Exercise 4.6

(a) The difinitions are

$$a_{\vec{p}} \equiv \frac{1}{\sqrt{2}}(a_{1\vec{p}} + i a_{2\vec{p}})\,, \quad b_{\vec{p}} \equiv \frac{1}{\sqrt{2}}(a_{1\vec{p}} - i a_{2\vec{p}})\,.$$

$$a^\dagger_{\vec{p}} = \frac{1}{\sqrt{2}}(a^\dagger_{1\vec{p}} - i a^\dagger_{2\vec{p}})\,, \quad b^\dagger_{\vec{p}} = \frac{1}{\sqrt{2}}(a^\dagger_{1\vec{p}} + i a^\dagger_{2\vec{p}})\,.$$

First, we note

$$[a_{\vec{p}}, a_{\vec{p}'}] = [b_{\vec{p}}, b_{\vec{p}'}] = [a_{\vec{p}}, b_{\vec{p}'}] = 0\,,$$

since only annihilation operators are involved and they all commute. Similarly,

$$[a^\dagger_{\vec{p}}, a^\dagger_{\vec{p}'}] = [b^\dagger_{\vec{p}}, b^\dagger_{\vec{p}'}] = [a^\dagger_{\vec{p}}, b^\dagger_{\vec{p}'}] = 0\,.$$

Using $[a_{k\vec{p}}, a^\dagger_{k'\vec{p}'}] = \delta_{kk'}\delta_{\vec{p},\vec{p}'}$, we have

$$[a_{\vec{p}}, a^\dagger_{\vec{p}'}] = \frac{1}{2}[a_{1\vec{p}} + i a_{2\vec{p}}, a^\dagger_{1\vec{p}'} - i a^\dagger_{2\vec{p}'}] = \frac{1}{2}(\underbrace{[a_{1\vec{p}}, a^\dagger_{1\vec{p}'}]}_{\delta_{\vec{p},\vec{p}'}} + \underbrace{[a_{2\vec{p}}, a^\dagger_{2\vec{p}'}]}_{\delta_{\vec{p},\vec{p}'}} = \delta_{\vec{p},\vec{p}'}\,,$$

$$[b_{\vec{p}}, b^\dagger_{\vec{p}'}] = \frac{1}{2}[a_{1\vec{p}} - i a_{2\vec{p}}, a^\dagger_{1\vec{p}'} + i a^\dagger_{2\vec{p}'}] = \frac{1}{2}(\underbrace{[a_{1\vec{p}}, a^\dagger_{1\vec{p}'}]}_{\delta_{\vec{p},\vec{p}'}} + \underbrace{[a_{2\vec{p}}, a^\dagger_{2\vec{p}'}]}_{\delta_{\vec{p},\vec{p}'}} = \delta_{\vec{p},\vec{p}'}\,.$$

Also,

$$[a_{\vec{p}}, b^\dagger_{\vec{p}'}] = \frac{1}{2}[a_{1\vec{p}} + i a_{2\vec{p}}, a^\dagger_{1\vec{p}'} + i a^\dagger_{2\vec{p}'}] = \frac{1}{2}(\underbrace{[a_{1\vec{p}}, a^\dagger_{1\vec{p}'}]}_{\delta_{\vec{p},\vec{p}'}} - \underbrace{[a_{2\vec{p}}, a^\dagger_{2\vec{p}'}]}_{\delta_{\vec{p},\vec{p}'}} = 0\,.$$

Taking the hermitian conjugate of this,

$$[b_{\vec{p}'}, a^\dagger_{\vec{p}}] = 0\,.$$

(b) The non-hermitian fields are written as

$$\phi \equiv \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)\,, \quad \pi \equiv \frac{1}{\sqrt{2}}(\pi_1 - i\pi_2)\,.$$

$$\phi^\dagger = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2)\,, \quad \pi^\dagger = \frac{1}{\sqrt{2}}(\pi_1 + i\pi_2)\,.$$

In the following, primed fields are understood to be functions of $(t, \vec{x}')$ and unprimed fields are functions of $(t, \vec{x})$. Since ϕ_k's all commute,

$$[\phi, \phi'] = [\phi^\dagger, \phi'^\dagger] = [\phi, \phi'^\dagger] = 0\,.$$

Similarly, since π_k's all commute,

$$[\pi, \pi'] = [\pi^\dagger, \pi'^\dagger] = [\pi, \pi'^\dagger] = 0\,.$$

Now,

$$[\phi, \pi'] = \frac{1}{2}[\phi_1 + i\phi_2, \pi'_1 - i\pi'_2] = \frac{1}{2}(\underbrace{[\phi_1, \pi'_1]}_{i\delta^3(\vec{x}-\vec{x}')} + \underbrace{[\phi_2, \pi'_2]}_{i\delta^3(\vec{x}-\vec{x}')}) = i\delta^3(\vec{x}-\vec{x}')\,.$$

Taking the hermitian conjugate of this, one obtains

$$[\pi'^\dagger, \phi^\dagger] = -i\delta^3(\vec{x}-\vec{x}') \quad \rightarrow \quad [\phi^\dagger, \pi'^\dagger] = i\delta^3(\vec{x}-\vec{x}')\,.$$

Somewhat non-trivial is

$$[\phi^\dagger, \pi'] = \frac{1}{2}[\phi_1^\dagger - i\phi_2^\dagger, \pi'_1 - i\pi'_2] = \frac{1}{2}(\underbrace{[\phi_1^\dagger, \pi'_1]}_{i\delta^3(\vec{x}-\vec{x}')} - \underbrace{[\phi_2^\dagger, \pi'_2]}_{i\delta^3(\vec{x}-\vec{x}')}) = 0\,.$$

Taking the hermitial conjugate of this,

$$[\phi, \pi'^\dagger] = 0\,.$$

Exercise 4.7

(a) The Lagrangian density is

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi = \dot{\phi}^\dagger \dot{\phi} - \vec{\nabla}\phi^\dagger \cdot \vec{\nabla}\phi - m^2 \phi^\dagger \phi\,.$$

The fields conjuagate to ϕ and $\phi^\dagger$ are

$$\pi \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}^\dagger\,, \quad \pi^\dagger \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}^\dagger} = \dot{\phi}\,.$$

Then, the Hamiltonian density is (at this point, we do not care ordering)

$$\begin{aligned} \mathcal{H} &= \pi\dot{\phi} + \pi^\dagger \dot{\phi}^\dagger - \mathcal{L} = 2\dot{\phi}^\dagger\dot{\phi} - \mathcal{L} \\ &= \dot{\phi}^\dagger\dot{\phi} + \vec{\nabla}\phi^\dagger \cdot \vec{\nabla}\phi + m^2 \phi^\dagger \phi\,. \end{aligned}$$

On the other hand,

$$\begin{aligned} \dot{\phi}^\dagger\dot{\phi} &= \frac{1}{2}(\dot{\phi}_1 - i\dot{\phi}_2)(\dot{\phi}_1 + i\dot{\phi}_2) = \frac{1}{2}(\dot{\phi}_1^2 + \dot{\phi}_2^2) \\ \vec{\nabla}\phi^\dagger \cdot \vec{\nabla}\phi &= \frac{1}{2}(\vec{\nabla}\phi_1 - i\vec{\nabla}\phi_2)\cdot(\vec{\nabla}\phi_1 + i\vec{\nabla}\phi_2) = \frac{1}{2}(\vec{\nabla}\phi_1\cdot\vec{\nabla}\phi_1 + \vec{\nabla}\phi_2\cdot\vec{\nabla}\phi_2) \\ \phi^\dagger\phi &= \frac{1}{2}(\phi_1 - i\phi_2)(\phi_1 + i\phi_2) = \frac{1}{2}(\phi_1^2 + \phi_2^2). \end{aligned}$$

Then, the Hamiltonian density becomes

$$\begin{aligned} \mathcal{H} &= \frac{1}{2}(\dot{\phi}_1^2 + \vec{\nabla}\phi_1\cdot\vec{\nabla}\phi_1 + m^2\phi_1^2) \\ &\quad + \frac{1}{2}(\dot{\phi}_2^2 + \vec{\nabla}\phi_2\cdot\vec{\nabla}\phi_2 + m^2\phi_2^2). \end{aligned}$$

When $\phi_{1,2}$ are regarded as independent, it was shown in the text that the total Hamiltonian density becomes the sum of those of each fields, which is nothing but the above.

(b) First, we modify the expression for $\mathcal{H}$:

$$\begin{aligned} \mathcal{H} &= \dot{\phi}^\dagger\dot{\phi} + \underbrace{\vec{\nabla}\phi^\dagger\cdot\vec{\nabla}\phi}_{\underbrace{\vec{\nabla}\cdot(\phi^\dagger\vec{\nabla}\phi)}_{\to 0} - \phi^\dagger \underbrace{\nabla^2\phi}_{\ddot{\phi} + \cancel{m^2}\phi\,:\ \text{by the K-G eq.}}} + \cancel{m^2}\phi^\dagger\phi = \dot{\phi}^\dagger\dot{\phi} - \phi^\dagger\ddot{\phi} \\ &= \phi^\dagger\, i\overleftrightarrow{\partial_0}(i\partial_0\phi)\,. \end{aligned}$$

Using the momentum expansion of the field (recovering the implicit normal ordering),

$$
\begin{aligned}
H &= \int d^3x \phi^\dagger \, i \overleftrightarrow{\partial}_0 (i\partial_0 \phi) \\
&= \int d^3x \Big[\sum_{\vec{p}} (a^\dagger_{\vec{p}} e^*_{\vec{p}} + b_{\vec{p}} e_{\vec{p}}) \Big] i \overleftrightarrow{\partial}_0 \Big[\sum_{\vec{p}'} p^{0\prime} (a_{\vec{p}'} e_{\vec{p}'} - b^\dagger_{\vec{p}'} e^*_{\vec{p}'}) \Big] \\
&= \int d^3x \sum_{\vec{p},\vec{p}'} p^{0\prime} \Big(a^\dagger_{\vec{p}} a_{\vec{p}'} \underbrace{e^*_{\vec{p}} i \overleftrightarrow{\partial}_0 e_{\vec{p}'}}_{\to \delta_{\vec{p},\vec{p}'}} - a^\dagger_{\vec{p}} b^\dagger_{\vec{p}'} \underbrace{e^*_{\vec{p}} i \overleftrightarrow{\partial}_0 e^*_{\vec{p}'}}_{\to 0} \\
&\qquad + b_{\vec{p}} a_{\vec{p}'} \underbrace{e_{\vec{p}} i \overleftrightarrow{\partial}_0 e_{\vec{p}'}}_{\to 0} - b_{\vec{p}} b^\dagger_{\vec{p}'} \underbrace{e_{\vec{p}} i \overleftrightarrow{\partial}_0 e^*_{\vec{p}'}}_{\to -\delta_{\vec{p},\vec{p}'}} \Big) \\
&= : \sum_{\vec{p}} p^0 (a^\dagger_{\vec{p}} a_{\vec{p}} + b_{\vec{p}} b^\dagger_{\vec{p}}) := \sum_{\vec{p}} p^0 (a^\dagger_{\vec{p}} a_{\vec{p}} + b^\dagger_{\vec{p}} b_{\vec{p}})
\end{aligned}
$$

(c) Similarly using the momentum expansion in Q,

$$
\begin{aligned}
Q &= \int d^3x \phi^\dagger \, i \overleftrightarrow{\partial}_0 \phi \\
&= \int d^3x \Big[\sum_{\vec{p}} (a^\dagger_{\vec{p}} e^*_{\vec{p}} + b_{\vec{p}} e_{\vec{p}}) \Big] i \overleftrightarrow{\partial}_0 \Big[\sum_{\vec{p}'} (a_{\vec{p}'} e_{\vec{p}'} + b^\dagger_{\vec{p}'} e^*_{\vec{p}'}) \Big] \\
&= \int d^3x \sum_{\vec{p},\vec{p}'} \Big(a^\dagger_{\vec{p}} a_{\vec{p}'} \underbrace{e^*_{\vec{p}} i \overleftrightarrow{\partial}_0 e_{\vec{p}'}}_{\to \delta_{\vec{p},\vec{p}'}} + b_{\vec{p}} b^\dagger_{\vec{p}'} \underbrace{e_{\vec{p}} i \overleftrightarrow{\partial}_0 e^*_{\vec{p}'}}_{\to -\delta_{\vec{p},\vec{p}'}} \Big) \\
&= : \sum_{\vec{p}} (a^\dagger_{\vec{p}} a_{\vec{p}} - b_{\vec{p}} b^\dagger_{\vec{p}}) : \\
&= \sum_{\vec{p}} (a^\dagger_{\vec{p}} a_{\vec{p}} - b^\dagger_{\vec{p}} b_{\vec{p}}) \,.
\end{aligned}
$$