Exercise 3.1 (10 pnts)
First, we see that σ_i^2 is identity:

$$\begin{aligned}
\sigma_1^2 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1\,, \\
\sigma_2^2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1\,, \\
\sigma_3^2 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1\,.
\end{aligned}$$

We also see that $\sigma_i\sigma_j = i\sigma_k$ $(ijk$:cyclic):

$$\begin{aligned}
\sigma_1\sigma_2 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\sigma_3\,, \\
\sigma_2\sigma_3 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = i\sigma_1\,, \\
\sigma_3\sigma_1 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma_2\,.
\end{aligned}$$

Right-multiplying σ_i to $\sigma_k\sigma_i = i\sigma_j$ $(ijk$:cyclic):,

$$\sigma_k \underbrace{\sigma_i^2}_{1} = i\sigma_j\sigma_i \quad \to \quad \sigma_j\sigma_i = -i\sigma_k \quad (ijk : \text{cyclic})\,.$$

Together with $\sigma_i\sigma_j = i\sigma_k$ $(ijk$:cyclic), this means $\{\sigma_i, \sigma_j\} = 0$ $(i \neq j)$. This and $\sigma_i^2 = 1$ can be combined as

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij}\,.$$

On the other hand, $\sigma_i\sigma_j = i\sigma_k$ $(ijk$:cyclic) and $\{\sigma_i, \sigma_j\} = 0$ can be written as

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k\,.$$

Exercise 3.2
Summing $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$ and $[\sigma_i . \sigma_j] = 2i\epsilon_{ijk}\sigma_k$,

$$\left\{ \begin{array}{rcl} \sigma_i\sigma_j + \sigma_j\sigma_i &=& 2\delta_{ij} \\ \sigma_i\sigma_j - \sigma_j\sigma_i &=& 2i\epsilon_{ijk}\sigma_k \end{array} \right. \quad \rightarrow \quad \sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k \,.$$

Then,

$$\begin{aligned} (\vec{a}\cdot\vec{\sigma})(\vec{b}\cdot\vec{\sigma}) &= (a_i\sigma_i)(b_j\sigma_j) = a_ib_j\sigma_i\sigma_j \\ &= a_ib_j(\delta_{ij} + i\epsilon_{ijk}\sigma_k) = a_ib_i + i\sigma_k \underbrace{\epsilon_{ijk}a_ib_j}_{(\vec{a}\times\vec{b})_k} \\ &= \vec{a}\cdot\vec{b} + i\vec{\sigma}\cdot(\vec{a}\times\vec{b}) \,. \end{aligned}$$

Applying this to $\vec{a} = \vec{b} = \hat{a}$,

$$(\hat{a}\cdot\vec{\sigma})^2 = \hat{a}^2 = 1 \,.$$

Using this and a definition of matrix exponentiation,

$$\begin{aligned} e^{i\vec{a}\cdot\vec{\sigma}} &= e^{ia(\hat{a}\cdot\vec{\sigma})} \\ &= 1 + ia(\hat{a}\cdot\vec{\sigma}) + \frac{1}{2!}(ia(\hat{a}\cdot\vec{\sigma}))^2 + \frac{1}{3!}(ia(\hat{a}\cdot\vec{\sigma}))^3 + \cdots \\ &= 1 + ia(\hat{a}\cdot\vec{\sigma}) - \frac{1}{2!}a^2 - \frac{1}{3!}ia^3(\hat{a}\cdot\vec{\sigma}) + \cdots \\ &= (1 - \frac{a^2}{2!} + \cdots) + i(\hat{a}\cdot\vec{\sigma})(a - \frac{a^3}{3!} + \cdots) \\ &= \cos a + i(\hat{a}\cdot\vec{\sigma})\sin a \,. \end{aligned}$$

Exercise 3.3

We note that $\{\alpha_i, \alpha_j\} = 0$ $(i \neq j)$ and $\alpha_i^2 = 1$ are equivalent to

$$\{\alpha_i, \alpha_j\} = 2\delta_{ij}\,.$$

We also have

$$\{\alpha_i, \beta\} = 0\,, \quad \beta^2 = 1\,.$$

For $(\mu, \nu = 1, 2, 3)$,

$$\begin{aligned}
\{\gamma^i, \gamma^j\} &= \{\beta\alpha_i, \beta\alpha_j\} = \beta\,\overbrace{\alpha_i\,\beta}^{-\beta\alpha_i}\,\alpha_j + \beta\,\overbrace{\alpha_j\,\beta}^{-\beta\alpha_j}\,\alpha_i \\
&= -\underbrace{\beta^2}_{1}\underbrace{(\alpha_i\alpha_j + \alpha_j\alpha_i)}_{2\delta_{ij}} = 2g^{ij}\,.
\end{aligned}$$

For $\mu = 0$ and $\nu = i$ ($\mu = i$, $\nu = 0$ is included since $\{\gamma^\mu, \gamma^\nu\} = \{\gamma^\nu, \gamma^\mu\}$)

$$\begin{aligned}
\{\gamma^0, \gamma^i\} &= \{\beta, \beta\alpha_i\} = \beta^2\alpha_i + \overbrace{\beta\alpha_i}^{-\alpha_i\beta}\,\beta \\
&= \alpha_i - \alpha_i = 0 = 2g^{0i}\,.
\end{aligned}$$

For $\mu = \nu = 0$,

$$\{\gamma^0, \gamma^0\} = 2{\gamma^0}^2 = 2\beta^2 = 2 = 2g^{00}\,.$$

Thus, we have shown that

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\,.$$