Exercise 3.7 (20 pnts)
1. Weyl representation:

$$\gamma^0 = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}.$$

The space part is the same as the Direac representation; thus, $\{\gamma^i, \gamma^j\} = 2g^{ij}$ is already shown. The rest is to show ${\gamma^0}^2 = 1$ and $\{\gamma^0, \gamma^i\} = 0$:

$${\gamma^0}^2 = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix}\begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} = 1.$$

and

$$\begin{aligned} \{\gamma^0, \gamma^i\} &= \gamma^0\gamma^i + \gamma^i\gamma^0 \\ &= \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}\begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix} \\ &= \begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix} + \begin{pmatrix} -\sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} = 0. \end{aligned}$$

Thus, the γ matrixes in the Weyl representation indeed satisfies $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$.

2. The γ matrixes in the Majorana representation (denoted as γ_M^μ) can be written using those in the Dirac representation as

$$\gamma_M^0 = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix} = \gamma^0\gamma^2, \quad \gamma_M^1 = \begin{pmatrix} i\sigma_3 & 0 \\ 0 & i\sigma_3 \end{pmatrix} = i\gamma_5\gamma^0\gamma^3,$$

$$\gamma_M^2 = \begin{pmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{pmatrix} = -\gamma^2, \quad \gamma_M^3 = \begin{pmatrix} -i\sigma_1 & 0 \\ 0 & -i\sigma_1 \end{pmatrix} = -i\gamma_5\gamma^0\gamma^1.$$

We have to show that ${\gamma_M^0}^2 = 1$, ${\gamma_M^i}^2 = -1$, and they all anticommute, which is equivalent to $\{\gamma_M^\mu, \gamma_M^\nu\} = 2g^{\mu\nu}$. With $k = 1$ or 3,

$${\gamma_M^0}^2 = \gamma^0\gamma^2\gamma^0\gamma^2 = -{\gamma^0}^2{\gamma^2}^2 = 1, \quad {\gamma_M^2}^2 = {\gamma^2}^2 = -1.$$

$${\gamma_M^k}^2 = -\gamma_5\gamma^0\gamma^k\gamma_5\gamma^0\gamma^k = {\gamma_5}^2{\gamma^0}^2{\gamma^k}^2 = -1.$$

Anticommutations (again with $k = 1$ or 3):

$$\{\gamma^0\gamma^2, \gamma_5\gamma^0\gamma^k\} = \underbrace{\gamma^0\gamma^2\gamma_5\gamma^0}_{{\gamma^0}^2\gamma^2\gamma_5}\gamma^k + \gamma_5\underbrace{\gamma^0\gamma^k\gamma^0}_{-{\gamma^0}^2\gamma^k}\gamma^2 = \gamma^2\gamma_5\gamma^k - \underbrace{\gamma_5\gamma^k\gamma^2}_{\gamma^2\gamma_5\gamma^k} = 0.$$

$$\{\gamma^0\gamma^2, \gamma^2\} = \gamma^0\gamma^2\gamma^2 + \gamma^2\gamma^0\gamma^2 = \gamma^0{\gamma^2}^2 - \gamma^0{\gamma^2}^2 = 0.$$

$$\{\gamma^2, \gamma_5\gamma^0\gamma^k\} = \gamma^2\gamma_5\gamma^0\gamma^k + \underbrace{\gamma_5\gamma^0\gamma^k\gamma^2}_{-\gamma^2\gamma_5\gamma^0\gamma^k} = 0$$

and

$$\{\gamma_5\gamma^0\gamma^1, \gamma_5\gamma^0\gamma^3\} = \underbrace{\gamma_5\gamma^0\gamma^1\gamma_5\gamma^0\gamma^3}_{-\gamma_5{}^2\gamma^{0\,2}\gamma^1\gamma^3} + \underbrace{\gamma_5\gamma^0\gamma^3\gamma_5\gamma^0\gamma^1}_{-\gamma_5{}^2\gamma^{0\,2}\gamma^3\gamma^1} = -(\gamma^1\gamma^3 + \gamma^3\gamma^1) = 0\,.$$

Thus, we see that $\{\gamma_M^\mu, \gamma_M^\nu\} = 0$ $(\mu \neq \nu)$. Putting all together we have shown $\{\gamma_M^\mu, \gamma_M^\nu\} = 2g^{\mu\nu}$.

Exercise 3.8
(a) We will use a shorthand for sin and cos: $s_\phi \equiv \sin\phi$, $c_\phi \equiv \cos\phi$ etc.
Using the formula $e^{i\vec{a}\cdot\vec{\sigma}} = c_a + i(\hat{a}\cdot\vec{\sigma})s_a$ and

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

the rotation matrix $u(\theta,\phi)$ can be written as

$$\begin{aligned}
u(\theta,\phi) &= e^{-i\frac{\phi}{2}\sigma_z} e^{-i\frac{\theta}{2}\sigma_y} = (c_{\frac{\phi}{2}} - i\sigma_z s_{\frac{\phi}{2}})(c_{\frac{\theta}{2}} - i\sigma_y s_{\frac{\theta}{2}}) \\
&= \begin{pmatrix} c_{\frac{\phi}{2}} - i s_{\frac{\phi}{2}} & 0 \\ 0 & c_{\frac{\phi}{2}} + i s_{\frac{\phi}{2}} \end{pmatrix} \begin{pmatrix} c_{\frac{\theta}{2}} & -s_{\frac{\theta}{2}} \\ s_{\frac{\theta}{2}} & c_{\frac{\theta}{2}} \end{pmatrix} \\
&= \underbrace{\begin{pmatrix} e^{-i\frac{\phi}{2}} & 0 \\ 0 & e^{i\frac{\phi}{2}} \end{pmatrix}}_{e^{-i\frac{\phi}{2}}\begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix}} \begin{pmatrix} c_{\frac{\theta}{2}} & -s_{\frac{\theta}{2}} \\ s_{\frac{\theta}{2}} & c_{\frac{\theta}{2}} \end{pmatrix} = e^{-i\frac{\phi}{2}} \begin{pmatrix} c_{\frac{\theta}{2}} & -s_{\frac{\theta}{2}} \\ e^{i\phi} s_{\frac{\theta}{2}} & e^{i\phi} c_{\frac{\theta}{2}} \end{pmatrix}.
\end{aligned}$$

Drop the overall phase $e^{-i\frac{\phi}{2}}$ and apply this to $|\uparrow\rangle$ and $|\downarrow\rangle$ to form χ_+ and χ_-:

$$\chi_+ = \begin{pmatrix} c_{\frac{\theta}{2}} & -s_{\frac{\theta}{2}} \\ e^{i\phi} s_{\frac{\theta}{2}} & e^{i\phi} c_{\frac{\theta}{2}} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} c_{\frac{\theta}{2}} \\ e^{i\phi} s_{\frac{\theta}{2}} \end{pmatrix},$$

$$\chi_- = \begin{pmatrix} c_{\frac{\theta}{2}} & -s_{\frac{\theta}{2}} \\ e^{i\phi} s_{\frac{\theta}{2}} & e^{i\phi} c_{\frac{\theta}{2}} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -s_{\frac{\theta}{2}} \\ e^{i\phi} c_{\frac{\theta}{2}} \end{pmatrix}.$$

Now write them in terms of $s_x = s_\theta c_\phi$, $s_y = s_\theta s_\phi$. Using $s_z = c_\theta$, $2c_\theta^2 = 1 + c_{2\theta}$, $2s_\theta^2 = 1 - c_{2\theta}$, and noting $|s_+| = s_\theta$,

$$c_{\frac{\theta}{2}} = \sqrt{\frac{1+c_\theta}{2}} = \begin{cases} \sqrt{\dfrac{(1+c_\theta)^2}{2(1+c_\theta)}} = \dfrac{1+s_z}{\sqrt{2(1+s_z)}} \\[2ex] \sqrt{\dfrac{1-c_\theta^2}{2(1-c_\theta)}} = \dfrac{s_\theta}{\sqrt{2(1-s_z)}} \end{cases}$$

$$s_{\frac{\theta}{2}} = \sqrt{\frac{1-c_\theta}{2}} = \begin{cases} \sqrt{\dfrac{(1-c_\theta)^2}{2(1-c_\theta)}} = \dfrac{1-s_z}{\sqrt{2(1-s_z)}} \\[2ex] \sqrt{\dfrac{1-c_\theta^2}{2(1+c_\theta)}} = \dfrac{s_\theta}{\sqrt{2(1+s_z)}} \end{cases}$$

Since $s_+ = s_\theta(c_\phi + i s_\phi) = s_\theta e^{i\phi}$,

$$e^{i\phi} s_{\frac{\phi}{2}} = \frac{s_+}{\sqrt{2(1+s_z)}} , \quad e^{i\phi} c_{\frac{\phi}{2}} = \frac{s_+}{\sqrt{2(1-s_z)}} .$$

Then, χ_+ and χ_- can be written as

$$\chi_+ = \frac{1}{\sqrt{2(1+s_z)}} \begin{pmatrix} 1+s_z \\ s_+ \end{pmatrix} , \quad \chi_- = \frac{1}{\sqrt{2(1-s_z)}} \begin{pmatrix} s_z - 1 \\ s_+ \end{pmatrix} .$$

Since the rotation $u(\theta, \phi)$ is a unitary matrix, the norm is conserved; namely, it is automatically unity.

(b) Using the explicit expression of $\vec{s} \cdot \vec{\sigma}$

$$\vec{s} \cdot \vec{\sigma} = \begin{pmatrix} s_z & s_- \\ s_+ & -s_z \end{pmatrix} ,$$

the projection operators are

$$P_+ = \frac{1 + \vec{s} \cdot \vec{\sigma}}{2} = \frac{1}{2} \begin{pmatrix} 1+s_z & s_- \\ s_+ & 1-s_z \end{pmatrix} , \quad P_- = \frac{1 - \vec{s} \cdot \vec{\sigma}}{2} = \frac{1}{2} \begin{pmatrix} 1-s_z & -s_- \\ -s_+ & 1+s_z \end{pmatrix}$$

Applying $P_\pm$ to any vector, say $|\uparrow\rangle$, should give $\chi_\pm$ up to a constant:

$$\chi_+ \propto P_+ \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1+s_z \\ s_+ \end{pmatrix} , \quad \chi_- \propto P_- \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1-s_z \\ -s_+ \end{pmatrix} ,$$

which is consistent with (a).