Problem 3.3

(a) Using the explicit γ matrices in the Dirac representation,

$$\not{p} = E\gamma^0 - p^i\gamma^i = E\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} - p^i\begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} = \begin{pmatrix} E & -\vec{p}\cdot\vec{\sigma} \\ \vec{p}\cdot\vec{\sigma} & -E \end{pmatrix}.$$

(b) With $c \equiv \sqrt{E+m}$ and $(\vec{p}\cdot\vec{\sigma})^2 = \vec{p}^2 = E^2 - m^2$,

$$\begin{aligned} \not{p}u &= c\begin{pmatrix} E & -\vec{p}\cdot\vec{\sigma} \\ \vec{p}\cdot\vec{\sigma} & -E \end{pmatrix}\begin{pmatrix} \chi \\ \dfrac{\vec{p}\cdot\vec{\sigma}}{E+m}\chi \end{pmatrix} = c\begin{pmatrix} E\chi - \dfrac{(\vec{p}\cdot\vec{\sigma})^2}{E+m}\chi \\ (\vec{p}\cdot\vec{\sigma})\chi - E\dfrac{\vec{p}\cdot\vec{\sigma}}{E+m}\chi \end{pmatrix} \\ &= c\begin{pmatrix} \Big(E - \dfrac{E^2-m^2}{E+m}\Big)\chi \\ \Big(1 - \dfrac{E}{E+m}\Big)(\vec{p}\cdot\vec{\sigma})\chi \end{pmatrix} = c\begin{pmatrix} m\chi \\ m\dfrac{\vec{p}\cdot\vec{\sigma}}{E+m}\chi \end{pmatrix} = mu\,. \end{aligned}$$

Thus, we have $(\not{p} - m)u = 0$.

Similarly,

$$\begin{aligned} \not{p}v &= c\begin{pmatrix} E & -\vec{p}\cdot\vec{\sigma} \\ \vec{p}\cdot\vec{\sigma} & -E \end{pmatrix}\begin{pmatrix} \dfrac{\vec{p}\cdot\vec{\sigma}}{E+m}\chi \\ \chi \end{pmatrix} = c\begin{pmatrix} E\dfrac{\vec{p}\cdot\vec{\sigma}}{E+m}\chi - (\vec{p}\cdot\vec{\sigma})\chi \\ \dfrac{(\vec{p}\cdot\vec{\sigma})^2}{E+m}\chi - E\chi \end{pmatrix} \\ &= c\begin{pmatrix} \Big(\dfrac{E}{E+m} - 1\Big)(\vec{p}\cdot\vec{\sigma})\chi \\ \Big(\dfrac{E^2-m^2}{E+m} - E\Big)\chi \end{pmatrix} = c\begin{pmatrix} -m\dfrac{\vec{p}\cdot\vec{\sigma}}{E+m}\chi \\ -m\chi \end{pmatrix} = -mv\,. \end{aligned}$$

Namely, $(\not{p} + m)v = 0$. Note that in the above χ could be any 2-component vector.

Problem 3.5

(a) Denote the gamma matrixes in the Weyl representation as γ_W^μ:

$$\gamma_W^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_W^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}.$$

The transformation of γ^0 is

$$V \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} V^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Since the elements are all real, we take the 2×2 unitary matrix V to be real, namely, orthogonal matrix which is a rotation in 2-dimensional space:

$$V = \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \quad \rightarrow \quad V^{-1} = \begin{pmatrix} c & s \\ -s & c \end{pmatrix},$$

where $c = \cos\theta$ and $s = \sin\theta$. Trying this to γ^0,

$$\begin{pmatrix} c & -s \\ s & c \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} c & s \\ -s & c \end{pmatrix} = \begin{pmatrix} c^2 - s^2 & 2sc \\ 2sc & s^2 - c^2 \end{pmatrix} = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Namely,

$$\cos 2\theta = 0, \quad \sin 2\theta = 1,$$

or,

$$2\theta = \frac{1}{2}\pi + 2n\pi, \quad \rightarrow \quad \theta = \frac{1}{4}\pi + n\pi.$$

Taking different n only changes overall sign. Trying $n = 1$, we have $c = 1/\sqrt{2}, s = 1/\sqrt{2}$, and

$$V = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad \rightarrow \quad V^{-1} = V^\dagger = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

It keeps the form of γ^i the same:

$$V\gamma^i V^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} = \gamma_W^i.$$

(b) Write the boost matrix S in the Dirac representation as

$$S = c \begin{pmatrix} 1 & \sigma_P \\ \sigma_P & 1 \end{pmatrix} \text{ with } \quad c \equiv \sqrt{\frac{E+m}{2m}}, \; \sigma_P \equiv \frac{\vec{p} \cdot \vec{\sigma}}{E+m}.$$

Then, S in the Weyl representation becomes

$$VSV^{-1} = \frac{c}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & \sigma_P \\ \sigma_P & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = c \begin{pmatrix} 1 - \sigma_P & 0 \\ 0 & 1 + \sigma_P \end{pmatrix}:$$

$$\text{namely,} \qquad S_W = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} 1 - \frac{\vec{p}\cdot\vec{\sigma}}{E+m} & 0 \\ 0 & 1 + \frac{\vec{p}\cdot\vec{\sigma}}{E+m} \end{pmatrix} .$$

(c) In the Weyl representation, the generators of boost and rotations are in general given by (i, j, k: cyclic)

$$B^{0i} = \frac{1}{2}\gamma_W^0 \gamma_W^i = \frac{1}{2}\begin{pmatrix} -\sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} ,$$

$$B^{ij} = \frac{1}{2}\gamma^j \gamma^k = \frac{1}{2}\begin{pmatrix} -\sigma_i\sigma_j & 0 \\ 0 & -\sigma_i\sigma_j \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -i\sigma_k & 0 \\ 0 & -i\sigma_k \end{pmatrix} = B_k^r .$$

Then, the general Lorentz transformation can be written as

$$\begin{aligned} S_W &= \exp\left(\xi_i B^{0i} + \theta_k B_k^r\right) \\ &= \exp\begin{pmatrix} \frac{1}{2}(-\vec{\xi}\cdot\vec{\sigma} - i\vec{\theta}\cdot\vec{\sigma}) & 0 \\ 0 & \frac{1}{2}(\vec{\xi}\cdot\vec{\sigma} - i\vec{\theta}\cdot\vec{\sigma}) \end{pmatrix} \\ &= \begin{pmatrix} \exp(-\vec{\xi}\cdot\frac{\vec{\sigma}}{2} - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}) & 0 \\ 0 & \exp(\vec{\xi}\cdot\frac{\vec{\sigma}}{2} - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}) \end{pmatrix} . \end{aligned}$$

Thus, the top half ϕ_R and the bottom half ϕ_L of the 4-spinor transforms independently in the Weyl representation. Corresponding generators are

$$\begin{aligned} G_i &= -\frac{\sigma_i}{2} \quad H_i = -i\frac{\sigma_i}{2} \quad \text{for } \phi_R , \\ G_i &= \frac{\sigma_i}{2} \quad\;\; H_i = -i\frac{\sigma_i}{2} \quad \text{for } \phi_L . \end{aligned}$$

(d) In the Weyl representation, the massless Dirac equation is

$$i\gamma_W^\mu \partial_\mu \psi_W = 0 \quad \text{with} \quad \psi_W = \begin{pmatrix} \phi_L \\ \phi_R \end{pmatrix} .$$

Using the explicit expressions for γ_W^μ,

$$i\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\partial_0 + \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}\partial_i\right]\begin{pmatrix} \phi_L \\ \phi_R \end{pmatrix} = 0 .$$

$$\rightarrow \quad \begin{cases} i(\partial_0 + \vec{\sigma}\cdot\nabla)\phi_R = 0 , \\ i(\partial_0 - \vec{\sigma}\cdot\nabla)\phi_L = 0 . \end{cases}$$