Exercise 3.10

Writing the spin signs as $\eta = \pm 1$, the two equalities to prove are

$$u_{\vec{p},\eta\vec{s}}\,\overline{u_{\vec{p},\eta\vec{s}}} = 2m\,\Lambda_+(p)\Sigma_\eta(s)\,, \qquad v_{\vec{p},\eta\vec{s}}\,\overline{v_{\vec{p},\eta\vec{s}}} = -2m\,\Lambda_-(p)\Sigma_\eta(s)\,.$$

In proving each of these, we will show that the left side and the right side of equality result in the same vector when applied to any of the compelte orthonormal basis $(u_{\vec{p},\eta\vec{s}}, v_{\vec{p},\eta\vec{s}})$ $(\eta = \pm 1)$.

Using the associativity $(a\bar{b})c = a(\bar{b}c)$ $(a, b, c$ are spinors) and the orthonormalities $\bar{u}_{\vec{p},\eta\vec{s}}\,u_{\vec{p},\eta'\vec{s}} = 2m\,\delta_{\eta,\eta'}$ and $\bar{u}_{\vec{p},\eta\vec{s}}\,v_{\vec{p},\eta'\vec{s}} = 0$, we have

$$(u_{\vec{p},\eta\vec{s}}\,\overline{u_{\vec{p},\eta\vec{s}}})u_{\vec{p},\eta'\vec{s}} = u_{\vec{p},\eta\vec{s}}\,\underbrace{(\overline{u_{\vec{p},\eta\vec{s}}}\,u_{\vec{p},\eta'\vec{s}})}_{2m\,\delta_{\eta,\eta'}} = 2m\,\delta_{\eta,\eta'}\,u_{\vec{p},\eta\vec{s}}\,,$$

$$(u_{\vec{p},\eta\vec{s}}\,\overline{u_{\vec{p},\eta\vec{s}}})v_{\vec{p},\eta'\vec{s}} = u_{\vec{p},\eta\vec{s}}\,\underbrace{(\overline{u_{\vec{p},\eta\vec{s}}}\,v_{\vec{p},\eta'\vec{s}})}_{0} = 0\,.$$

On the other hand, using the properties of the energy and spin projection operators $\Sigma_\eta(s)w_{\vec{p},\eta'\vec{s}} = \delta_{\eta,\eta'}w_{\vec{p},\eta'\vec{s}}$ $(w = u, v)$, $\Lambda_+(p)u_{\vec{p},\eta'\vec{s}} = u_{\vec{p},\eta'\vec{s}}$, and $\Lambda_+(p)v_{\vec{p},\eta'\vec{s}} = 0$,

$$2m\,\Lambda_+(p)\,\underbrace{\Sigma_\eta(s)u_{\vec{p},\eta'\vec{s}}}_{\delta_{\eta,\eta'}u_{\vec{p},\eta'\vec{s}}} = 2m\,\delta_{\eta,\eta'}\,\underbrace{\Lambda_+(p)u_{\vec{p},\eta'\vec{s}}}_{u_{\vec{p},\eta'\vec{s}}} = 2m\,\delta_{\eta,\eta'}u_{\vec{p},\eta'\vec{s}}\,,$$

$$2m\,\Lambda_+(p)\,\underbrace{\Sigma_\eta(s)v_{\vec{p},\eta'\vec{s}}}_{\delta_{\eta,\eta'}v_{\vec{p},\eta'\vec{s}}} = 2m\,\delta_{\eta,\eta'}\,\underbrace{\Lambda_+(p)v_{\vec{p},\eta'\vec{s}}}_{0} = 0\,.$$

Thus, the 4×4 matrixes $u_{\vec{p},\eta\vec{s}}\,\overline{u_{\vec{p},\eta\vec{s}}}$ and $2m\,\Lambda_+(p)\Sigma_\eta(s)$ behave the same way when they are applied to the complete orthonormal basis: thus, they are the same matrixes.

Similarly,

$$(v_{\vec{p},\eta\vec{s}}\,\overline{v_{\vec{p},\eta\vec{s}}})u_{\vec{p},\eta'\vec{s}} = v_{\vec{p},\eta\vec{s}}\,(\overline{v_{\vec{p},\eta\vec{s}}}\,u_{\vec{p},\eta'\vec{s}}) = 0\,,$$

$$(v_{\vec{p},\eta\vec{s}}\,\overline{v_{\vec{p},\eta\vec{s}}})v_{\vec{p},\eta'\vec{s}} = v_{\vec{p},\eta\vec{s}}\,(\overline{v_{\vec{p},\eta\vec{s}}}\,v_{\vec{p},\eta'\vec{s}}) = 2m\,\delta_{\eta,\eta'}\,v_{\vec{p},\eta\vec{s}}\,,$$

and

$$2m\,\Lambda_-(p)\Sigma_\eta(s)u_{\vec{p},\eta'\vec{s}} = 2m\,\delta_{\eta,\eta'}\Lambda_-(p)u_{\vec{p},\eta'\vec{s}} = 0\,,$$

$$2m\,\Lambda_-(p)\Sigma_\eta(s)v_{\vec{p},\eta'\vec{s}} = 2m\,\delta_{\eta,\eta'}\Lambda_-(p)v_{\vec{p},\eta'\vec{s}} = 2m\,\delta_{\eta,\eta'}v_{\vec{p},\eta'\vec{s}}\,.$$

Thus, the 4×4 matrixes $v_{\vec{p},\eta\vec{s}}\,\overline{v_{\vec{p},\eta\vec{s}}}$ and $2m\,\Lambda_-(p)\Sigma_\eta(s)$ behaves the same way when they are applied to the complete orthonormal basis: thus, they are the same matrixes.

Problem 3.4

(a) For the time reversal Lorentz transformation

$$\Lambda = T = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix},$$

the relation $S^{-1}\gamma^\mu S = \Lambda^\mu{}_\nu \gamma^\nu$ reads

$$S^{-1}\gamma^0 S = -\gamma^0\,, \quad S^{-1}\gamma^i S = \gamma^i\,.$$

To verify that $S = \gamma^1\gamma^2\gamma^3$ indeed satisfy the relations, we first write

$$S = \gamma^1\gamma^2\gamma^3 = -i\gamma^0\gamma_5\,.$$

Then, using $\{\gamma_5, \gamma^\mu\} = 0$ and ${\gamma^0}^2 = {\gamma_5}^2 = 1$,

$$S^2 = -\gamma^0\gamma_5\gamma^0\gamma_5 = {\gamma^0}^2{\gamma_5}^2 = 1 \quad \to \quad S^{-1} = S\,.$$

($\bar{S} = S^{-1}$ works for proper and orthochronous transformations or for space inversion, but it does not hold in this case.)

Check the time component:

$$S^{-1}\gamma^0 S = (-i\gamma^0\gamma_5)\gamma^0(-i\gamma^0\gamma_5) = -\gamma^0\gamma_5{\gamma^0}^2\gamma_5 = -\gamma^0{\gamma_5}^2 = -\gamma^0\,.$$

Check the space component:

$$S^{-1}\gamma^i S = (-i\gamma^0\gamma_5)\gamma^i(-i\gamma^0\gamma_5) = -\gamma^0\gamma_5\gamma^i\gamma^0\gamma_5 = -{\gamma^0}^2\gamma_5\gamma^i\gamma_5 = {\gamma_5}^2\gamma^i = \gamma^i\,.$$

(b) In the Dirac representation,

$$S = -i\gamma^0\gamma_5 = -i\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = i\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

(c) We now apply this transformation to the electron solution

$$\psi'(x') = S\psi(x) \quad \text{with } u_{\vec{p},\vec{s}}\, e^{-ip\cdot x} \quad \text{and } x' = (t', \vec{x}') = Tx = (-t, \vec{x})\,,$$

where $\vec{p}$ in $-ip\cdot x$ and $\vec{p}$ in $u_{\vec{p},\vec{s}}$ are the same, and

$$p^\mu = (p^0, \vec{p}) \quad \text{with} \quad p^0 \equiv \sqrt{\vec{p}^2 + m^2} \geq 0\,.$$

Using the expression of $u_{\vec{p},\vec{s}}$ in the Dirac representation and dropping the overall constant,

$$\psi'(x') = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}\begin{pmatrix} \chi_+ \\ \dfrac{\vec{p}\cdot\vec{\sigma}}{E+m}\chi_+ \end{pmatrix} e^{-i(p^0 t - \vec{p}\cdot\vec{x})} = \begin{pmatrix} \dfrac{-\vec{p}\cdot\vec{\sigma}}{E+m}\chi_+ \\ \chi_+ \end{pmatrix} e^{i(p^0 t' + \vec{p}\cdot\vec{x}')}\,,$$

where we used the space-time variable x' following the definition of the transformation. Comparing this with the positron solutions, we can write

$$\psi'(x') = v_{-\vec{p},-\vec{s}}\, e^{ip'\cdot x'} \quad \text{with} \quad p'^{\mu} = (\sqrt{(-\vec{p})^2 + m^2}, -\vec{p}) \,.$$

Namely, this is a positron solution of the Dirac equation with momentum $-\vec{p}$ and physical spin $-\vec{s}$.