

Formula Sheets

0. Metric

$$g^{\mu\nu} = 0 \ (\mu \neq \nu), \quad g^{00} = -g^{11} = -g^{22} = -g^{33} = 1$$

$$a^\mu = (a^0, \vec{a}), \quad a_\mu = (a^0, -\vec{a})$$

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \nabla \right)$$

$$\partial_\mu a^\mu(x) = \frac{\partial}{\partial t} a^0(x) + \nabla \cdot \vec{a}(x)$$

1. Fields and equations of motion

$$\text{(spin-0)} \quad \phi(x) = \sum_{\vec{p}} [a_{\vec{p}} e_{\vec{p}}(x) + a_{\vec{p}}^\dagger e_{\vec{p}}^*(x)], \quad (\partial^2 + m^2)\phi = 0$$

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2) \quad \text{(neutral)}$$

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi \quad \text{(charged; with } a_{\vec{p}}^\dagger \rightarrow b_{\vec{p}}^\dagger \text{ above)}$$

$$\text{(spin-1/2)} \quad \psi(x) = \sum_{\vec{p}, s} [a_{\vec{p}, s} f_{\vec{p}, s}(x) + b_{\vec{p}, s}^\dagger g_{\vec{p}, s}(x)], \quad (i\partial\!\!\!/ - m)\psi = 0$$

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi$$

$$\text{(spin-1)} \quad A^\mu(x) = \sum_{\vec{p}\lambda} [a_{\vec{p}\lambda} h_{\vec{p}\lambda}^\mu(x) + a_{\vec{p}\lambda}^\dagger h_{\vec{p}\lambda}^{\mu*}(x)], \quad \partial_\nu F^{\mu\nu} + m^2 A^\mu = 0$$

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m^2}{2} A_\mu A^\mu \quad \text{(neutral)}$$

$$\mathcal{L} = -\frac{1}{2} F_{\mu\nu}^\dagger F^{\mu\nu} + m^2 A_\mu^\dagger A^\mu \quad \text{(charged; with } a_{\vec{p}\lambda}^\dagger \rightarrow b_{\vec{p}\lambda}^\dagger \text{ above)}$$

$$F^{\mu\nu} \equiv \partial^\nu A^\mu - \partial^\mu A^\nu$$

$$\left(e_{\vec{p}}(x) = \frac{e^{-ipx}}{\sqrt{2p^0V}}, \quad f_{\vec{p}, s}(x) = u_{\vec{p}, s} e_{\vec{p}}(x), \quad g_{\vec{p}, s}(x) = v_{\vec{p}, s} e_{\vec{p}}^*(x), \quad h_{\vec{p}\lambda}^\mu(x) = \epsilon_{\vec{p}\lambda}^\mu e_{\vec{p}}(x) \right)$$

2. Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_i \sigma_j = i\sigma_k \quad (i, j, k \text{ cyclic}), \quad \sigma_i^2 = 1$$

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k, \quad \{\sigma_i, \sigma_j\} = 2\delta_{ij}$$

$$(\vec{a} \cdot \vec{\sigma})(\vec{b} \cdot \vec{\sigma}) = \vec{a} \cdot \vec{b} + i\vec{\sigma} \cdot (\vec{a} \times \vec{b})$$

$$e^{i\vec{a} \cdot \vec{\sigma}} = \cos a + i\hat{a} \cdot \vec{\sigma} \sin a, \quad (a = |\vec{a}|, \hat{a} = \vec{a}/a)$$

3. Gamma matrices

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \quad \gamma_5 = \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3, \quad \{\gamma^5, \gamma^\mu\} = 0$$

$$\sigma_{\mu\nu} = \frac{i}{2} [\gamma_\mu, \gamma_\nu]$$

$$\gamma_0^2 = 1, \quad \gamma_i^2 = -1 \quad (i = 1, 2, 3), \quad \gamma_5^2 = 1$$

$$\bar{\gamma}_\mu = \gamma_\mu, \quad \bar{\gamma}_5 = -\gamma_5, \quad \gamma_\mu^\dagger = \gamma^\mu, \quad \gamma_5^\dagger = \gamma_5$$

$$\overline{\sigma^{\mu\nu}} = \sigma^{\mu\nu}, \quad \overline{\gamma_5 \gamma^\mu} = \gamma_5 \gamma^\mu$$

[Dirac representation]

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

[Weyl representation]

$$\gamma^0 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

4. Dirac spinors

$$p^\mu = (p^0, \vec{p}), \quad |\vec{s}| = 1, \quad s^\mu = (0, \vec{s}) \quad (\text{in rest frame})$$

$$p \cdot s = 0 \quad (\text{in any frame})$$

$$(\not{p} - m)u_{\vec{p},s} = 0, \quad (\not{p} + m)v_{\vec{p},s} = 0$$

$$\bar{u}_{\vec{p},s}(\not{p} - m) = 0, \quad \bar{v}_{\vec{p},s}(\not{p} + m) = 0$$

$$\begin{aligned} \bar{u}_{\vec{p},s_1} u_{\vec{p},s_2} &= 2m\delta_{s_1 s_2} & \bar{v}_{\vec{p},s_1} v_{\vec{p},s_2} &= -2m\delta_{s_1 s_2} \\ \bar{u}_{\vec{p},s_1} v_{\vec{p},s_2} &= 0 & (s_{1,2} = \pm 1) \end{aligned}$$

$$\begin{aligned} u_{\vec{p},s_1}^\dagger u_{\vec{p},s_2} &= 2E\delta_{s_1 s_2} & v_{-\vec{p},s_1}^\dagger v_{-\vec{p},s_2} &= 2E\delta_{s_1 s_2} \\ u_{\vec{p},s_1}^\dagger v_{-\vec{p},s_2} &= 0 & (s_{1,2} = \pm 1) \end{aligned}$$

$$u_{\vec{p},s} \bar{u}_{\vec{p},s} = (\not{p} + m) \frac{1 + \gamma_5 \not{s}}{2}, \quad v_{\vec{p},s} \bar{v}_{\vec{p},s} = (\not{p} - m) \frac{1 + \gamma_5 \not{s}}{2}$$

$$\sum_s u_{\vec{p},s} \bar{u}_{\vec{p},s} = (\not{p} + m), \quad \sum_s v_{\vec{p},s} \bar{v}_{\vec{p},s} = (\not{p} - m)$$

[Explicit expressions in the Dirac representation]

$$u_{\vec{p},\pm} = \sqrt{E+m} \begin{pmatrix} 1 \\ \frac{\vec{p} \cdot \vec{\sigma}}{E+m} \end{pmatrix} \chi_\pm, \quad v_{\vec{p},\pm} = \sqrt{E+m} \begin{pmatrix} \frac{\vec{p} \cdot \vec{\sigma}}{E+m} \\ 1 \end{pmatrix} \chi_\mp$$

$$(\vec{s} \cdot \vec{\sigma}) \chi_\pm = \pm \chi_\pm, \quad \chi_{s_1}^\dagger \chi_{s_2} = \delta_{s_1 s_2} \quad (s_{1,2} = \pm)$$

$$\chi_+ = \frac{1}{\sqrt{2(1+s_z)}} \begin{pmatrix} 1+s_z \\ s_+ \end{pmatrix}, \quad \chi_- = \frac{1}{\sqrt{2(1-s_z)}} \begin{pmatrix} s_z-1 \\ s_+ \end{pmatrix} \quad (s_\pm = s_x \pm i s_y)$$

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (\vec{s} = \hat{z})$$

5. Spin-1 polarization vectors

$$\epsilon_{0\lambda}^\mu = (0, \hat{e}_{\bar{0}\lambda}) \quad \hat{e}_{\lambda=1,2,3} = \hat{e}_x, \hat{e}_y, \hat{e}_z \quad (\text{in rest frame}), \quad \epsilon_{\bar{p}\lambda} \cdot p = 0 \quad (\text{in any frame})$$

$$\epsilon_{\bar{p}1}^\mu = (0, \hat{e}_{\bar{p}1}), \quad \epsilon_{\bar{p}2}^\mu = (0, \hat{e}_{\bar{p}2}), \quad \epsilon_{\bar{p}3}^\mu = \left(\frac{|\vec{p}|}{m}, \frac{p^0}{m} \hat{p} \right) \quad (\text{in any frame})$$

$$\epsilon_{\bar{p}\pm}^\mu \equiv \mp \frac{1}{\sqrt{2}} (\epsilon_{\bar{p}1}^\mu \pm i\epsilon_{\bar{p}2}^\mu), \quad \epsilon_{\bar{p}0}^\mu \equiv \epsilon_{\bar{p}3}^\mu \quad (\text{helicity basis})$$

Polarization sums

$$(m \neq 0) \quad \sum_{\lambda} \epsilon_{\bar{p}\lambda}^\mu \epsilon_{\bar{p}\lambda}^{\nu*} = -g^{\mu\nu} + \frac{p^\mu p^\nu}{m^2} \quad (\lambda = 1, 2, 3 \text{ or } \pm, 0)$$

$$(m = 0) \quad \begin{cases} \sum_{\lambda} \epsilon_{\bar{p}\lambda}^i \epsilon_{\bar{p}\lambda}^{j*} = \delta_{ij} - \hat{p}_i \hat{p}_j \\ \sum_{\lambda} \epsilon_{\bar{p}\lambda}^\mu \epsilon_{\bar{p}\lambda}^{\nu*} = -g^{\mu\nu} + \frac{\bar{k}^\mu k^\nu + k^\mu \bar{k}^\nu}{\bar{k} \cdot k} \\ \sum_{\lambda} \epsilon_{\bar{p}\lambda}^\mu \epsilon_{\bar{p}\lambda}^{\nu*} = -g^{\mu\nu} \quad (\text{QED}) \end{cases} \quad \left(\begin{array}{l} \lambda = 1, 2 \text{ or } \pm. \text{ Coulomb gauge: } \epsilon_{\bar{p}\lambda}^0 = 0 \\ k = (k^0, \vec{k}), \bar{k} = (k^0, -\vec{k}) \end{array} \right)$$

6. Trace theorems and related formul

$$\not{a} \not{b} + \not{b} \not{a} = 2a \cdot b \quad \not{a} \not{a} = a^2$$

$$\text{Tr} \gamma_{i_1} \dots \gamma_{i_{2n+1}} = 0 \quad (i's = 0, 1, 2, 3 : \text{ odd number of } \gamma's. \gamma_5 \text{ count as } 0.)$$

$$\text{Tr} \gamma_{i_1} \dots \gamma_{i_{2n}} = \text{Tr} \gamma_{i_{2n}} \dots \gamma_{i_1} \quad (\text{reverse order})$$

$$\text{Tr} 1 = 4$$

$$\text{Tr} \not{a} \not{b} = 4a \cdot b \quad \text{Tr} \gamma_\mu \gamma_\nu = 4g_{\mu\nu}$$

$$\text{Tr} \not{a} \not{b} \not{c} \not{d} = 4(a \cdot b c \cdot d - a \cdot c b \cdot d + a \cdot d b \cdot c) \quad \text{Tr} \gamma_\mu \gamma_\nu \gamma_\alpha \gamma_\beta = 4(g_{\mu\nu} g_{\alpha\beta} - g_{\mu\alpha} g_{\nu\beta} + g_{\mu\beta} g_{\nu\alpha})$$

$$\begin{aligned} \text{Tr} \not{a}_1 \not{a}_2 \dots \not{a}_{2n} &= \sum_{i=2}^{2n} (-1)^i a_1 \cdot a_i \text{Tr} \not{a}_2 \dots \not{a}_{i-1} \not{a}_{i+1} \dots \not{a}_{2n} \\ &= a_1 \cdot a_2 \text{Tr} \not{a}_3 \not{a}_4 \dots \not{a}_{2n} - a_1 \cdot a_3 \text{Tr} \not{a}_2 \not{a}_4 \dots \not{a}_{2n} \\ &\quad + \dots + a_1 \cdot a_{2n} \text{Tr} \not{a}_2 \not{a}_3 \dots \not{a}_{2n-1} \end{aligned}$$

$$\text{Tr} \gamma_5 = 0$$

$$\text{Tr} \gamma_5 \not{a} \not{b} = 0 \quad \text{Tr} \gamma_5 \gamma_\mu \gamma_\nu = 0$$

$$\text{Tr} \gamma_5 \not{a} \not{b} \not{c} \not{d} = 4i \epsilon^{\mu\nu\alpha\beta} a_\mu b_\nu c_\alpha d_\beta \quad \text{Tr} \gamma_5 \gamma_\mu \gamma_\nu \gamma_\alpha \gamma_\beta = 4i \epsilon_{\mu\nu\alpha\beta}$$

$$(\epsilon_{0123} \equiv +1)$$

$$\gamma_\mu \gamma^\mu = 4$$

$$\gamma_\mu \not{a} \gamma^\mu = -2\not{a} \quad \gamma_\mu \gamma_\alpha \gamma^\mu = -2\gamma_\alpha$$

$$\gamma_\mu \not{a} \not{b} \gamma^\mu = 4a \cdot b \quad \gamma_\mu \gamma_\alpha \gamma_\beta \gamma^\mu = 4g_{\alpha\beta}$$

$$\gamma_\mu \not{a} \not{b} \not{c} \gamma^\mu = -2\not{c} \not{b} \not{a} \quad \gamma_\mu \gamma_\alpha \gamma_\beta \gamma_\sigma \gamma^\mu = -2\gamma_\sigma \gamma_\beta \gamma_\alpha$$

$$g_{\mu\nu} g^{\mu\nu} = 4 \quad \epsilon^{\mu\nu\alpha\beta} \epsilon_{\mu\nu\rho\sigma} = 2(g_\sigma^\alpha g_\rho^\beta - g_\rho^\alpha g_\sigma^\beta)$$

$$\epsilon^{\mu\nu\alpha\beta} \epsilon_{\mu\nu\alpha\sigma} = -6g^\beta_\sigma \quad \epsilon^{\mu\nu\alpha\beta} \epsilon_{\mu\nu\alpha\beta} = -24$$

7. Cross sections and decay rates

$$S_{fi} = \frac{(2\pi)^4 \delta^4(\sum_i p_i - \sum_f p_f)}{\sqrt{\prod_i (2p_i^0 V) \prod_j (2p_j^0 V)}} \mathcal{M} \quad (\mathcal{M}: \text{Lorentz-invariant matrix element})$$

$$d\Phi_n = \delta^4(\sum_i p_i - \sum_f p_f) \prod_f \frac{d^3 p_f}{(2\pi)^3 2p_f^0} \quad (\text{n-body Lorentz-invariant phase space})$$

(a) Cross sections (general: $p_1 + p_2 = \sum_f q_f$)

$$d\sigma = \frac{(2\pi)^4}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} |\overline{\mathcal{M}}|^2 d\Phi_n$$

$$\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2} = \begin{cases} m_1 |\vec{p}_2| & (1 \text{ at rest}) \\ M |\vec{p}| & (\text{c.m. } M: \text{total c.m. energy}) \end{cases}$$

2-body final state ($p_1 + p_2 = p_3 + p_4$: all masses could be different)

$$\frac{d\sigma}{d\Omega} = \frac{|\overline{\mathcal{M}}|^2}{(8\pi M)^2} \frac{|\vec{p}_f|}{|\vec{p}_i|} \quad (\text{c.m. system})$$

$$\frac{d\sigma}{dt} = \frac{|\overline{\mathcal{M}}|^2}{16\pi \lambda(s, m_1^2, m_2^2)} \quad (\text{no azimuth dependence. any frame})$$

$$\text{where } \begin{cases} s = (p_1 + p_2)^2 \\ t = (p_3 - p_1)^2 \end{cases} \quad \lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx$$

$$\frac{d\sigma}{d\Omega_3} = \frac{|\vec{p}_3| |\overline{\mathcal{M}}|^2}{64\pi^2 m_1 |\vec{p}_2| [m_1 + E_2(1 - \frac{\beta_2}{\beta_3} \cos \theta_3)]} \quad (\text{Lab frame; 1 at rest})$$

(b) Decay rates (c.m. system)

$$d\Gamma = \frac{(2\pi)^4}{2M} |\overline{\mathcal{M}}|^2 d\Phi_n \quad (M: \text{parent mass})$$

$$\frac{d\Gamma}{d\Omega} = \frac{|\vec{p}|}{32\pi^2 M^2} |\overline{\mathcal{M}}|^2 \quad (2\text{-body})$$

$$\Gamma = \frac{|\vec{p}|}{8\pi M^2} |\overline{\mathcal{M}}|^2 \quad (2\text{-body, parent not polarized})$$

$$d\Gamma = \frac{|\overline{\mathcal{M}}|^2}{(2\pi)^3 8M} dE_1 dE_2 = \frac{|\overline{\mathcal{M}}|^2}{(2\pi)^3 32M^3} ds_{23} ds_{31} \quad (3\text{-body, parent not polarized})$$

8. Feynman rules (tree level)

External legs and propagators: (f : fermion, \bar{f} : anti-fermion)

	initial state	final state	propagator
(spin-0)	1	1	$\frac{i}{p^2 - m^2 + i\epsilon}$
(spin-1/2)	$u_{\vec{p},s}(f) \bar{v}_{\vec{p},s}(\bar{f})$	$\bar{u}_{\vec{p},s}(f) v_{\vec{p},s}(\bar{f})$	$\frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon} \equiv \frac{i}{\not{p} - m + i\epsilon}$
(spin-1)	$\epsilon_{\vec{p}\lambda}^\mu$	$\epsilon_{\vec{p}\lambda}^{\mu*}$	$\frac{i(-g^{\mu\nu} + \frac{p^\mu p^\nu}{m^2})}{p^2 - m^2 + i\epsilon}$ ($m \neq 0$) $\frac{-ig^{\mu\nu}}{p^2 + i\epsilon}$ (photon)

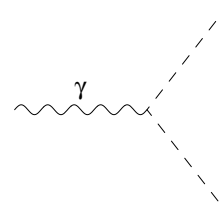
Vertices:

$$(e = 0.303 \ (e^2 = 4\pi\alpha), \ g = 0.652)$$

scalar-photon

$$-ie(p_1 - p_2)$$

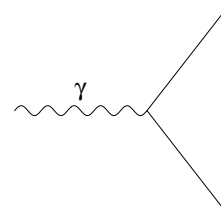
($p_{1,2}$: 4-momenta flowing into the vertex)



fermion-photon

$$-iq\gamma_\mu$$

(q : charge of the fermion. $q = e$ if electron)



fermion-W

$$\frac{ig}{2\sqrt{2}}\gamma_\mu(1 - \gamma_5) V_{ij}$$

(V_{ij} : CKM matrix element; = 1 for leptons.)

