Exercise 4.1

(a) Noting that $[a, a^\dagger] = 1$ leads to $aa^\dagger = 1 + N$, and using $N{a^\dagger}^n|0\rangle = n{a^\dagger}^n|0\rangle$,

$$\begin{aligned}\langle n|n\rangle &= \frac{1}{n!}\langle 0|a^n {a^\dagger}^n|0\rangle = \frac{1}{n!}\langle 0|a^{n-1}\underbrace{aa^\dagger}_{1+N=1+(n-1)=n}{a^\dagger}^{n-1}|0\rangle \\ &= \frac{n}{n!}\langle 0|a^{n-1}{a^\dagger}^{n-1}|0\rangle\end{aligned}$$

Rpeating the procedure results in

$$\langle n|n\rangle = \frac{n!}{n!}\langle 0|0\rangle = 1\,.$$

(b) We have

$$|n\rangle = \frac{1}{\sqrt{n!}}{a^\dagger}^n|0\rangle\,, \quad |n+1\rangle = \frac{1}{\sqrt{(n+1)!}}{a^\dagger}^{n+1}|0\rangle\,.$$

Then,

$$a^\dagger|n\rangle = \frac{1}{\sqrt{n!}}{a^\dagger}^{n+1}|0\rangle = \frac{\sqrt{n+1}}{\sqrt{(n+1)!}}{a^\dagger}^{n+1}|0\rangle = \sqrt{n+1}|n+1\rangle\,.$$

Also,

$$|n-1\rangle = \frac{1}{\sqrt{(n-1)!}}{a^\dagger}^{n-1}|0\rangle\,.$$

Then,

$$a|n\rangle = \frac{1}{\sqrt{n!}}a{a^\dagger}^n|0\rangle = \frac{1}{\sqrt{n!}}\underbrace{aa^\dagger}_{1+\underbrace{N}_{n-1}=n}{a^\dagger}^{n-1}|0\rangle = \frac{\sqrt{n}}{\sqrt{(n-1)!}}{a^\dagger}^{n-1}|0\rangle = \sqrt{n}|n-1\rangle\,.$$

Exercise 4.2 Matrix representation of fermionic oscillator.
The operator a can be written as a 2 by 2 matrix as

$$a = \begin{pmatrix} \langle 0|a|0\rangle & \langle 0|a|1\rangle \\ \langle 1|a|0\rangle & \langle 1|a|1\rangle \end{pmatrix} .$$

Using the relations given, we obtain

$$\langle 0|\underbrace{a|0\rangle}_{0} = 0, \qquad \langle 0|\underbrace{a|1\rangle}_{|0\rangle} = 1 ,$$
$$\langle 1|\underbrace{a|0\rangle}_{0} = 0, \qquad \underbrace{\langle 1|a}_{0}|1\rangle = 0 .$$

Namely,

$$a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \to a^\dagger = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} .$$

Then, the anti-commutation relations can be verified explicitly,

$$\{a, a^\dagger\} = aa^\dagger + a^\dagger a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 ,$$

and

$$a^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 = 0 , \quad \to \quad a^{\dagger 2} = 0 ,$$

thus,

$$\{a, a\} = \{a^\dagger, a^\dagger\} = 0 .$$