Problem 1.1

(a) In the original frame, the energy momentum are

$$\begin{pmatrix} E \\ P_x = P\cos\theta \\ P_y = P\sin\theta \\ P_z = 0 \end{pmatrix} .$$

Boosting in x direction by velocity β tranforms this to

$$\begin{pmatrix} E' \\ P'_x \end{pmatrix} = \begin{pmatrix} \gamma & \eta \\ \eta & \gamma \end{pmatrix} \begin{pmatrix} E \\ P_x \end{pmatrix}$$
$$\begin{pmatrix} P'_y \\ P'_z \end{pmatrix} = \begin{pmatrix} P_y \\ P_z \end{pmatrix} ,$$

or

$$\begin{pmatrix} E' \\ P'_x \\ P'_y \\ P'_z \end{pmatrix} = \begin{pmatrix} \gamma E + \eta P\cos\theta \\ \eta E + \gamma P\cos\theta \\ p\sin\theta \\ 0 \end{pmatrix} .$$

(b) Using the above and $\gamma^2 - \eta^2 = 1$,

$$\begin{aligned} E'^2 - \vec{P}'^2 &= (\gamma E + \eta P\cos\theta)^2 - (\eta E + \gamma P\cos\theta)^2 - P^2\sin^2\theta \\ &= (\gamma^2 - \eta^2)E^2 + (\eta^2 - \gamma^2)P^2\cos^2\theta - P^2\sin^2\theta \\ &= E^2 - P^2\cos^2\theta - P^2\sin^2\theta \\ &= E^2 - P^2 . \end{aligned}$$

(c) Using (a) and $\beta_0 = P/E$ in $\tan\theta' = P'_y/P'_x$,

$$\tan\theta' = \frac{P\sin\theta}{\eta E + \gamma P\cos\theta} = \frac{\beta_0\sin\theta}{\eta + \gamma\beta_0\cos\theta} ,$$

or with $\gamma = 1/\sqrt{1-\beta^2}$ and $\eta = \beta\gamma$,

$$\tan\theta' = \frac{\sqrt{1-\beta^2}\sin\theta}{\beta/\beta_0 + \cos\theta} .$$

Problem 1.2

(a) The 4-momentum conservation $P = P_1 + P_2$ leads to

$$P - P_1 = P_2 \quad \rightarrow \quad (P - P_1)^2 = P_2^2\,.$$

Expanding it and using $P^2 = M^2$ and $P_i^2 = m_i^2$,

$$P^2 - 2P\cdot P_1 + P_1^2 = P_2^2 \quad \rightarrow \quad M^2 - 2ME_1 + m_1^2 = m_2^2\,,$$

where we have used $P = (E, \vec{0})$ in the rest frame of the parent. Then E_1 is

$$E_1 = \frac{M^2 - m_2^2 + m_1^2}{2M}\,,$$

and its momentum p is

$$\begin{aligned} p^2 &= E_1^2 - m_1^2 = \frac{(M^2 - m_2^2 + m_1^2)^2}{4M^2} - m_1^2 \\ &= \frac{M^4 + m_1^4 + m_2^4 - 2M^2m_1^2 - 2M^2m_2^2 - 2m_1^2m_2^2}{4M^2} \\ &= \frac{\lambda(M^2, m_1^2, m_2^2)}{4M^2},. \end{aligned}$$

Taking the square root gives the answer.

(b) Suppose the energy and momentum of daughter 1 in the rest frame of the parent are e and p, repectively. Then, from (a), $e = (M^2 - m_2^2 + m_1^2)/(2M)$ and $p = \sqrt{\lambda(M^2, m_1^2, m_2^2)}/(2M)$. The its energy in the lab frame E_1 is

$$E_1 = \gamma e + \eta p \cos\theta\,.$$

The maximum and minimum of E_1 corresponds to $\cos\theta = +1$ and -1, respectively:

$$E_{1\max} = \gamma e + \eta p\,, \quad E_{1\min} = \gamma e - \eta p\,.$$

The decay is uniform in 4π steradian. The expression for the solid angle element $d\omega = 2\pi d\cos\theta$ indicates that $\cos\theta$ will distribute uniformly from -1 to $+1$. Thus, the energy will distribute uniformly from $E_{1\min}$ to $E_{1\max}$. Normalizing the distribution in the range of $E_{1\max} - E_{1\min} = 2\eta p$,

$$f(E_1)dE_1 = \frac{dE_1}{2\eta p}\,.$$