Exercise 4.3

The momentum expansions of $\phi(x)$ and $\pi(x)$ are

$$\phi(x) = \sum_{\vec{p}} (a_{\vec{p}} e_{\vec{p}}(x) + a^\dagger_{\vec{p}} e^*_{\vec{p}}(x)) \,, \quad \pi(x) = \sum_{\vec{p}} (-ip^0)(a_{\vec{p}} e_{\vec{p}}(x) - a^\dagger_{\vec{p}} e^*_{\vec{p}}(x)) \,.$$

Using the commutation relations among $a_{\vec{p}}$'s and $a^\dagger_{\vec{p}}$'s and with

$$x \equiv (t, \vec{x}) \,, \qquad x' \equiv (t, \vec{x}') \,,$$

the equal-time commutator of $\phi(x)$ and $\pi(x')$ is

$$\begin{aligned}
[\phi(x), \pi(x')] &= \sum_{\vec{p}\vec{p}'} (-ip^{0\prime})[a_{\vec{p}} e_{\vec{p}}(x) + a^\dagger_{\vec{p}} e^*_{\vec{p}}(x), a_{\vec{p}'} e_{\vec{p}'}(x') - a^\dagger_{\vec{p}'} e^*_{\vec{p}'}(x')] \\
&= \sum_{\vec{p}\vec{p}'} (-ip^{0\prime}) \Big(\underbrace{[a^\dagger_{\vec{p}}, a_{\vec{p}'}]}_{-\delta_{\vec{p},\vec{p}'}} e^*_{\vec{p}}(x) e_{\vec{p}'}(x') - \underbrace{[a_{\vec{p}}, a^\dagger_{\vec{p}'}]}_{\delta_{\vec{p},\vec{p}'}} e_{\vec{p}}(x) e^*_{\vec{p}'}(x') \Big) \\
&= i \sum_{\vec{p}} p^0 e^*_{\vec{p}}(x) e_{\vec{p}}(x') + i \sum_{\vec{p}} p^0 e_{\vec{p}}(x) e^*_{\vec{p}}(x')
\end{aligned}$$

In the last line, we see that the second term is the complex conjugate of the first (apart from i). The first term is

$$\begin{aligned}
\sum_{\vec{p}} p^0 e^*_{\vec{p}}(x) e_{\vec{p}}(x') &= \sum_{\vec{p}} p^0 \frac{e^{ip\cdot x}}{\sqrt{2p^0V}} \frac{e^{-ip\cdot x'}}{\sqrt{2p^0V}} = \sum_{\vec{p}} \frac{p^0}{2p^0V} e^{ip^0t - i\vec{p}\cdot\vec{x}} e^{-ip^0t + i\vec{p}\cdot\vec{x}'} \\
&= \frac{1}{2V} \underbrace{\sum_{\vec{p}} e^{i\vec{p}\cdot(\vec{x}' - \vec{x})}}_{V\delta^3(\vec{x} - \vec{x}')} = \frac{1}{2}\delta^3(\vec{x} - \vec{x}')
\end{aligned}$$

Then,

$$[\phi(x), \pi(x')] = i \left[\frac{1}{2}\delta^3(\vec{x} - \vec{x}') + \Big(\frac{1}{2}\delta^3(\vec{x} - \vec{x}') \Big)^* \right] = i\delta^3(\vec{x} - \vec{x}') \,.$$

Similarly,

$$\begin{aligned}
[\phi(x), \phi(x')] &= \sum_{\vec{p}\vec{p}'} [a_{\vec{p}} e_{\vec{p}}(x) + a^\dagger_{\vec{p}} e^*_{\vec{p}}(x), a_{\vec{p}'} e_{\vec{p}'}(x') + a^\dagger_{\vec{p}'} e^*_{\vec{p}'}(x')] \\
&= \sum_{\vec{p}\vec{p}'} \Big(\underbrace{[a^\dagger_{\vec{p}}, a_{\vec{p}'}]}_{-\delta_{\vec{p},\vec{p}'}} e^*_{\vec{p}}(x) e_{\vec{p}'}(x') + \underbrace{[a_{\vec{p}}, a^\dagger_{\vec{p}'}]}_{\delta_{\vec{p},\vec{p}'}} e_{\vec{p}}(x) e^*_{\vec{p}'}(x') \Big) \\
&= \sum_{\vec{p}} \Big(e_{\vec{p}}(x) e^*_{\vec{p}}(x') - e^*_{\vec{p}}(x) e_{\vec{p}}(x') \Big) \,.
\end{aligned}$$

The second term can be seen to be the complex conjugate of the first. On the other hand, the first term is

$$\sum_{\vec{p}} e_{\vec{p}}(x)e^*_{\vec{p}}(x') = \sum_{\vec{p}} \frac{e^{-ip\cdot x}}{\sqrt{2p^0V}}\frac{e^{ip\cdot x'}}{\sqrt{2p^0V}} = \sum_{\vec{p}} \frac{1}{2p^0V} e^{i\vec{p}\cdot(\vec{x}-\vec{x}\,')} \,.$$

This is real since relabeling $\vec{p} \to -\vec{p}$ changes it to its complex conjugate while the relabeling should not change the value. (Here, we note that $p^0 \equiv \sqrt{\vec{p}^2 + m^2}$ does not change under $\vec{p} \to -\vec{p}$.) Thus, the first and second terms cancel out and we have

$$[\phi(x), \phi(x')] = 0 \,.$$

The last commutator is

$$\begin{aligned}
[\pi(x), \pi(x')] &= \sum_{\vec{p}\vec{p}\,'} (-ip^0)(-ip^{0\prime})[a_{\vec{p}}e_{\vec{p}}(x) - a^\dagger_{\vec{p}}e^*_{\vec{p}}(x), a_{\vec{p}\,'}e_{\vec{p}\,'}(x') - a^\dagger_{\vec{p}\,'}e^*_{\vec{p}\,'}(x')] \\
&= \sum_{\vec{p}\vec{p}\,'} (p^0p^{0\prime})\Big(\underbrace{[a^\dagger_{\vec{p}}, a_{\vec{p}\,'}]}_{-\delta_{\vec{p},\vec{p}\,'}} e^*_{\vec{p}}(x)e_{\vec{p}\,'}(x') + \underbrace{[a_{\vec{p}}, a^\dagger_{\vec{p}\,'}]}_{\delta_{\vec{p},\vec{p}\,'}} e_{\vec{p}}(x)e^*_{\vec{p}\,'}(x')\Big) \\
&= \sum_{\vec{p}} {p^0}^2 \Big(e_{\vec{p}}(x)e^*_{\vec{p}}(x') - e^*_{\vec{p}}(x)e_{\vec{p}}(x')\Big)
\end{aligned}$$

Again, the second term is seen to be the complex conjugate of the first. The first term is

$$\sum_{\vec{p}} {p^0}^2 e_{\vec{p}}(x)e^*_{\vec{p}}(x') = \sum_{\vec{p}} {p^0}^2 \frac{e^{-ip\cdot x}}{\sqrt{2p^0V}}\frac{e^{ip\cdot x'}}{\sqrt{2p^0V}} = \sum_{\vec{p}} \frac{p^0}{2V} e^{i\vec{p}\cdot(\vec{x}-\vec{x}\,')} \,,$$

which is real since relabeling $\vec{p} \to -\vec{p}$ changes it to its complex conjugate while the relabeling should not change the value. Thus, the first and second terms cancel out, and we have

$$[\pi(x), \pi(x')] = 0 \,.$$

Exercise 4.4

Using the momentum expansions of $\phi(x)$

$$\phi(x) = \sum_{\vec{p}} (a_{\vec{p}} e_{\vec{p}}(x) + a^{\dagger}_{\vec{p}} e^{*}_{\vec{p}}(x))$$

the total momentum becomes (normal ordering is implicit)

$$\begin{aligned}
\vec{P} &= \frac{1}{2}\int d^3x \phi\, i\overleftrightarrow{\partial}_0(-i\vec{\nabla}\phi) \\
&= \frac{1}{2}\int d^3x \sum_{\vec{q}} (a_{\vec{q}} e_{\vec{q}}(x) + a^{\dagger}_{\vec{q}} e^{*}_{\vec{q}}(x)) i\overleftrightarrow{\partial}_0 \underbrace{(-i\vec{\nabla}) \sum_{\vec{p}} (a_{\vec{p}} e_{\vec{p}}(x) + a^{\dagger}_{\vec{p}} e^{*}_{\vec{p}}(x))}_{\sum_{\vec{p}} \vec{p}\,(a_{\vec{p}} e_{\vec{p}}(x) - a^{\dagger}_{\vec{p}} e^{*}_{\vec{p}}(x))} \\
&= \frac{1}{2}\int d^3x \sum_{\vec{q},\vec{p}} (a_{\vec{q}} e_{\vec{q}}(x) + a^{\dagger}_{\vec{q}} e^{*}_{\vec{q}}(x)) i\overleftrightarrow{\partial}_0\, \vec{p}\,(a_{\vec{p}} e_{\vec{p}}(x) - a^{\dagger}_{\vec{p}} e^{*}_{\vec{p}}(x))\,,
\end{aligned}$$

and using the orthonormality of $e_{\vec{p}}(x)$,

$$\begin{aligned}
&= \frac{1}{2}\int d^3x \sum_{\vec{q},\vec{p}} \vec{p}\Big(a_{\vec{q}} a_{\vec{p}} \underbrace{e_{\vec{q}}(x) i\overleftrightarrow{\partial}_0 e_{\vec{p}}(x)}_{\to 0} - a^{\dagger}_{\vec{q}} a^{\dagger}_{\vec{p}} \underbrace{e^{*}_{\vec{q}}(x) i\overleftrightarrow{\partial}_0 e^{*}_{\vec{p}}(x)}_{\to 0} \\
&\qquad - a_{\vec{q}} a^{\dagger}_{\vec{p}} \underbrace{e_{\vec{q}}(x) i\overleftrightarrow{\partial}_0 e^{*}_{\vec{p}}(x)}_{\to -\delta_{\vec{q},vp}} + a^{\dagger}_{\vec{q}} a_{\vec{p}} \underbrace{e^{*}_{\vec{q}}(x) i\overleftrightarrow{\partial}_0 e_{\vec{p}}(x)}_{\to \delta_{\vec{q},vp}}\Big) \\
&= \frac{1}{2}\sum_{\vec{p}} \vec{p}\,\underbrace{(: a_{\vec{p}} a^{\dagger}_{\vec{p}} + a^{\dagger}_{\vec{p}} a_{\vec{p}} :)}_{2a^{\dagger}_{\vec{p}} a_{\vec{p}}} = \sum_{\vec{p}} \vec{p}\, a^{\dagger}_{\vec{p}} a_{\vec{p}}\,.
\end{aligned}$$