Problem 4.3

The Lagrangian density for the Dirac field can be written as

$$\begin{aligned}\mathcal{L} &= \bar{\psi}(i\partial\!\!\!/ - m)\psi \\ &= \psi^\dagger\gamma^0\gamma^\mu\partial_\mu\psi - m\psi^\dagger\gamma^0\psi \\ &= \psi_b^* i(\gamma^0\gamma^\mu)_{ba}\partial_\mu\psi_a - m\psi_b^*(\gamma^0)_{ba}\psi_a\,.\end{aligned}$$

Applying the Euler-Lagrange equation

$$\frac{\partial\mathcal{L}}{\partial\phi_k} = \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_k)}$$

with $\phi_k = \psi_a$ (regarding ψ_a and ψ_a^* as independent),

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\psi_a} &= -m\psi_b^*(\gamma^0)_{ba} \\ \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_a)} &= \partial_\mu\psi_b^* i(\gamma^0\gamma^\mu)_{ba}\,;\end{aligned}$$

namely,

$$-m\psi_b^*(\gamma^0)_{ba} = \partial_\mu\psi_b^* i(\gamma^0\gamma^\mu)_{ba}$$

or,

$$-m\bar{\psi} = i\partial_\mu\bar{\psi}\gamma^\mu \qquad \to \qquad \bar{\psi}(i\overleftarrow{\partial\!\!\!/} + m) = 0\,,$$

which is the Dirac equation for $\bar{\psi}$.

Now, with $\phi_k = \psi_b^*$,

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\psi_b^*} &= i(\gamma^0\gamma^\mu)_{ba}\partial_\mu\psi_a - m(\gamma^0)_{ba}\psi_a \\ \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_b^*)} &= 0\,.\end{aligned}$$

namely,

$$i\gamma^0\gamma^\mu\partial_\mu\psi - m\gamma^0\psi = 0\,.$$

Multiplying γ^0 from the left, we obtain

$$(i\partial\!\!\!/ - m)\psi = 0\,.$$