Belle Note 269
July 1999

Space-time Profile of e^+e^- Interaction With Crossing Angles

Hitoshi Yamamoto

Dept. of Physics and Astronomy
The University of Hawaii
2505 Correa Rd., Honolulu, HI 96822, USA
e-mail: hitoshi@uhheph.phys.hawaii.edu

Abstract

We investigate the space-time profile as well as the luminosity factor of beam-beam interactions when the beams collide with finite crossing angles. Cases with and without crab cavity are considered. The widths and offsets of interaction profile are estimated using gaussian approximation. Effects of timing error, different beam sizes for e^+ and e^-, as well as the waist effect are also included.

1 Introduction

The space distribution of interactions in e^+e^- colliders are needed in variety of studies including vertex constraint of charged particle tracking. On the other hand, the time profile of the interaction has implication on time-of-flight measurements. The shape of the interaction profile can be directly measured if the measurement resolution is good enough, but often it is not the case. It is thus desirable to predict the ineraction profile from the physical bunch dimensions that are obtained from accelerator information. The problem becomes non-trivial when the two beams cross with a finite crossing angle with or without crab crossing. Also, often two beams have different dimensions, and the resulting effects need to be included. We will also consider the possibility that the two beams are not exactly in synchronized or transversely offset. As we will see, the off-timing or transverse displacement has no effect on the interaction profile even though it reduces luminosity.

2 Luminosity formulas

When a distribution of electrons with density ρ_- passes through a distribution of positrons with density ρ_+, the interaction per unit time per unit volume n_0 is given by

$$n_0 = v_{\rm rel}\, \rho_- \rho_+ \sigma(s)\,, \qquad \text{(collinear)} \tag{1}$$

where $\sigma(s)$ is the cross section of the interaction evaluated at the corresponding C.M. energy squared s, and $v_{\rm rel}$ is the relative velocity of e^+ and e^- measured in the laboratory. This formula is valid even when the two beams have different velocities (or different energies) as long as they are collinear.

If the two beams are not collinear, the above expression needs to be modified. In order to derive the correct formula, we start from a head-on collision case [Figure 1 (a)] where the $e^+(e^-)$ beam has energy $p_0^+(p_0^-)$. In Figure 1 (b) is shown the configuration after boosting it in the x direction with velocity β. We assume that the electron mass is much smaller than the energy scale of interest, and use a unit system where $c = 1$. The energies of e^- and e^+ are increased by the factor $\gamma = 1/\sqrt{1-\beta^2}$ and the angle from the z axis is given by

$$\cos\theta = \frac{1}{\gamma}\,, \qquad \tan\theta = \eta\,, \qquad \sin\theta = \beta\,, \tag{2}$$

where $\eta \equiv \beta\gamma$. Now focus on an 'event' where a single electron projectile is moving with velocity $(v_x^+, v_y^+, v_z^+) = (\sin\theta, 0, \cos\theta)$ inside a cloud of positrons moving with velocity $(v_x^-, v_y^-, v_z^-) = (\sin\theta, 0, -\cos\theta)$. Let $\rho_\pm^*$ be the density of $e^\pm$ in the vicinity of this 'event' evaluated in the rest frame of each bunch. The density in other frame is increased by the Lorentz contraction. The density in the head-on collision case

(a) is then $\rho^0_\pm = \gamma^*_\pm \rho^*_\pm$, where $\gamma^*_\pm$ is the γ-factor of $e^\pm$. On the other hand, the corresponding Lorentz contraction factor in the x-boosted frame (b) is $\gamma p_0^\pm/m_e = \gamma\gamma^*_\pm$ which means that the density in that frame is $\rho_\pm = \gamma\gamma^*_\pm\rho^*_\pm$. Namely, the densities in frame (a) and those in frame (b) are related by

$$\rho_\pm = \gamma \rho^0_\pm \,. \tag{3}$$

In the x-boosted frame (b), what is the number of interaction for the given electron to interact with positron within a time window of $\Delta t = t_2 - t_1$, or between the space-time point w_1 (defined as the position of the given electron at the start of the time window t_1) and w_2 (defined as the position of the given electron at the end of the time window t_2)? We will boost these two events back to the head-on frame where the number of interaction is easily evaluated. The 4-vector $\Delta w \equiv w_2 - w_1$ is given by

$$\Delta w \equiv \begin{pmatrix} \Delta t \\ \Delta x \\ \Delta y \\ \Delta z \end{pmatrix} = \begin{pmatrix} \Delta t \\ \vec{v}^+ \Delta t \end{pmatrix} = \begin{pmatrix} \Delta t \\ \sin\theta\,\Delta t \\ 0 \\ \cos\theta\,\Delta t \end{pmatrix} . \tag{4}$$

In the head-on frame, $\Delta w'$ is then given by

$$\begin{pmatrix} \Delta t' \\ \Delta x' \end{pmatrix} = \begin{pmatrix} \gamma & -\eta \\ -\eta & \gamma \end{pmatrix} \begin{pmatrix} \Delta t \\ \Delta x \end{pmatrix} = \begin{pmatrix} \Delta t/\gamma \\ 0 \end{pmatrix} , \tag{5}$$

where we have used $\Delta x = \sin\theta \Delta t = \beta \Delta t$. Then the number of interaction in the head-on frame between the two events, n_{12}, is the product of the length of the travel, the density of target, and the cross section (with $v_{\rm rel} = 2$ and $\rho^0_+ = \rho_+/\gamma$)

$$n_{12} = (v_{\rm rel}\Delta t')\,\rho^0_+\,\sigma(s) = 2\,\frac{\Delta t}{\gamma}\,\frac{\rho_+}{\gamma}\,\sigma(s)\,. \tag{6}$$

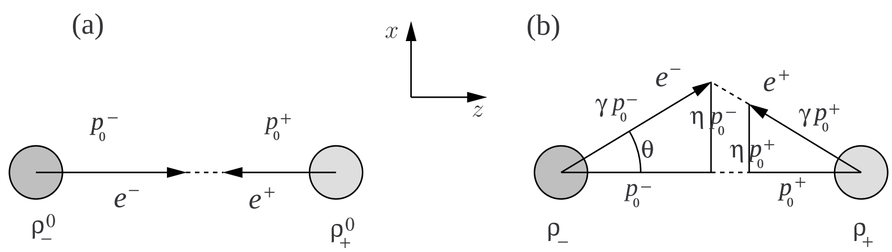

Figure 1: The luminosity formula for the case where two beams do not collide head-on, the configuration (b), can be obtained by transversely boosting the head-on configuration (a) by velocity β. ($\eta \equiv \beta\gamma$.)

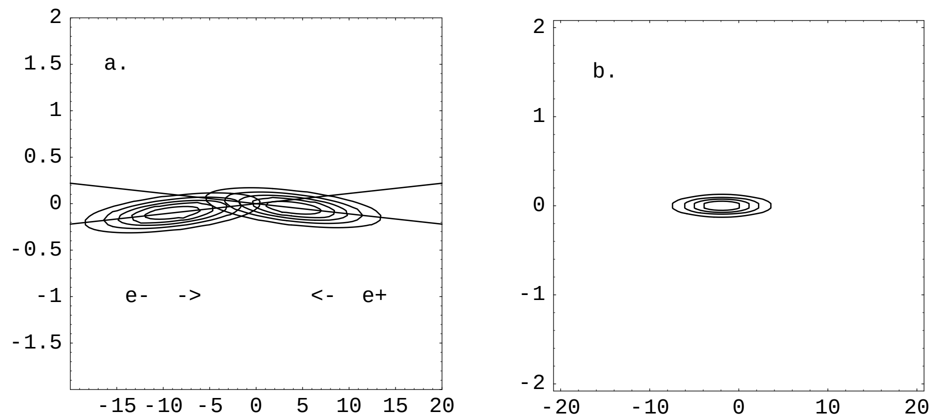

Figure 2: Finite-angle collision without crab crossing. The configuration shown corresponds to $\sigma_z = 5$ mm, $\sigma_x = 0.1$ mm, $t = -4$ and the electron bunch is delayed by $a = 5$. The crossing angle is ± 11 mrad. Each bunch is shown in (a), and the product of the two is shown in (b). Unit is mm.

The number of interactions between two events on the path of a particle is the same in any frame. In the x-boosted frame (b), the number of projectile in a unit volume is ρ_-, thus the number of interaction per unit time per unit volume in the x-boosted frame is

$$n = \frac{n_{12}\rho_-}{\Delta t} = \frac{2}{\gamma^2}\,\rho_-\rho_+\,\sigma(s) = 2\cos^2\theta\,\rho_-\rho_+\,\sigma(s)\,. \quad (x\text{-boosted frame}) \tag{7}$$

Or, the luminosity density $\ell(w)$ at a given space-time point w is given by

$$\ell(w) \equiv \frac{n}{\sigma(s)} = 2\cos^2\theta\,\rho_-\rho_+\,. \tag{8}$$

When we compare the luminosity for different crossing angles, we will fix the center of mass energy, and thus $\sigma(s)$, and vary the angle θ, while the beam sizes in the laboratory frame are kept the same. This means that we vary the beam energies for different crossing angles.

3 Finite-angle collision without crab crossing

We first study the case where two beams collide with a finite crossing angle without using a crab cavity, an example of which is shown in Figure 2. We recall that $c = 1$, and time is measured in mm. We include the possibility that the two beams do not collide at the exact crossing point of the design orbits. Specifically, we assume that the electron bunch is delayed relative to the positron bunch by a. When one of the beam is transversely shifted, it is equivalent to a such timing offset.

If the crossing angle were zero, the densities of positron and electron at a space-time point $w = (x, y, z, t)$ is given by

$$\rho^0_\pm(w) = A \exp\Big(-\frac{1}{2} f^0_\pm(w)\Big)\,, \tag{9}$$

with

$$\begin{aligned} f^0_-(w) &= \frac{x^2}{\sigma_x^2} + \frac{(z - t + a)^2}{\sigma_z^2} \\ f^0_+(w) &= \frac{x^2}{\sigma_x^2} + \frac{(z + t)^2}{\sigma_z^2}\,. \end{aligned} \tag{10}$$

The constant A does not depend on w nor on a. We will be interested in the widths and means of the distributions, and not the absolute rates except for the relative luminosities when the crossing angle θ or the timing offset a is varied. Thus, the actual value of the constant A is irrelevant. We have assumed that the beam shape is gaussian. We also assume that the design orbits are in the xz plane and the y profile is independent of x, z, t; then, the y-dependence decouples from the rest and can be ignored. Later, we will discuss the waist effect which violates this assumption. The crossing angle is incorporated by rotating the electron bunch by an angle θ and the positron bunch by $-\theta$. Namely, $\rho_\pm = A \exp(-\frac{1}{2} f_\pm)$ with

$$\begin{aligned} f_-(x, z, t) &= f^0_-(\cos\theta\, x - \sin\theta\, z, \quad \sin\theta\, x + \cos\theta\, z, t) \\ f_+(x, z, t) &= f^0_+(\cos\theta\, x + \sin\theta\, z, -\sin\theta\, x + \cos\theta\, z, t)\,. \end{aligned} \tag{11}$$

For example, the design values for KEK-B are[1]

$$\sigma_x = 0.077\,\mathrm{mm}, \quad \sigma_z = 5\,\mathrm{mm}, \quad \theta = 0.011\,\mathrm{rad}\,; \tag{12}$$

namely, σ_x/σ_z and θ are of the same order.

The luminosity density is then obtained by (8):

$$\ell(x, z, t) = 2A^2 \cos^2\theta\, \exp\Big(-\frac{1}{2} g(x, z, t)\Big) \tag{13}$$

with

$$\begin{aligned} g(z, x, t) &= \frac{(\cos\theta\, x - \sin\theta\, z)^2 + (\cos\theta\, x + \sin\theta\, z)^2}{\sigma_x^2} \\ &+ \frac{(\sin\theta\, x + \cos\theta\, z - t + a)^2 + (-\sin\theta\, x + \cos\theta\, z + t)^2}{\sigma_z^2}\,. \end{aligned} \tag{14}$$

This is a multivariate gaussian in the space of (x, z, t). Due to cancellation of cross terms, there is no xz term, indicating that the gaussian is not tilted in the xz-plane.

In general, the width σ and mean μ of a gaussian distribution

$$\exp\left[-\frac{1}{2}(ax^2 + bx + c)\right] \tag{15}$$

are given by

$$\sigma = \frac{1}{\sqrt{a}}\,, \quad \mu = -\frac{b}{2a}\,. \tag{16}$$

For the distribution (14) at given z and t, the width and mean of x can be readily extracted:

$$\sigma'_x(z,t) = \frac{1}{\sqrt{2}}\frac{\sigma_z}{\sigma_c}\sigma_x\,, \quad \mu_x(z,t) = \left(t - \frac{a}{2}\right)\frac{\sigma_x^2}{\sigma_c^2}\sin\theta\,, \tag{17}$$

where

$$\sigma_c \equiv \sqrt{\sigma_z^2\cos^2\theta + \sigma_x^2\sin^2\theta}\,. \tag{18}$$

We have used σ' to denote a width of luminosity as opposed to the bunch width. For KEK-B, we have $\sigma_c \sim \sigma_z$. The motion of mean x, $\mu_x(z,t)$, is finite but practically zero. If the crossing angle is zero, the x width is exactly the x width of bunch divided by $\sqrt{2}$. The factor $1/\sqrt{2}$ comes from the fact that the luminosity is proportional to the product of the two gaussians. This argument holds for the y width which are decoupled from other dimensions:

$$\sigma'_y = \frac{1}{\sqrt{2}}\sigma_y\,. \tag{19}$$

For given x and t, the width and mean of z are

$$\sigma'_z(x,t) = \frac{1}{\sqrt{2}}\frac{\sigma_x}{\sigma_s}\sigma_z\,, \quad \mu_z(x,t) = -\frac{a}{2}\frac{\sigma_x^2}{\sigma_s^2}\cos\theta\,, \tag{20}$$

where

$$\sigma_s \equiv \sqrt{\sigma_x^2\cos^2\theta + \sigma_z^2\sin^2\theta}\,. \tag{21}$$

We have $\sigma_x\cos\theta \sim \sigma_z\sin\theta$ for KEK-B; in fact, numerically $\sigma_s \sim 1.23\,\sigma_x$ for the design values and thus $\sigma'_z(x,t) \sim 0.57\sigma_z$. The extra suppression by the factor σ_x/σ_s in addition to the $1/\sqrt{2}$ factor is caused by the crossing angle. Similarly, for a given space point (x, z), the width and mean of the time distribution of interaction is

$$\sigma'_t(x,z) = \frac{1}{\sqrt{2}}\sigma_z\,, \quad \mu_t(x,z) = \frac{a}{2} + x\sin\theta\,. \tag{22}$$

Inspecting (17), (20), and (22), we notice that the widths do not depend on time as the bunches cross. They are also independent of the timing delay a. The average

z is shifted proportionally to a, but it is independent of x and t. It indicates that when integrated over x and t, the z width will stay the same. The average t is a function of x, thus the time width will be wider when integrated over x and z.

Integration of (13) over t can be easily accomplished with the following formula:

$$\int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}(ax^2+bx+c)} = \sqrt{\frac{2\pi}{a}} e^{-\frac{1}{2}\left(c-\frac{b^2}{4a}\right)} . \tag{23}$$

The result is a diagonal multivariate gaussian in xz-plane (namely, there is no xz term), and the widths and means of x and z can be identified as before. The result for x is

$$\sigma'_x = \frac{1}{\sqrt{2}} \frac{\sigma_x}{\cos\theta} , \quad \mu_x = 0 . \tag{24}$$

As expected, the width is larger than that for a given time t. The mean is averaged over to become zero. For z, we have

$$\sigma'_z = \frac{1}{\sqrt{2}} \frac{\sigma_x}{\sigma_s} \sigma_z , \quad \mu_z = -\frac{a}{2} \frac{\sigma_x^2}{\sigma_s^2} \cos\theta , \tag{25}$$

which are the same as those at a given time t as expected.

Similarly, one can integrate (13) over x and z to investigate the overall time profile. The resulting width and mean are

$$\sigma'_t = \frac{1}{\sqrt{2}} \frac{\sigma_c}{\cos\theta} , \quad \mu_t = \frac{a}{2} . \tag{26}$$

The width of the overall time distribution is slightly larger than that at a fixed point (x, z) which was expected since the average time depended on x. The timing shift is simply $a/2$ which is the same as the naive expectation for the head-on case with small point-like bunches.

Integrating over all three variables, one obtains the total luminosity per bunch crossing:

$$L \equiv \int dxdzdt\, \ell(x, z, t) . \tag{27}$$

Define the luminosity factor $r(\theta, a)$ by

$$r(\theta, a) = \frac{L(\theta, a)}{L(\theta = 0, a = 0)} . \tag{28}$$

Performing the integral, the luminosity factor is found to be

$$r(\theta, a) = \exp\Big(-\frac{a^2 \sin^2\theta}{4\sigma_s^2} \Big) \frac{\sigma_x}{\sigma_s} \cos\theta . \tag{29}$$

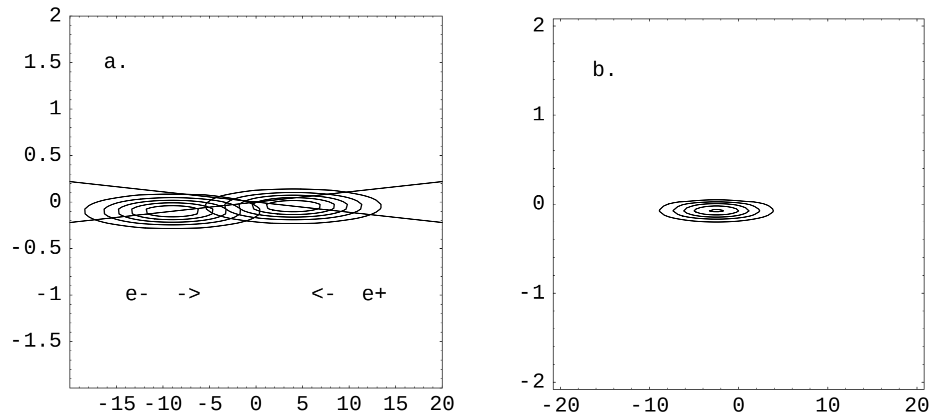

Figure 3: FInite-angle collision with crab crossing. The configuration shown corresponds to $\sigma_z = 5$ mm, $\sigma_x = 0.1$ mm, $t = -4$ and the electron bunch is delayed by $a = 5$. The crossing angle is ± 11 mrad. Each bunch is shown in (a), and the product of the two is shown in (b).

When $\theta = 0$, we have $\sigma_s = \sigma_x$ and thus $r = 1$; namely, for a head-on collision, there is no luminosity loss regardless of the timing offset a. The exponential represents the luminosity loss due to the timing offset, and the ratio σ_x/σ_s represents the luminosity loss due to the crossing angle without crab crossing. For $a = 0$ and $\theta << 1$, the formula above reduces to

$$r(a, \theta) \sim \frac{\sigma_x}{\sqrt{\sigma_x^2 + \sigma_z^2 \theta^2}} \quad (a = 0, \theta \ll 1)\,, \tag{30}$$

which agrees with the result of Ref [2].

4 Finite-angle collision with crab crossing

In crab crossing, a rotational oscillation in the xz-plane is induced for each bunch such that, when they collide, they cross head-on while moving transversely. Such configuration is shown in Figure 3. To express it analytically, we start from the simple function

$$h_0(x, z) = \frac{x^2}{\sigma_x^2} + \frac{z^2}{\sigma_z^2}\,, \tag{31}$$

then apply an transformation

$$\begin{aligned} h_-(x, z) &= h_0\Big(x - \sin\theta\,(t - a), z - \cos\theta\,(t - a)\Big) \\ h_+(x, z) &= h_0\Big(x - \sin\theta\, t, z + \cos\theta\, t\Big) \end{aligned} \tag{32}$$

and exponentiate them to define the particle densities as

$$\rho_\pm = A\,\exp\Big(-\frac{1}{2}h_\pm\Big)\,. \tag{33}$$

The crossing angle is still θ, and the electron bunch is delayed relative to the positron bunch by a as before. The luminosity density is then given by (8) and the rest is identical to the previous section. The result is listed below: before the integration over time, the interaction profile is a gaussian with all three axes aligned with the coordinate axes where

$$\sigma'_x(z,t) = \frac{\sigma_x}{\sqrt{2}}\,, \quad \mu_x(z,t) = \Big(t - \frac{a}{2}\Big)\sin\theta\,, \tag{34}$$

$$\sigma'_z(x,t) = \frac{\sigma_z}{\sqrt{2}}\,, \quad \mu_z(x,t) = -\frac{a}{2}\cos\theta\,, \tag{35}$$

$$\sigma'_t(x,z) = \frac{1}{\sqrt{2}}\frac{\sigma_x}{\sigma_s}\,\sigma_z\,, \quad \mu_t(x,z) = \frac{a}{2} + \frac{\sigma_z^2}{\sigma_s^2}\,x\sin\theta\,, \tag{36}$$

where σ_s is defined by (21) as before. From (34) and (35), one can see that the interaction profile is a gaussian of fixed x and z widths moving in the $+x$ direction with velocity $\sin\theta$. There is no motion in z direction. After integrating over t, we still have a gaussian aligned with the coordinate axes with

$$\sigma'_x = \frac{1}{\sqrt{2}}\frac{\sigma_s}{\cos\theta}\,, \quad \mu_x = 0\,, \tag{37}$$

$$\sigma'_z = \frac{\sigma_z}{\sqrt{2}}\,, \quad \mu_z = -\frac{a}{2}\cos\theta\,. \tag{38}$$

The x width became larger since the interaction profile is moving in x direction, while the z width did not change since $\mu_z(x,t)$ is a constant.

Integrating over x and z, the time profile is

$$\sigma'_t = \frac{1}{\sqrt{2}}\frac{\sigma_z}{\cos\theta}\,, \quad \mu_t = \frac{a}{2}\,, \tag{39}$$

and the luminosity factor is obtained by integrating over all three variables:

$$r(\theta,a) = \exp\Big(-\frac{a^2\sin^2\theta}{4\sigma_x^2}\Big)\cos\theta\,, \tag{40}$$

which does not contain the 'crossing angle factor' σ_x/σ_s, which is the main purpose of the crab crossing.

5 If bunch sizes differ for e^- and e^+

When the electron and positron bunches have different bunch sizes, the expressions for the interaction profile become quite complicated. One may simply form linear averages of the two bunch sizes

$$\sigma_i = \frac{\sigma_i^- + \sigma_i^+}{2} \quad (i = x, z)\,, \tag{41}$$

where $\sigma_i^\pm$ are the $e^\pm$ bunch sizes, and blindly use them in the expressions obtained earlier. The approximations are reasonably good when two sizes are nearly equal. In some cases, such as the x width in a head-on collision, the smaller of the two sizes is weighted more. However, in some other cases, such as z width in the same situation, the larger of the two sizes is weighted more. Also, the cancellation of xz terms in (14) does not occur if two beam sizes are different and the gaussian will not be aligned with the laboratory coordinates. This is true for both with and without crab cavity.

Here, we will list the correct expressions for the overall x, z, and t widths, where 'overall' means that after integrating over all other variables. First, define two kinds of averages:

$$\sigma_i^{a2} \equiv \frac{1}{2}(\sigma_i^{-2} + \sigma_i^{+2}) \quad (i = x, y, z)\,, \tag{42}$$

and

$$\frac{1}{\sigma_i^{b2}} \equiv \frac{1}{2}\left(\frac{1}{\sigma_i^{-2}} + \frac{1}{\sigma_i^{+2}}\right) \quad (i = x, y, z)\,, \tag{43}$$

which satisfy

$$\sigma_i^a \sigma_i^b = \sigma_i^- \sigma_i^+ \quad (i = x, z)\,. \tag{44}$$

Then, for crossing angle without crab cavity, the overall widths are found to be

$$\text{(without crab)} \quad \begin{aligned} \sigma'_x &= \frac{1}{\sqrt{2}} \frac{\sigma_x^a}{\sigma_s^a} \sqrt{\sigma_x^{b2} + \sigma_z^{a2} \tan^2\theta}\,, \\ \sigma'_z &= \frac{1}{\sqrt{2}} \frac{\sigma_x^a}{\sigma_s^a}\, \sigma_z^a\,, \\ \sigma'_t &= \frac{1}{\sqrt{2}} \frac{\sqrt{\sigma_x^{a2}\sigma_z^{a2} + (\sigma_x^{-2} - \sigma_z^{-2})(\sigma_x^{+2} - \sigma_z^{+2}) \cos^2\theta \sin^2\theta}}{\sigma_s^a \cos\theta}\,, \end{aligned} \tag{45}$$

with

$$\sigma_s^a \equiv \sqrt{\sigma_x^a \cos^2\theta + \sigma_z^a \sin^2\theta}\,. \tag{46}$$

For crossing angle with crab cavity, we have

$$\sigma'_x = \frac{1}{\sqrt{2}} \sqrt{\sigma_x^{b2} + \sigma_z^{a2} \tan^2\theta}$$

$$(\text{crab}) \quad \sigma'_z = \frac{1}{\sqrt{2}}\sigma^a_z\,, \qquad (47)$$
$$\sigma'_t = \frac{1}{\sqrt{2}}\frac{\sigma^a_z}{\cos\theta}\,.$$

The above widths are for the distribution projected on to the axis of interest after integrated over all other vairiables. This was necessary since there is non zero xz term after integration over t. Another interesting parameter then is the tilt of the axis of the interaction profile after integrated over t. We express it in the form of

$$x = \lambda z \qquad (48)$$

where the origin is taken to be the center of the multi-variate gaussian. It is extracted in the approximation of small θ and $\sigma_z \gg \sigma_x$ by

$$\lambda = -\frac{C_{xz}}{2C_{x^2}}\,. \qquad (49)$$

where $C_{x^2,xz}$ is the coefficient of x^2 or xz term in the exponent of the gaussian. For the case of crossing without crab cavity, this slope paramter is given by

$$(\text{without crab}) \quad \lambda = \frac{\sigma^{-2}_x - \sigma^{+2}_x}{\sigma^{-2}_x + \sigma^{+2}_x}\tan\theta\,. \qquad (50)$$

It should be noted that there may be an intrinsic tilt of the major axis of bunch with respect to the direction of motion in the laboratory frame due to the transverse force excerted on a bunch during finite angle crossing. This tilt angle, however, is estimated to be less than 0.2 mrad for the operating point of KEK-B [1]. For the case of crab crossing, λ is given by

$$(\text{crab}) \quad \lambda = \frac{\sigma^{+2}_z - \sigma^{-2}_z}{\sigma^{+2}_z + \sigma^{-2}_z}\tan\theta\,. \qquad (51)$$

We see that the tilt angle is never greater than the crossing angle itself, and is suppressed by the asymmetry parameter of the x width squared for the case of crossing without crab cavity, and for crab crossing it is suppressed by the asymmetry parameter of z width squared.

For completeness, we give the fomulas for the luminosity per bunch crossing:

$$(\text{without crab}) \quad L_{\text{bunch}} = n^+n^-e^{-\frac{a^2\sin^2\theta}{4\sigma^{a2}_x}}\frac{\cos\theta}{4\pi\sigma^a_s\sigma^a_y}\,, \qquad (52)$$

$$(\text{with crab}) \quad L_{\text{bunch}} = n^+n^-e^{-\frac{a^2\sin^2\theta}{4\sigma^{a2}_s}}\frac{\cos\theta}{4\pi\sigma^a_x\sigma^a_y}\,, \qquad (53)$$

where $n^\pm$ is the number of particles in one $e^\pm$ bunch. One sees that if one replaces σ^a_x's in the luminosity formula with crab by σ^a_s one obtains that without crab.

6 Waist effects

The beam size in x or y at a given point along the orbit is given by

$$\sigma_i(s) = \sqrt{\beta_i(s)\epsilon_i + (\eta_i(s)\delta)^2} \quad (i = x, y)\,, \tag{54}$$

where $\beta_i(s)$ is the amplitude function, ϵ_i is the emittance, $\eta_i(s)$ is the dispersion or off-energy function, δ is the fractional energy spread, and s is the longitudinal distance from some reference point on the orbit. Near IP, the off-energy function is usually small and can be ignored. The beam width is thus changing as $\sqrt{\beta_{x,y}(s)}$ in the absence of the energy spread term. The beta function varies quite rapidly near IP as [2]

$$\beta_i(s) = \beta_i^* \left(1 + \frac{s^2}{\beta_i^{*2}}\right) \quad (i = x, y)\,, \tag{55}$$

where β_i^* is the minimum value of the beta function which is assumed to occur at $s = 0$. Namely, at $s = \beta_i^*$ the beam width is $\sqrt{2}$ times its minimum value. In the ideal case, the minimum is at IP where the design orbits of the two beams cross and we have $s = z$. We assume that such is the case. We also assume that the crossing angle θ is small in this section. When the beam is expanded due to a large value of the beta function, the density of the particles becomes less and as a result the luminosity density becomes smaller. The effects this has on the interaction profile can sometimes be quite large. This associated effects are called 'waist effects' or 'hourglass effects'.

For KEK-B, the design values are $\beta_y^* = 1$ cm and $\beta_x^* = 33$ cm which should be compared to the longitudinal beam size of $\sigma_z = 0.5$ cm. Typically, $\beta_x^* \gg \sigma_z$ and the waist effect cause by the variation of σ_x along z is usually negligible. Thus, the gaussian density distribution so far used in the luminosity profile estimations should be corrected by the variation of σ_y along z, which leads to the factor $\sqrt{\beta_y^*/\beta_y(z)}$ for each beam. In particular, the vertical size of the interaction is now

$$\sigma'_y(z) = \sigma'_y \sqrt{1 + \frac{z^2}{\beta_y^{*2}}}\,, \tag{56}$$

where σ'_y is the vertical width without the waist effect, or the vertical size at $z = 0$. This effectively reduces the longitudinal size of the interaction profile. On the other hand, the overall vertical size is enlarged by the very effect of the vertical enlargement of the bunch away from IP.

Let $\ell(z)$ be the luminoisty density at z after integarted over t and x. For a given number of particles, the luminosity scales as inverse of the cross sectional area. Assuming that β_y^* is the same for both beams, the correction factor for the

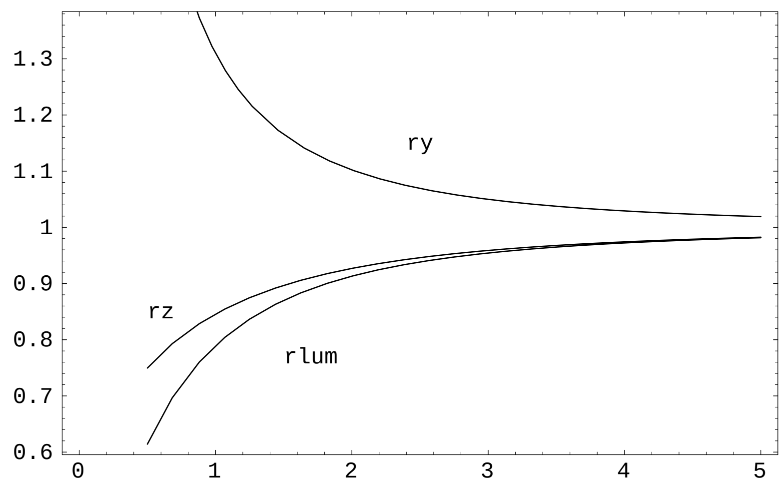

Figure 4: Various correction factors for the waist effect plotted as functions of $b \equiv \beta_y^*/\sigma_z'$. r_z is for z width, r_y is for y width, and r_{lum} is for luminosity.

luminosity density as a function of z is then $\sqrt{\beta_y^*/\beta_y(z)}$. Namely, $\ell(z)$ is now

$$\ell(z) \propto \exp\Big(-\frac{z^2}{2\sigma_z'^2}\Big)\left(1+\frac{z^2}{\beta_y^{*2}}\right)^{-\frac{1}{2}} \tag{57}$$

where σ_z', as before, is the z-width of the interaction profile estimated without the waist effect. This is no longer a gaussian where the gaussian tail is suppressed by the correction factor. When the experimental shape of z profile is to be compared with theory, the appropriate expression to be used would be (57) with σ_z' given by (25) or (38). The rms of this distribution can be given in a closed form:

$$\sigma_z^{\text{rms}} = r_z \sigma_z' \tag{58}$$

with

$$r_z^2 = \sqrt{\pi} e^{-\frac{b^2}{4}} \frac{U\Big(\frac{1}{2}, 0, \frac{b^2}{2}\Big)}{K_0\Big(\frac{b^2}{4}\Big)}\,, \tag{59}$$

where U is the confluent hypergeometric function, K is a Bessel function, and

$$b \equiv \frac{\beta_y^*}{\sigma_z'}\,. \tag{60}$$

The correction factor r_z as a function of b is plotted in Figure 4. The KEK-B design value corresponds to $b \sim 3$ which leads to $\sigma_z^{\text{rms}} \sim 0.96\sigma_z'$. For CESR, $b \sim 1.4$ and the correction factor is $\sim$0.89.

The vertical size is estimated by taking the average of ${\sigma'_y}^2(z)$ (56) over the corrected z profile (57), which also can be written in a closed form:

$$\sigma_y^{\mathrm{rms}2} = \frac{\int_{-\infty}^{\infty} dz\, \sigma'_y(z)^2\, \ell(z)}{\int_{-\infty}^{\infty} dz\, \ell(z)} = \frac{2\sqrt{\pi}}{b^2} e^{-\frac{b^2}{4}} \frac{U(-\frac{1}{2}, 0, \frac{b^2}{2})}{K_0(\frac{b^2}{4})} {\sigma'_y}^2 . \tag{61}$$

For $b \sim 3$, we have $\sigma_y^{\mathrm{rms}} \sim 1.15\sigma'_y$. The distribution is not a gaussian; it has an enhanced tail due to the enlargement of the beam size at large $|z|$.

The interaction profiles in the previous sections are gaussian with all three axes aligned with the coordinate axes regardless of whether the distribution was at a given time or after integrated over time. The waist effect is to enlarge the vertical size according to (56) and weight the distribution as a function of z according to (57). At a given z, the x width does not change by enlarging the vertical size, nor does it depend on the weighting since the x width is independent of z. Thus, the expressions of σ'_x obtained in the previous sections are valid even with the waist effect (as long as $\beta^*_x \gg \sigma_z$).

Another quantity of interest is the luminosity loss factor due to the waist effect. It is given by

$$r_{\mathrm{lum}}(\beta^*_y) \equiv \frac{\int_{-\infty}^{\infty} dz\, \ell(z)}{\int_{-\infty}^{\infty} dz\, \ell(z)\Big|_{\beta^*_y \to \infty}} = \frac{b}{\sqrt{2\pi}} e^{-\frac{b^2}{4}} K_0\Big(\frac{b^2}{4}\Big) , \tag{62}$$

which is roughly the inverse of the vertical expansion factor r_y.

References

[1] 'KEKB B-factory Design Report,' KEK Report 95-7 (1995).

[2] M. Sands, 'The Physics of Electron Strange ring, An Introduction,' SLAC-121 (1970).