^{\mu_1}\cdots\gamma^{\mu_n} \quad (\mu_i = 0, 1, 2, 3 \quad \mu_1 < \cdots < \mu_n \quad n \leq 4)$$

where c is a constant and all γ^μ's are different. The axial vector $\gamma^\mu\gamma_5$ corresponds to $n = 3$:

$$\gamma^\mu\gamma_5 = \gamma^\mu i\gamma^0\gamma^1\gamma^2\gamma^3 = \pm i\gamma^\alpha\gamma^\beta\gamma^\gamma\,,$$

where α, β, γ are 3 numbers different from μ. Thus, up to a constant, if $n = 1$ it should be one of the 4 vectors γ^μ, if $n = 2$ it should be one of the 6 tensors $i\gamma^\mu\gamma^\nu$, if $n = 3$ it should be one of the 4 axial vectors $\gamma^\mu\gamma_5$, and if $n = 4$ it is the pseudoscalar γ_5. When the product of two Γ_i's are taken, the only way it reduces to the scalar is that all γ^μ's are paired to form $\gamma^{\mu 2}$'s; namely, only when the two Γ_i's are the same. Thus, for any Γ_i and Γ_j $(i \neq j)$, the product reduces to the above form with $n > 0$:

$$\Gamma_i\Gamma_j = c\Gamma_k \qquad (i \neq j, k \neq 1)\,.$$

(d) Suppose $\sum_{i=1}^{16} c_i\Gamma_i = 0$. Taking the trace and noting that $\mathrm{Tr}\Gamma_i = 0$ $(i \neq 1)$, we have

$$c_1\mathrm{Tr}\Gamma_1 = 0 \quad \rightarrow \quad c_1 = 0\,.$$

Multiply Γ_j ($j \neq 1$) to $\sum_{i=1}^{16} c_i \Gamma_i = 0$ and take the trace. Then the only non-zero term is $i = j$ since all other terms are proportional to a certain Γ_k ($k \neq 1$):

$$c_j \mathrm{Tr} \Gamma_j^2 = 0 \quad \rightarrow \quad c_j = 0 \, . \quad (j \neq 1)$$

Namely, all the coefficients become zero, and thus Γ_i ($i = 1, \ldots, 16$) are linearly independent.