BELLE note #463

# Flavor-untagged Lifetime Differences of $D^0$ Decays

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The mixing and $CP$-violation parameter of the neutral $D$ system, $y_{cp}$, is usually obtained from the effective decay rates of $K^+K^-$ and $K^\pm\pi^\mp$ final states. We obtain expressions for these effective decay rates without assuming that the semileptonic asymmetry parameter is small. We then estimate the error in measuring the average lifetime by the effective lifetime of the $K\pi$ mode. We also study the biases caused by fitting single exponentials to these decay distributions which are not single exponentials in reality.

## 1 Introduction

One way to probe mixing and $CP$ violation in the neutral $D$ system is to measure the flavor-untagged lifetime differences of $D^0$ mesons [1]. In particular, the often-used modes are the $K^\pm\pi^\mp$ and $K^+K^-$ final states [2, 3].

In this memo, we will start from the exact expression for the time-dependent decay rates and systematically evaluate approximate formulas assuming that $x, y$ and $\Delta$ are small but non-zero values in general. These parameters are defined as

$$x \equiv \frac{\delta m}{\gamma_+}, \quad y \equiv \frac{\gamma_-}{\gamma_+}, \quad \Delta \equiv \frac{|p|^2 - |q|^2}{|p|^2 + |q|^2}, \tag{1}$$

where

$$\delta m \equiv m_1 - m_2, \quad \gamma_\pm \equiv \frac{\gamma_1 \pm \gamma_2}{2}, \tag{2}$$

with $m_{1,2}$ and $\gamma_{1,2}$ being the mass and decay rates of the physical states $D_{1,2}$ given by

$$D_1 \equiv pD^0 + q\overline{D}^0, \quad D_2 \equiv pD^0 - q\overline{D}^0 \,. \tag{3}$$

The coefficients are normalized as

$$|p|^2 + |q|^2 = 1 \,. \tag{4}$$

If there is no $CP$ violation, we have $|p| = |q| = 1/\sqrt{2}$. In (3), we have assumed $CPT$ invariance in mixing; otherwise, we would have needed four instead of two ($p$ and $q$) complex coefficients. At this point, we do not need to specify which of $D_{1,2}$ is heavier.

The asymmetry between the $D^0$ and $\bar{D}^0$ contents in the $D_1$ or $D_2$ state is the asymmetry between $|p|^2$ and $|q|^2$, and is nothing but the $\Delta$ parameter defined above. Note that this asymmetry is the same for $D_1$ and $D_2$. The equivalent parameter in the neutral $K$ system can be measured as the asymmetry between positive and negative leptons in the semileptonic decays of $K_L$.

A pure $D_{1,2}$ state is an eigenstate of the total Hamiltonian; thus, it will physically stay as $D_{1,2}$, but its amplitude and phase will change over time:

$$D_1 \to D_1 e_1(t)\,, \quad D_2 \to D_2\, e_2(t)\,, \tag{5}$$

where

$$e_1(t) \equiv e^{-\frac{\gamma_1}{2}t - im_1 t}\,, \quad e_2(t) \equiv e^{-\frac{\gamma_2}{2}t - im_2 t}\,. \tag{6}$$

# 2 Basic formulas

In this section, we obtain the basic expressions for the time-dependent decay distributions of a pure $D_0$ or $\bar{D}^0$ at $t = 0$. No approximation will be made except for the $CPT$ invariance and the Wigner-Weisskopf formalism. In particular, $xt$ and $yt$ could be large and $CP$ may be violated in decay as well as in mixing.

Solving (3) for $D^0$ and $\bar{D}^0$, applying the time evolutions of $D_{1,2}$ given by (5), then re-expressing $D_{1,2}$ in terms of $D^0$ and $\bar{D}^0$, we obtain the time evolutions of $D^0$ and $\bar{D}^0$:

$$D^0 \to \frac{1}{2}\Big[D(e_1 + e_2) + \frac{q}{p}\bar{D}(e_1 - e_2)\Big]\,, \tag{7}$$

$$\bar{D}^0 \to \frac{1}{2}\Big[\bar{D}(e_1 + e_2) + \frac{p}{q}D(e_1 - e_2)\Big]\,. \tag{8}$$

For a final state $f$ and $\bar{f}$, we write the instantaneous decay amplitudes as

$$a \equiv Amp(D^0 \to f), \quad a' \equiv Amp(\bar{D}^0 \to f), \tag{9}$$

$$b \equiv Amp(\bar{D}^0 \to \bar{f}), \quad b' \equiv Amp(D^0 \to \bar{f}). \tag{10}$$

For $f = K^-\pi^+$, $a$ and $b$ are Cabibbo-favored decays and $a'$ and $b'$ are doubly Cabibbo-suppressed decays.

Let $A_{D\to f}(t)$ be the time-dependent decay amplitudes for a pure $D^0$ state at $t = 0$ to decay to a final state $f$ at time $t$. It can be obtained by replacing $D$ and $\bar{D}$ in (7) by $a$ and $a'$, respectively. Similarly, $A_{\bar{D}\to\bar{f}}(t)$ can be obtained by replacing $D$ and $\bar{D}$ in (8) by $b'$ and $b$, respectively. These 'favored' amplitudes can then be written as

$$A_{D\to f}(t) = \frac{a}{2}\big[(e_1 + e_2) + \alpha(e_1 - e_2)\big]\,, \tag{11}$$

$$A_{\bar{D}\to\bar{f}}(t) = \frac{b}{2}\big[(e_1 + e_2) + \beta(e_1 - e_2)\big]\,, \tag{12}$$

where

$$\alpha \equiv \frac{qa'}{pa}\,, \quad \beta \equiv \frac{pb'}{qb}\,. \tag{13}$$

For $f = K^-\pi^+$, the approximate values are $|\alpha| \sim |\beta| \sim 0.06$ which is the factor of double Cabibbo suppression.

The 'suppressed' modes are similarly expressed as

$$A_{\bar{D}\to f}(t) = \frac{a}{2}\frac{p}{q}\big[(e_1 - e_2) + \alpha(e_1 + e_2)\big]\,, \tag{14}$$

$$A_{D\to \bar{f}}(t) = \frac{b}{2}\frac{q}{p}\big[(e_1 - e_2) + \beta(e_1 + e_2)\big]\,. \tag{15}$$

Since we have not actually specified what is $f$ and what is $\bar{f}$, (11) is equivalent to (15), and (12) to (14). The difference is which amplitude is factored out, and it has to do with mere convenience.

Squaring these amplitudes gives the corresponding decay distributions:

$$\begin{aligned}\Gamma_{D\to f}(t) &= \frac{|a|^2}{2}e^{-\gamma_+ t}\Big\{(1+|\alpha|^2)\cosh\gamma_- t - 2\Re\alpha\,\sinh\gamma_- t\\ &\qquad +(1-|\alpha|^2)\cos\delta m t + 2\Im\alpha\,\sin\delta m t\Big\}\,, \end{aligned}\tag{16}$$

$$\begin{aligned}\Gamma_{\bar{D}\to \bar{f}}(t) &= \frac{|b|^2}{2}e^{-\gamma_+ t}\Big\{(1+|\beta|^2)\cosh\gamma_- t - 2\Re\beta\,\sinh\gamma_- t\\ &\qquad +(1-|\beta|^2)\cos\delta m t + 2\Im\beta\,\sin\delta m t\Big\}\,, \end{aligned}\tag{17}$$

for the 'favored' modes and

$$\begin{aligned}\Gamma_{\bar{D}\to f}(t) &= \frac{|a|^2}{2}\frac{|p|^2}{|q|^2}e^{-\gamma_+ t}\Big\{(1+|\alpha|^2)\cosh\gamma_- t - 2\Re\alpha\,\sinh\gamma_- t\\ &\qquad -(1-|\alpha|^2)\cos\delta m t - 2\Im\alpha\,\sin\delta m t\Big\}\,, \end{aligned}\tag{18}$$

$$\begin{aligned}\Gamma_{D\to \bar{f}}(t) &= \frac{|b|^2}{2}\frac{|q|^2}{|p|^2}e^{-\gamma_+ t}\Big\{(1+|\beta|^2)\cosh\gamma_- t - 2\Re\beta\,\sinh\gamma_- t\\ &\qquad -(1-|\beta|^2)\cos\delta m t - 2\Im\beta\,\sin\delta m t\Big\}\,, \end{aligned}\tag{19}$$

for the 'suppressed' modes. Note that, for each category, the second expression can be obtained from the first by the replacements $a \to b$, $\alpha \to \beta$, and $p \leftrightarrow q$. This can also be seen at the amplitude level (11) through (15).

Assuming that $D^0$ and $\bar{D}^0$ are generated in the same numbers, the untagged decay distributions are

$$\begin{aligned}\Gamma_{D,\bar{D}\to f}(t) &= \Gamma_{D\to f}(t) + \Gamma_{\bar{D}\to f}(t)\\ &= \frac{|a|^2}{2}\left(1 + \frac{|p|^2}{|q|^2}\right)e^{-\gamma_+ t}\Big\{\Big[(1+|\alpha|^2)\cosh\gamma_- t - 2\Re\alpha\,\sinh\gamma_- t\Big]\end{aligned}$$

$$-\Delta\left[(1-|\alpha|^2)\cos\delta m t+2\Im\alpha\,\sin\delta m t\right]\Big\}\,, \tag{20}$$

$$\begin{aligned}\Gamma_{D,\bar{D}\to\bar{f}}(t) &= \Gamma_{D\to\bar{f}}(t)+\Gamma_{\bar{D}\to\bar{f}}(t)\\ &= \frac{|b|^2}{2}\left(1+\frac{|q|^2}{|p|^2}\right)e^{-\gamma_+ t}\Big\{\left[(1+|\beta|^2)\cosh\gamma_- t-2\Re\beta\,\sinh\gamma_- t\right]\\ &\qquad +\Delta\left[(1-|\beta|^2)\cos\delta m t+2\Im\beta\,\sin\delta m t\right]\Big\}\,,\end{aligned} \tag{21}$$

where we have used the definition of $\Delta$ given in (1). These expressions for decay distributions are exact in the sense that they do not use approximations that $x$, $y$, or $\Delta$ are small.

When the $CP$ asymmetry in mixing $\Delta$ is zero, the untagged decay distribution is a linear combination of two exponentials $e^{-\gamma_1 t}$ and $e^{-\gamma_2 t}$, and when $\Delta$ is nonzero there are oscillating terms. We see that the oscillating terms in the untagged distributions are suppressed by the parameter $\Delta$.

# 3 Applications to $f = K^+K^-$ and $K^-\pi^+$

## 3.1 Effective decay rates

In the case of the neutral $D$ system, the parameters $x$, $y$, are small (% level or less) and it is not practical to measure the two separate exponentials or the oscillating terms for the untagged decay distributions. In practice, a single exponential is fit to the measured decay time distribution to obtain an 'effective decay rate'.

Suppose we have a function

$$f(t) = e^{-\gamma_1 t}+e^{-\gamma_2 t}. \tag{22}$$

The relative slope at $t=0$ $f'(0)/f(0)$ is the same as that of another function given by

$$g(t) = 2e^{-\frac{1}{2}(\gamma_1+\gamma_2)}\,. \tag{23}$$

These two functions are actually quite close when $\gamma_1\sim\gamma_2$ and $\gamma_- t$ is small. In fact,

$$\frac{f(t)}{g(t)} = \cosh\gamma_- t\sim 1+\frac{(\gamma_- t)^2}{2}\cdots \tag{24}$$

For $\gamma_- t < 0.1$, or for example $y = 0.01$ and less than 10 lifetimes, the deviation above is less than 0.005.

A toy Monte Carlo study has shown that when a single gaussian is fit to the function $f(t)$ above, the slope is close the average of $\gamma_1$ and $\gamma_2$, or $\gamma_+$, where the fit value is always smaller by approximately $2y^2$ relative. Namely, if one is to measure $y$ by comparing $\gamma_1$ with the effective decay rate of $f(t)$, the measured value of $y$ is shifted by a fraction $2y$ of $y$ itself as long as the upper limit of the fit is 5 to 30 lifetimes. As we will see, this is comparable to the shifts we are concerned with in the following sections to the extent $|y| \sim |\alpha|, |\beta|$. As the experimental limit on $y$ becomes smaller, the bias caused by fitting a single gaussian would become smaller. In the case of an oscillating term, the deviation of the fitted slope of a single exponential and the relative slope at $t = 0$, $f'(0)/f(0)$, was found to be of the same order as above. The sign of the shift is such that it is not as large as evaluated from the relative slope at $t = 0$. This is due to the fact that the osillating terms are bounded (i.e. $|\sin\theta|, |\cos\theta| \le 1$).

### 3.2 $f = K^+K^-$

In this section and the next, we will take the unit of time as the mean lifetime $1/\gamma_+$. Then,

$$\gamma_+ t \to t\,, \quad \gamma_- t \to yt\,, \quad \delta m\, t \to xt\,. \tag{25}$$

Here we have $f = \bar{f}$, so we can use either of (20) or (21); the result should be the same. What we are interested in is the relative slope at $t = 0$. We will thus express the decay distributions in the form $(1 + ct)$ (apart from $e^{-\gamma_+ t}$) taking terms only to the first order in $t$ (to be precise, in $yt$ and $xt$).

Thus, we start from (20), apply (25) and use the approximations

$$\cosh yt \to 1\,, \quad \sinh yt \to yt\,, \quad \cos xt \to 1\,, \quad \sin xt \to xt \tag{26}$$

to obtain

$$\begin{aligned} \Gamma_{D,\bar{D}\to KK}(t) = {}& \frac{|a|^2}{2}\left(1 + \frac{|p|^2}{|q|^2}\right)(1 + |\alpha|^2)e^{-t} \\ & \times\Big[1 - \Delta\,\frac{1-|\alpha|^2}{1+|\alpha|^2} - \frac{2|\alpha|}{1+|\alpha|^2}(y\cos\phi + \Delta\, x\sin\phi)t\Big]\,, \end{aligned} \tag{27}$$

where

$$\alpha \equiv \frac{qAmp(\bar{D}^0 \to K^+K^-)}{pAmp(D^0 \to K^+K^-)}\,, \tag{28}$$

and

$$\phi \equiv \arg\alpha\,. \tag{29}$$

Experimentally, we have $|Amp(\bar{D}^0 \to K^+K^-)| \sim |Amp(D^0 \to K^+K^-)|$ but the limit on $\Delta$ is essentially non-existent [4]. Thus, we first assume that $|\alpha|$ could deviate significantly from 1 and $\Delta$ from 0. We then write the above decay distribution as

$$\Gamma_{D,\bar{D}\to KK}(t) \propto e^{-(1+y_{cp})} \tag{30}$$

with

$$y_{cp} = \frac{2|\alpha|}{1 + |\alpha|^2 - \Delta(1 - |\alpha|^2)}(y\cos\phi + \Delta\, x\sin\phi)\,. \tag{31}$$

This parameter $y_{cp}$ may be defined as

$$y_{cp} \equiv \frac{\Gamma_{eff}(K^+K^-)}{\gamma_+} - 1\,, \tag{32}$$

where $\Gamma_{eff}$ indicates the effective decay rate. If $]p| \sim |q|$, then we have $\Delta \sim 0$ and $|\alpha| \sim 1$, and keeping terms to the first order in $\Delta$ and $|\alpha| - 1$, $y_{cp}$ can be written as

$$y_{cp} = y\cos\phi + \Delta\, x \sin\phi\,. \tag{33}$$

## 3.3 $f = K^-\pi^+$

The average decay rate $\gamma_+$ is often measured as the effective decay rate of the $K^-\pi^+$ final state. The goal of this section is to find out the accuracy of this method.

We take $f = K^-\pi^+$ in (20) and (21) and apply the change of time unit (25) and the approximation $xt, yt \ll 1$ or equivalently (26). The result is

$$\begin{aligned}
\Gamma_{D,\bar{D}\to K^-\pi^+}(t) &= \frac{|a|^2}{2}\left(1 + \frac{|p|^2}{|q|^2}\right)\Big[(1+|\alpha|^2) - \Delta\,(1-|\alpha|^2)\Big]e^{-t} \\
&\qquad \times\Big[1 - \frac{2(\Re\alpha\, y + \Delta\,\Im\alpha\, x)}{(1+|\alpha|^2) - \Delta\,(1-|\alpha|^2)}\, t\Big]\,, \qquad (34) \\
\Gamma_{D,\bar{D}\to K^+\pi^-}(t) &= \frac{|b|^2}{2}\left(1 + \frac{|q|^2}{|p|^2}\right)\Big[(1+|\beta|^2) + \Delta\,(1-|\beta|^2)\Big]e^{-t} \\
&\qquad \times\Big[1 - \frac{2(\Re\beta\, y - \Delta\,\Im\beta\, x)}{(1+|\beta|^2) + \Delta\,(1-|\beta|^2)}\, t\Big]\,, \qquad (35)
\end{aligned}$$

where the second can be obtained from the first by the replacements $a \to b$, $\alpha \to \beta$, $p \leftrightarrow q$, and $\Delta \to -\Delta$ (since $p$ and $q$ is exchanged). The parameters $\alpha$ and $\beta$ are explicitly given by

$$\alpha \equiv \frac{qAmp(\bar{D}^0 \to K^-\pi^+)}{pAmp(D^0 \to K^-\pi^+)}\,, \quad \beta \equiv \frac{pAmp(D^0 \to K^+\pi^-)}{qAmp(\bar{D}^0 \to K^+\pi^-)}\,, \tag{36}$$

and as mentioned earlier, the order of magnitude is $|\alpha| \sim |\beta| \sim 0.06$.

Writing in exponential form,

$$\Gamma_{D,\bar{D}\to K^-\pi^+}(t) \propto e^{-(1+y_{K^-\pi^+})t}\,, \quad y_{K^-\pi^+} = \frac{2(\Re\alpha\, y + \Delta\,\Im\alpha\, x)}{(1+|\alpha|^2) - \Delta\,(1-|\alpha|^2)}\,, \tag{37}$$

$$\Gamma_{D,\bar{D}\to K^+\pi^-}(t) \propto e^{-(1+y_{K^+\pi^-})t}\,, \quad y_{K^+\pi^-} = \frac{2(\Re\beta\, y - \Delta\,\Im\beta\, x)}{(1+|\beta|^2) + \Delta\,(1-|\beta|^2)}\,. \tag{38}$$

The parameters $y_{K^-\pi^+}$ and $y_{K^+\pi^-}$ are the fractional shift of the effective decay rate from $\gamma_+$. If $\Delta \sim 0$, these parameters are roughly,

$$y_{K^+\pi^-} \sim 2\Re\alpha\, y\,, \quad y_{K^-\pi^+} \sim 2\Re\beta\, y\,. \tag{39}$$

When the effective decay rate of $K^-\pi^+$ is used instead of $\gamma_+$ in evaluating $y_{cp}$ using the definition (32), the measured $y_{cp}$ is actually $y_{cp} - y_{K^-\pi^+}$. Since $\Re\alpha$ and $\Re\beta$ are of order 0.06 (or less) and $y_{cp}$ is of order $y$ (or less), the typical shift in the measured $y_{cp}$ is of order 10% of itself. In practice, one may measure the effective decay rate of the sum of the $K^-\pi^+$ and $K^+\pi^-$ samples. There is, however, no automatic cancellation between $y_{K^+\pi^-}$ and $y_{K^-\pi^+}$, and thus the typical shift in $y_{cp}$ is still of the same order.

# 4 Summary

In the case of the $K^+K^-$ final state, the bias caused by fitting a single exponential to the real distribution that includes two decay rates and oscillating terms is of order a few times $y$ or $\Delta \cdot x$ whichever is larger. Namely, if $y_{cp}$ is of order 3% the error is of order 10% of itself. The effective decay rate of the $K^\pm\pi^\mp$ mode does not exactly measure the average decay rate $\gamma_+$ with the bias that is roughly 0.1 y which leads to about 10% mismeasurement of $y_{cp}$. We also found an expression for $y_{cp}$ where $\Delta$ is not assumed to be small.

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