Measurement of the London Penetration Depth of Superconductors using a Spiral Coil for Next-Generation Accelerator Development

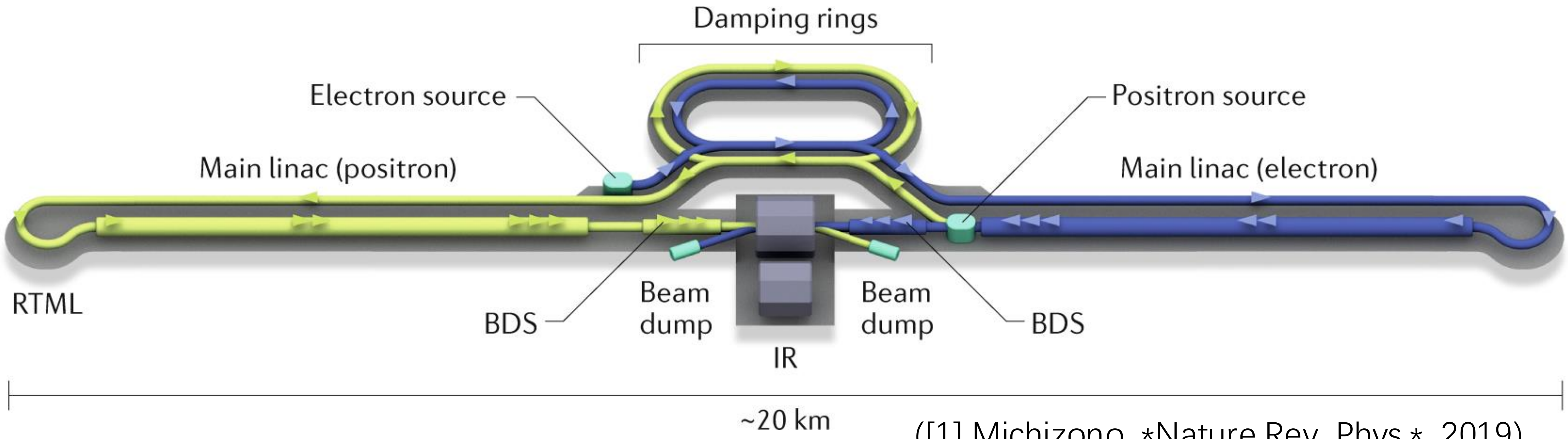
(次世代加速器開発のための螺旋コイルを用いた超伝導体のロンドン侵入長の測定)

2025

LI HE

ILC Experiment - Overview

- **The International Linear Collider (ILC) is a 250 GeV linear e+ e− collider, which is designed to provide a complete, high-precision picture of the Higgs boson and its interactions.**
- **The heart of the ILC accelerator consists of two superconducting Main Linacs that accelerate particle beams from 5 to 250 GeV. A key component of achieving this energy level is the Superconducting Radio Frequency (SRF) Cavity.**



([1] Michizono, *Nature Rev. Phys.*, 2019)

Introduction of SRF Technology

What is SRF cavities?

Maxwell's Equations :

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\begin{Bmatrix}\mathbf{E}\\ \mathbf{H}\end{Bmatrix} = 0$$

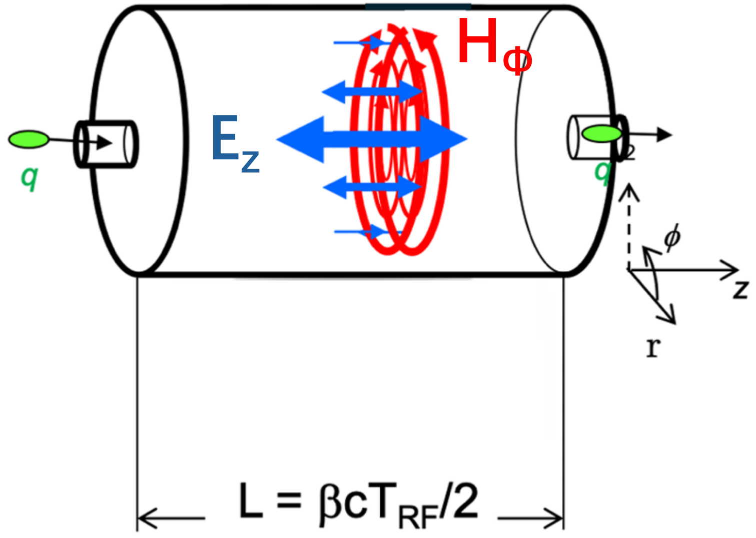
Accelerating Mode : TM_{010}

$$E_z = E_0 J_0\left(\frac{2.405r}{R}\right)e^{-i\omega t}$$

$$H_\phi = -i\frac{E_0}{\eta}J_1\left(\frac{2.405r}{R}\right)e^{-i\omega t}$$

$$\omega_{010} = \frac{2.405c}{R}$$

$$\eta = \sqrt{\frac{\mu_0}{\epsilon_0}}$$



1.3GHz nine-cell superconducting cavities made of Niobium

- **Shape is selected so that TM_{010} mode can efficiently transfer its energy to a charged particle**

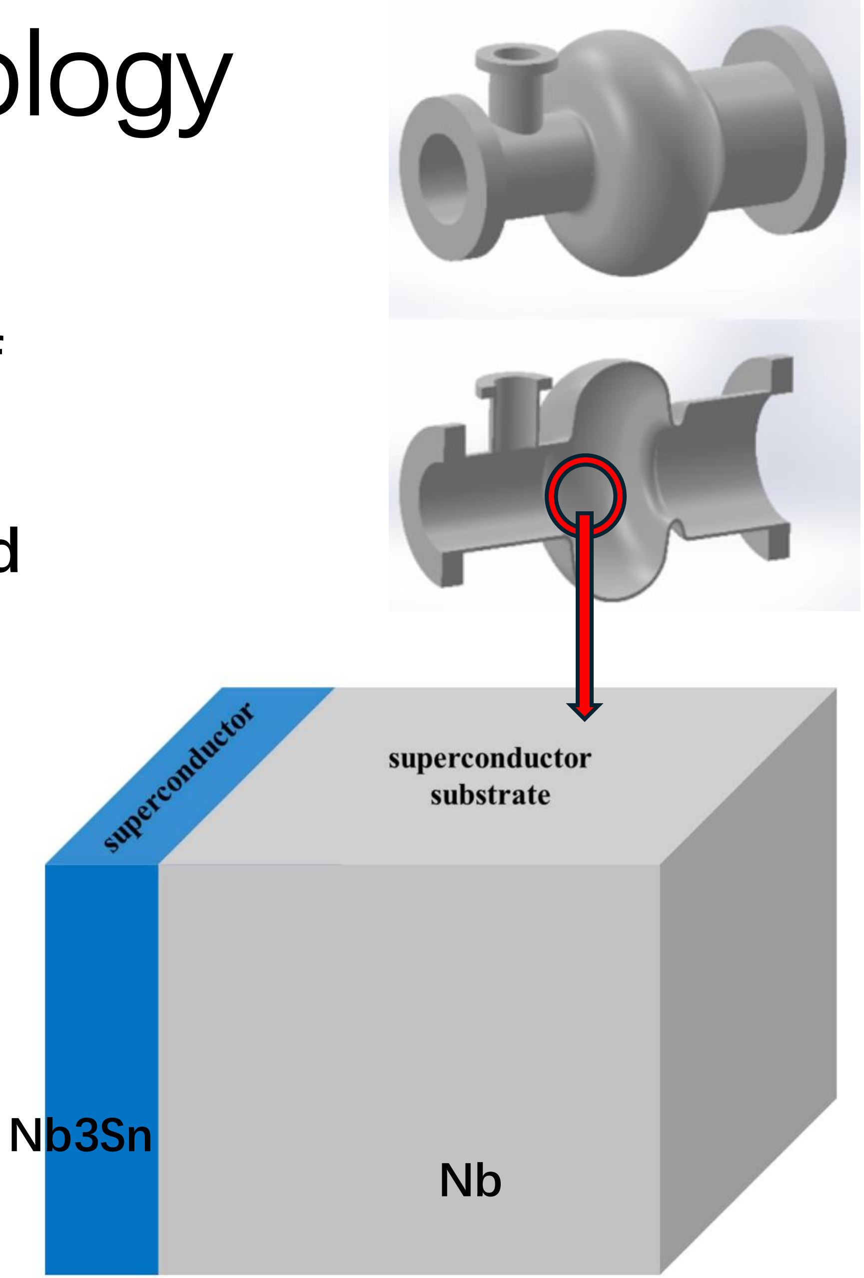
([2] ILC Collaboration, 2013)

Advanced SRF: Coating technology

- **In the recent development of advanced SRF accelerator, the bulk Nb cavity reached the limit of performance.**
- **Ideas to make inner-surface coatings to go beyond the performance of bulk Nb cavity.**

	Maximum Accelerating Gradient
Pure Nb Bulk	~35 MV/m
After Nb3Sn Coating	~60 MV/m (Expected value)

- **Capable of achieving higher Accelerating gradient.**



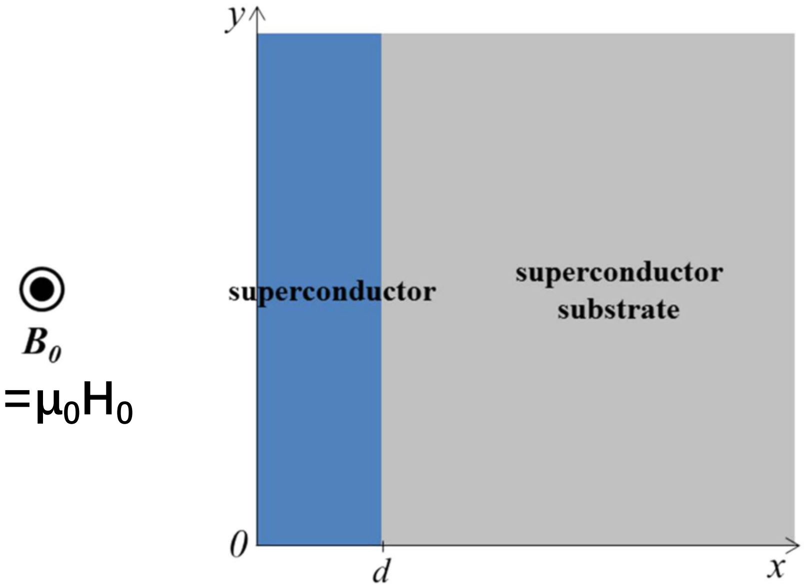
([3] Kubo, *Supercond. Sci. Technol.*, 2017)

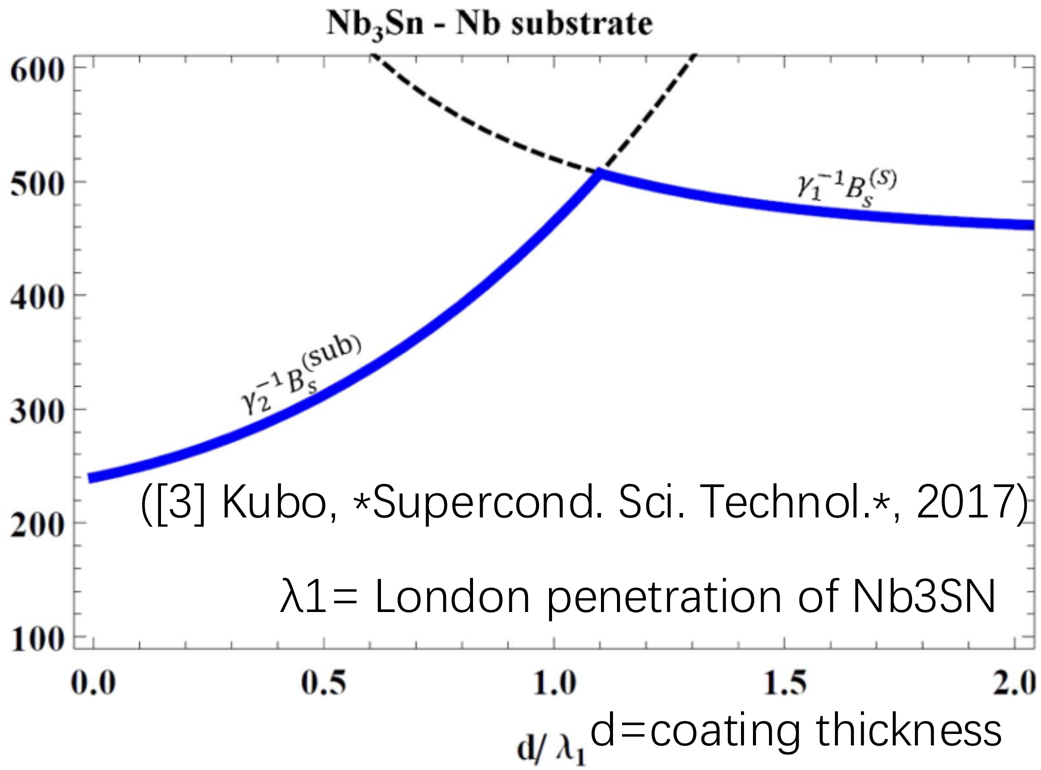
Why we want to know λ(0)?

$$gE_{acc} = \mu_0 H_0$$

- H_0 -> oscillating magnetic field
- g depends on the shape of cavities
- Eacc is the accelerating gradient.
- $H0_{max}$ is determined by the London penetration depths

$B_0 = \mu_0 H_0$





B_{0max}

([3] Kubo, *Supercond. Sci. Technol.*, 2017)

λ1= London penetration of Nb3SN

d=coating thickness

General Introduction of this Thesis

The Content of this Thesis

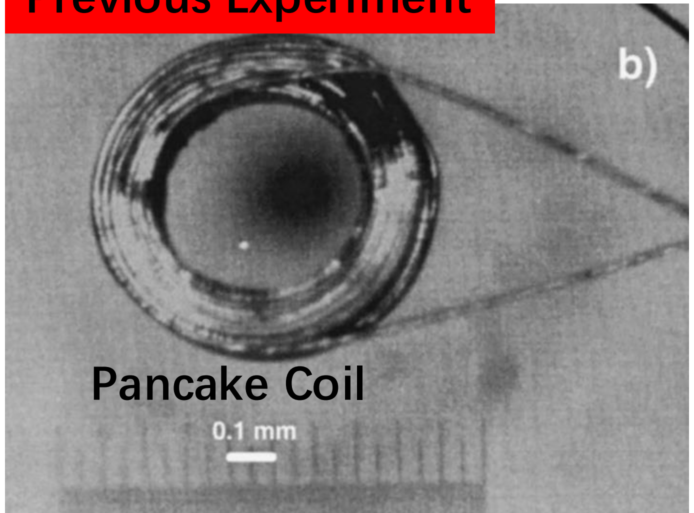
- **We used inductance technique to measure London penetration depth, which requires a coil.**

Novel Aspects :

- **The previously used pancake coil was replaced with a more easily manufactured spiral coil.**

- **A theoretical formula in the case of spiral coil, which express the resonant frequency f as a function of London penetration depth λ has been proposed.**

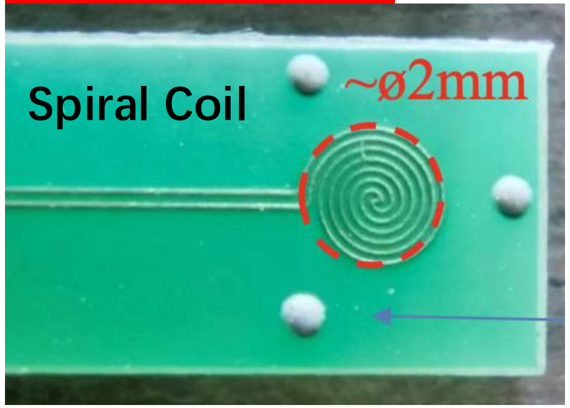
Previous Experiment



[4]A. Gauzzi et al., Rev. Sci. Instrum. 71, 2147 (2000)

- Non-Planar
- 0.6 mm diameter
- Hard to make

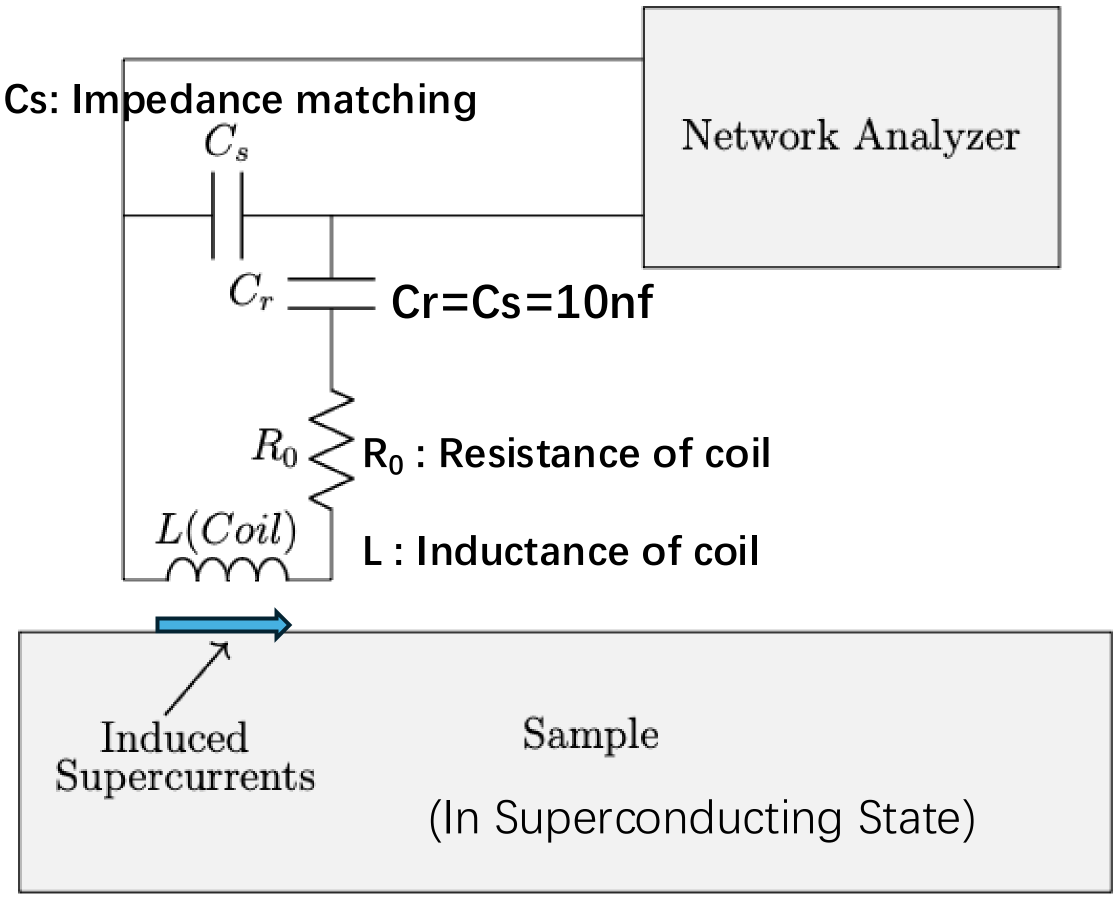
This Experiment

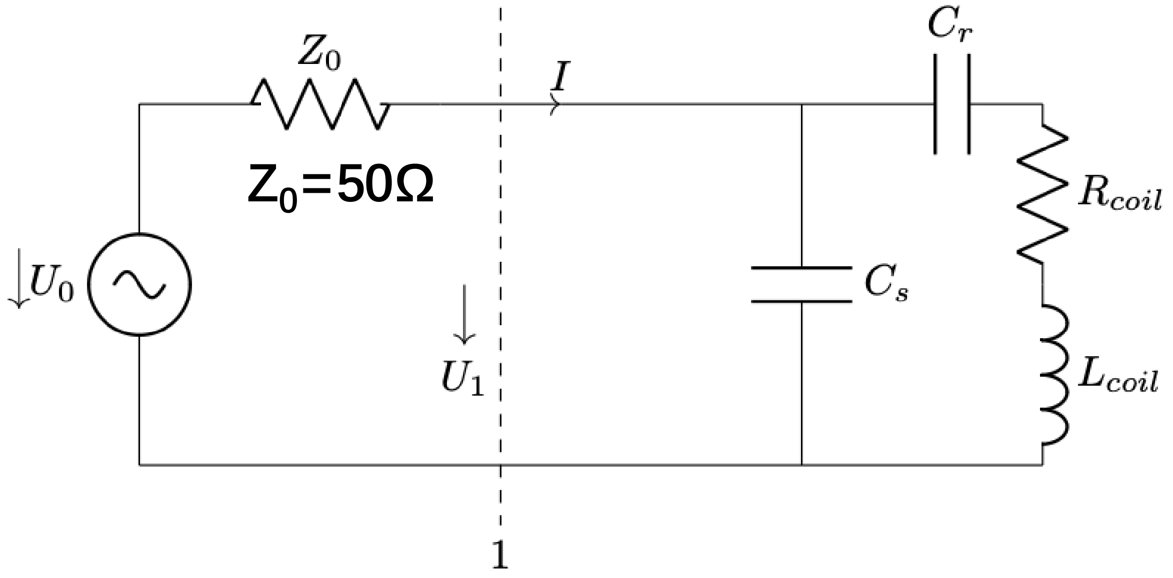


- A Planar Spiral coil, easy to fabricate!

Inductance Technique

- **How can we measure London Penetration depth?**





$$S_{11} = \frac{Z_{in}^{(1)} - Z_0}{Z_{in}^{(1)} + Z_0}$$

Z_{in}: Equivalent Impedance of resonant circuit

Z_0: Reference Impedance = 50Ω.

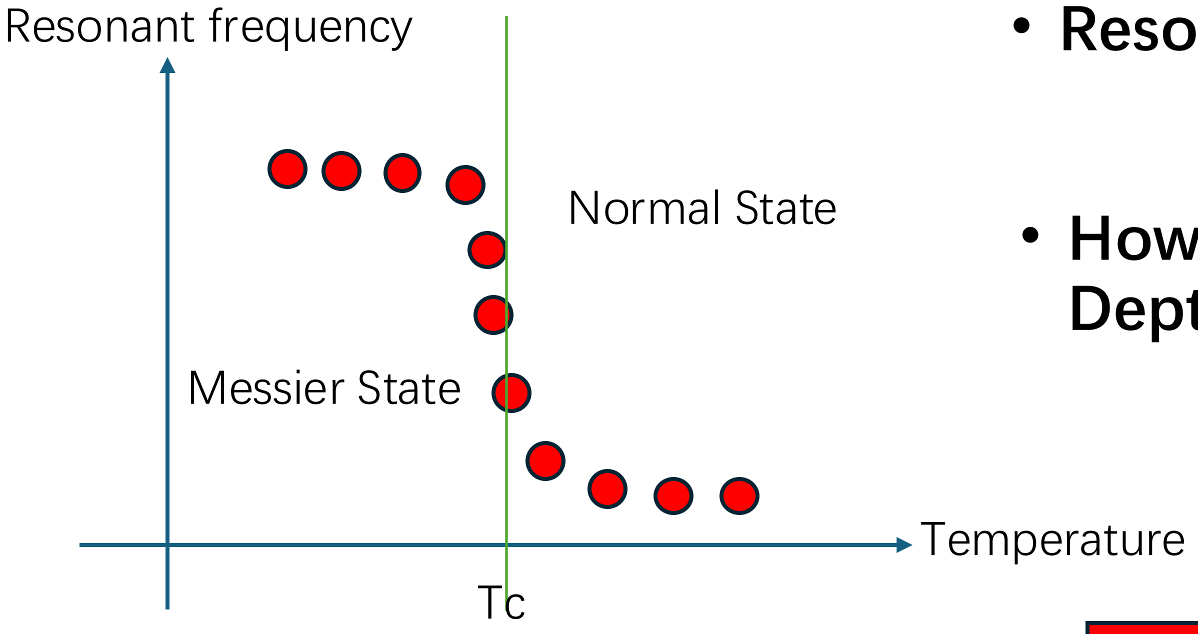
At resonance we observed:

LogMag[S_{11}] ~ -6 dB

The importance of theoretical formula f(λ)

- **What kind of raw data will we obtain?**

  - **Expected experimental result**



- **Resonant frequency f at different Temperatures**

- **How can we obtain London penetration Depth from raw data?**

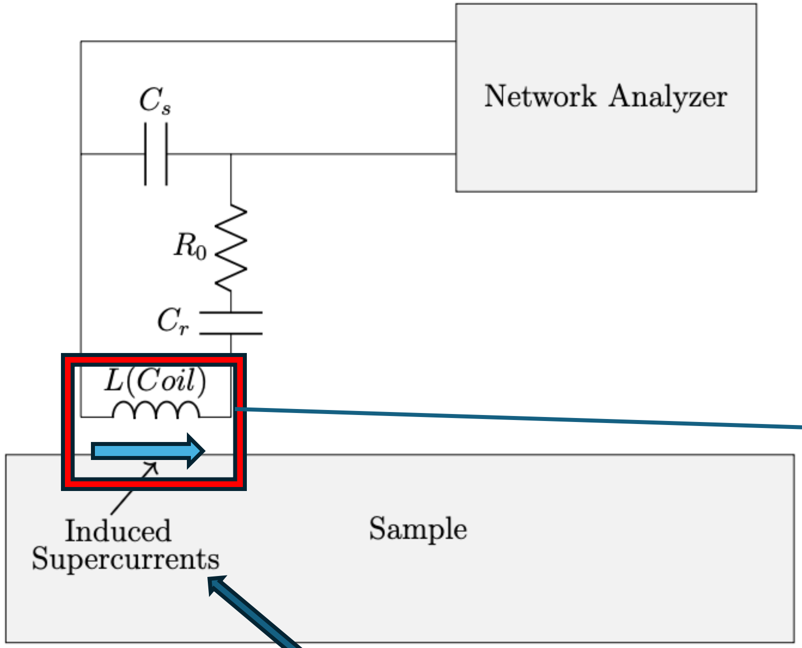
f(λ) ← **Focus of this thesis!**

Data set (f , T) → **Data set (λ , T)**

Theoretical/Simulation Part of this Thesis

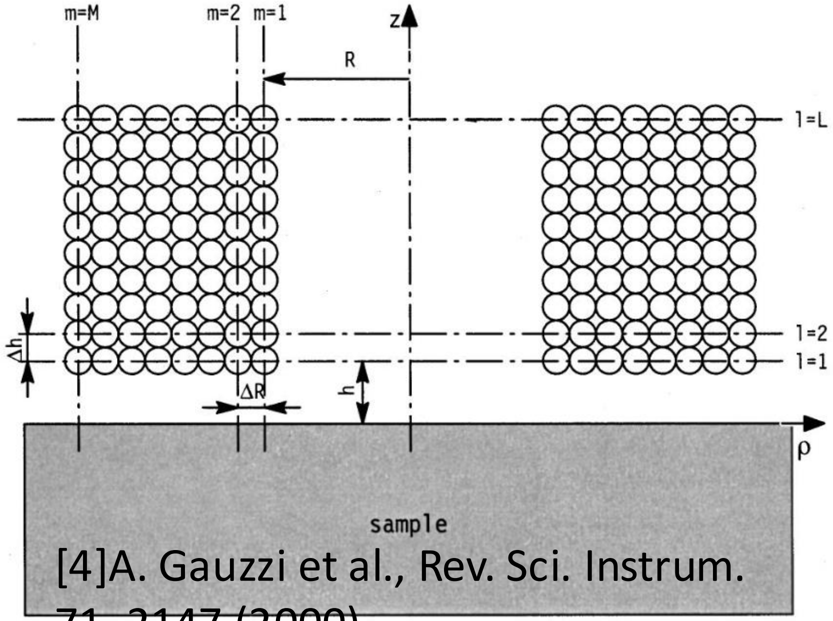
How to obtain Resonant frequency f(λ)

- **In order to get f(λ), we must obtain A(λ).**



- **Solving Maxwell's equations.**

- **Take pancake coil as an example.**



[4]A. Gauzzi et al., Rev. Sci. Instrum. 71, 2147 (2000)

$$\vec{J} = \vec{J}_{\text{coil}} + \vec{J}_{\text{sample}}$$

Supercurrent Density

$$J_{sample} = j_s = \frac{-1}{\mu_0 \lambda^2} \vec{A}$$

$$J_\theta(r, z) = I \sum_{l=0}^{L-1} \delta(z + h + l\Delta h) \sum_{m=0}^{M-1} \delta(r - R - m\Delta R)$$

Current density in Coil

Maxwell's Equations for Axis-Symmetric coil

$$\nabla^2 \vec{A} = -\mu_0 \vec{J}$$

- **MW's equations in Coulomb Gauge**

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{\partial^2}{\partial z^2}\right) A(r,z) = \boxed{-\mu_0 J_\theta} \quad (z \le 0),$$

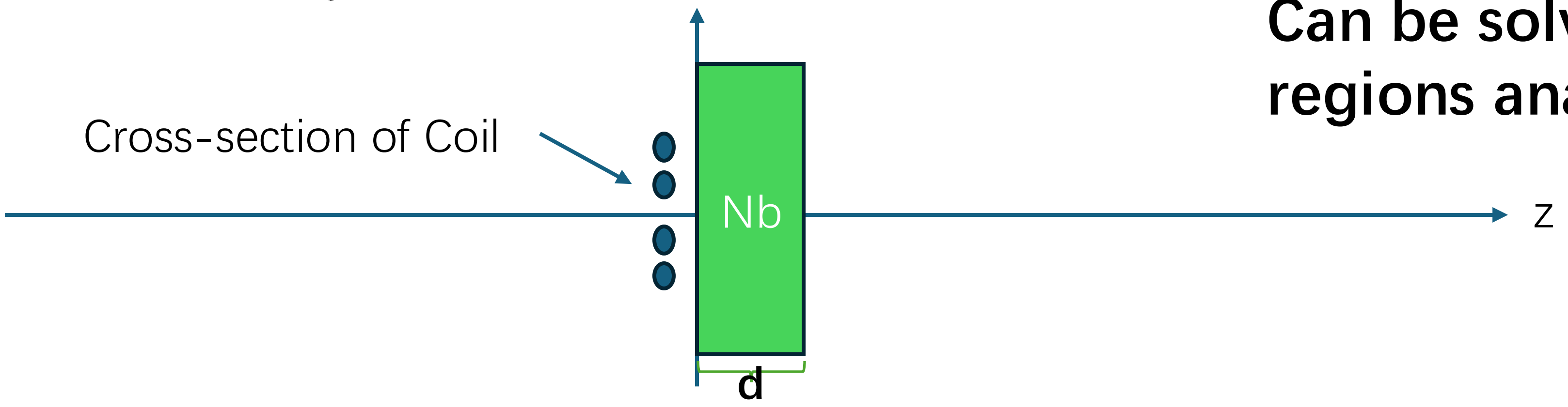
$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{\partial^2}{\partial z^2} - \boxed{\frac{1}{\ell^2}}\right) A(r,z) = 0 \quad (0 \le z \le d),$$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{\partial^2}{\partial z^2}\right) A(r,z) = 0 \quad (d \le z).$$

[5]T. Kubo, IPAC2016, p. 2164 (2016)

Axis-symmetric case

Can be solved at each regions analytically!

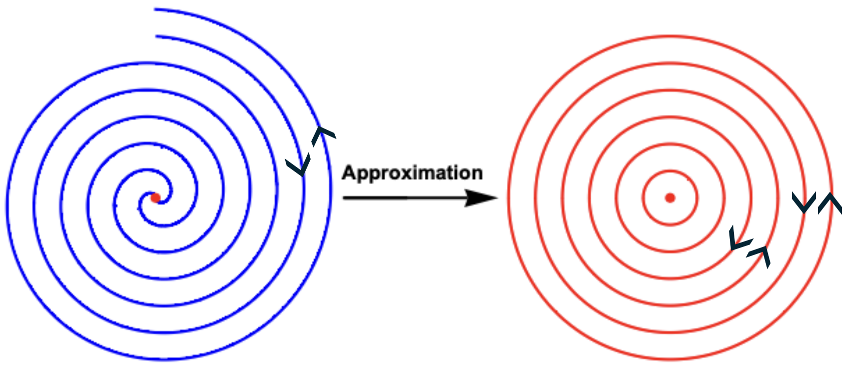


Approximation Model of Spiral Coil

But the spiral coil is not axis-symmetric!

([6]T. Kubo, internal research note, unpublished, 2021)

- **Make spiral coil axis-symmetric!**



The solution to z<0 region can be derived from both Hankel Transformation and the following two-dimensional Green Function:

$$G(r,r',z,z') = \frac{\mu_0}{2}\int r' e^{-s|z-z'|} J_1(sr) J_1(sr')\, ds$$

$$J_\theta(r,z) = I\delta(z+h)\sum_m (-1)^m \delta(r-r_m)$$

$$\tilde{A}(s,z) = \tilde{f}(s,z) + C_1(s)e^{sz}, \quad (z \le 0)$$

$$\tilde{A}(s,z) = C_2(s)e^{-\beta z} + C_3(s)e^{\beta z}, \quad (0 \le z \le d)$$

$$\tilde{A}(s,z) = C_4(s)e^{-s(z-d)}, \quad (d \le z)$$

$$\beta \equiv \lambda^{-1}\sqrt{1+s^2\lambda^2}$$

$$C_1 = \frac{(-1+e^{2\beta d})(s-\beta)(s+\beta)\,\tilde{f}(s,0)}{(-1+e^{2\beta d})\,s^2 + 2\,(1+e^{2\beta d})\,s\beta + (-1+e^{2\beta d})\,\beta^2}$$

$$\tilde{f}(s,z) \equiv \frac{\mu_0 I}{2}\sum_{m=0}^{M-1}(-1)^m r_m J_1(sr_m)\frac{1}{s}e^{-s|z+h|}$$

Derive Inductance L(λ) from A(λ)

The inductance of the spiral coil: $$L = \frac{Z - Z_0}{i\omega} = L_0 + \frac{1}{I}\oint_C \vec{A} \cdot dl$$

C is given by : $$\int_{\text{coil}} dl \quad \to \quad \sum_m \int dr \times r(-1)^m \delta(r - r_m) \int d\theta \int dz\, \delta(z + h)$$

$$L(\beta) = L_0 + i\pi\mu_0\omega \int_0^\infty ds \frac{\left(\frac{s}{\beta} - \frac{\beta}{s}\right)\mathcal{L}(s)}{2\coth(\beta d) + \left(\frac{s}{\beta} + \frac{\beta}{s}\right)}$$

$$\mathcal{L}(s) = \left\{ e^{-sh} \sum_{m=0}^{M-1} (-1)^m J_1(sr_m) \times r_m \right\}^2$$

When λ is around 30nm to 1000nm

([6]T. Kubo, internal research note, unpublished, 2021)

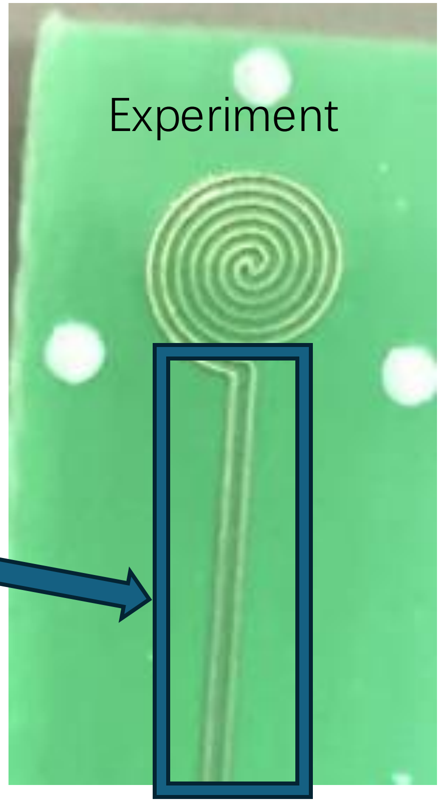
$$\beta \sim \frac{1}{\lambda} \qquad \frac{k}{\beta} - \frac{\beta}{k} = -\frac{1}{k\lambda} \qquad \frac{k}{\beta} + \frac{\beta}{k} = \frac{1}{k\lambda} \qquad \beta = \frac{1}{\lambda}\left(1 + s^2\lambda^2\right)^{\frac{1}{2}}$$

$$L(\lambda) = L_0 - \pi\mu_0 \int_0^\infty \frac{\mathcal{L}(s)}{1 + 2s\lambda\coth\left(\frac{d}{\lambda}\right)} ds$$

The Reasons for Correction/Calibration

- **Before we use L(λ), we need to correct and calibrate it.**
- **The theoretical formula is obtained by an approximation model**
- **The actual coil has cross-section.**
- **The actual coil has the straight-line part.**

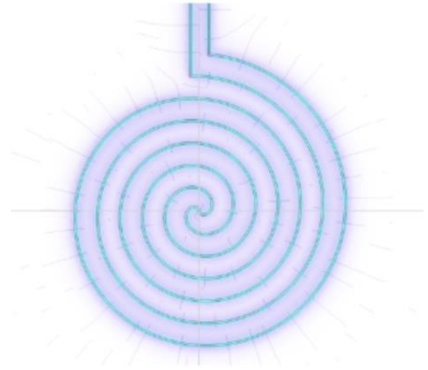
Theory



Correction of L(λ)

- **Inductance of spiral coil increases linearly as penetration depth increase**
- **We can use two parameters to correct the theoretical formula:**

$$L = a \cdot L_0 - c \cdot \pi\mu_0 \int_0^\infty \frac{\left\{e^{-sh}\sum_{m=0}^{M-1}(-1)^m J_1(sr_m)\times r_m\right\}^2}{1+2s\lambda\coth\left(\frac{d}{\lambda}\right)} ds$$



- **Spiral Coil defined in COMSOL**

- **Parameter a : Absolute Correction**
- **Parameter c : Relative Correction**

- **Table: Value of L_0**

Source of result	L_0
Experiment	21.9 nH (Coil1) / 22.5 nH (Coil2)
Simulation	21.54 nH
Theory	11.70 nH

Without Correction

- **Table: Fitting Result**

Parameter	Fitted Value
a	1.8442
c	0.5535

Absolute Calibration of f(λ)

- From parametrized L(λ) we can calculate resonant frequency f(λ)
- The calculated f(λ) still has discrepancy with experimental results.
- Reasons: 1. Inaccurate value of the capacitors. 2. Differences exist between two spiral coils.
- The absolute value of f(λ) will be calibrated with experimental data again

- We need a sample with known value of λ to do the absolute calibration!
- This time we used an assumption:

$$f(100nm)=f(8.903K)$$

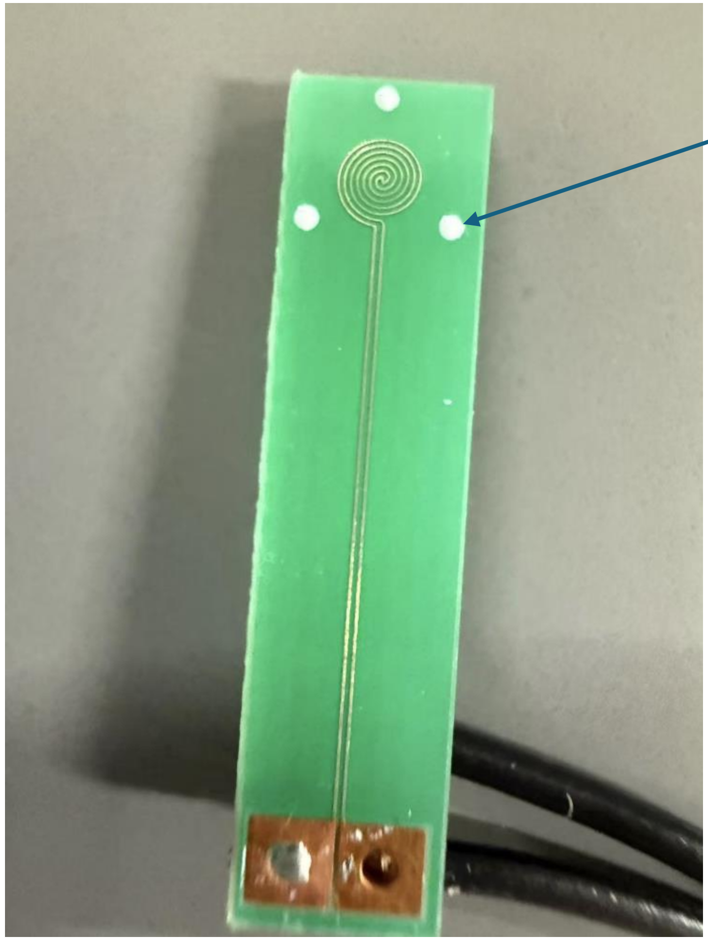
- Table: Resonant frequency f(λ) value

Source of result	f(100nm)
Experiment	17.5*10^7 (Coil1) 17.7*10^7 (Coil2)
Theory	18.3*10^7

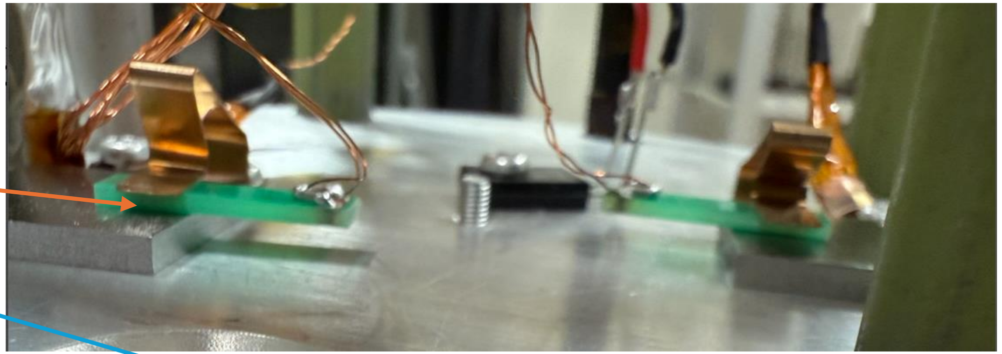
Experimental Part of this Thesis

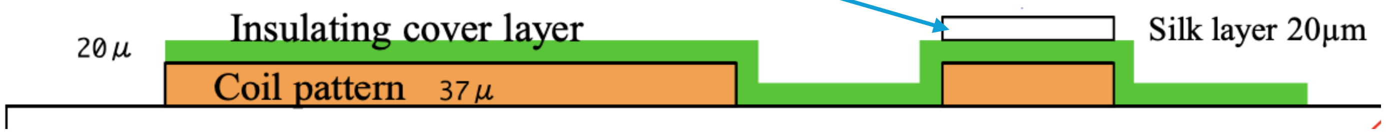
Measurement – Coil Board

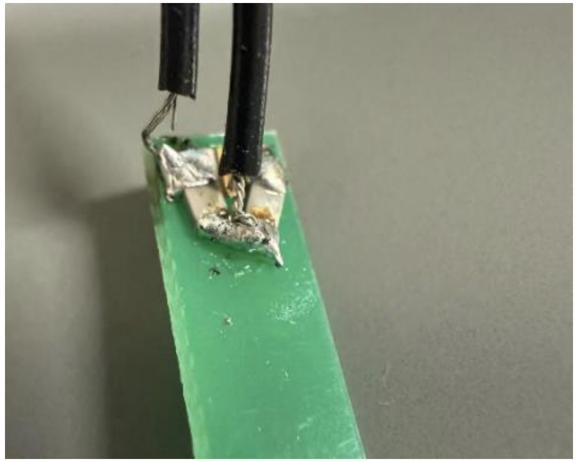
- Now we are finally ready to conduct the experiment!



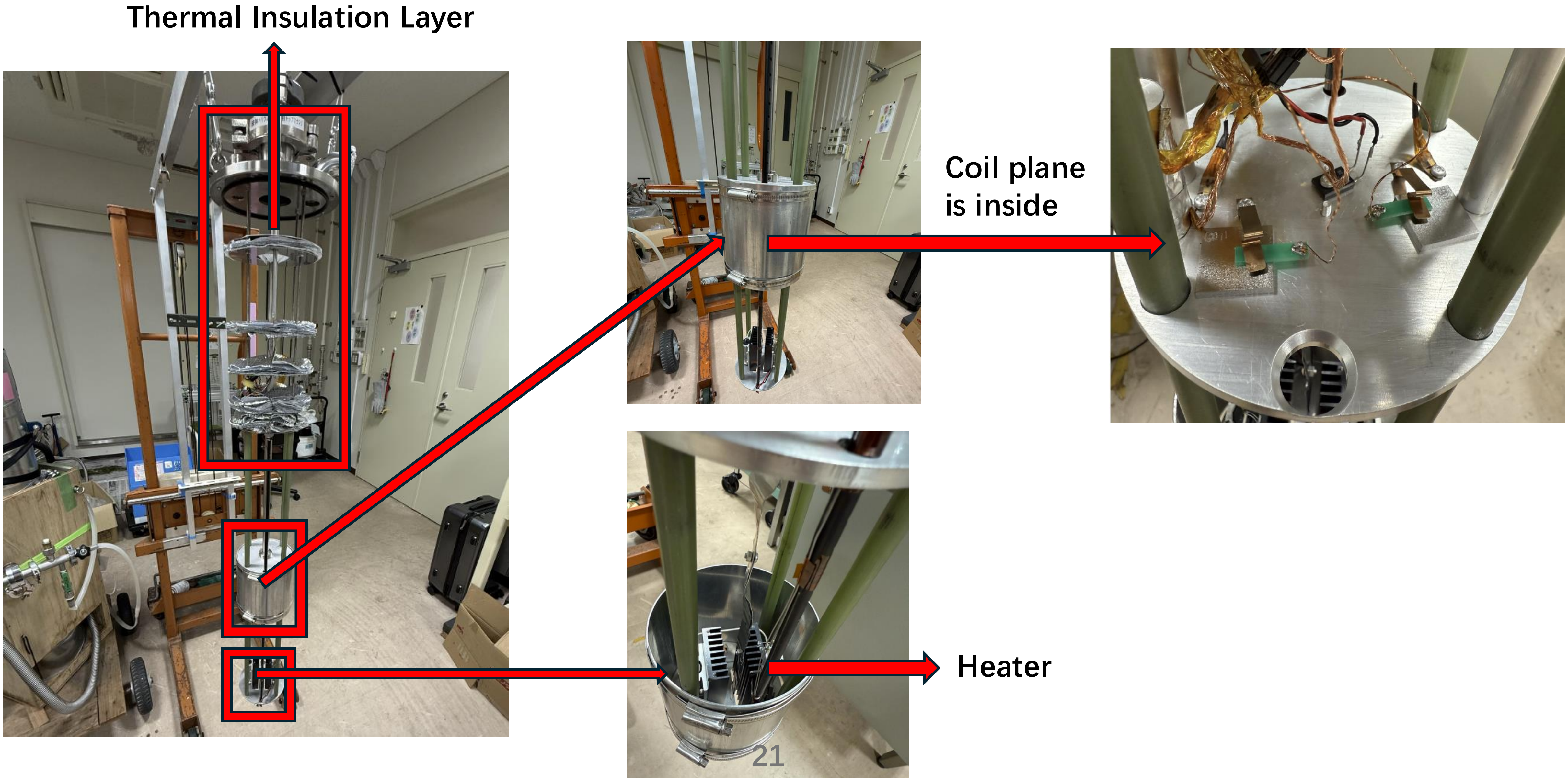
Gap was kept by bumps



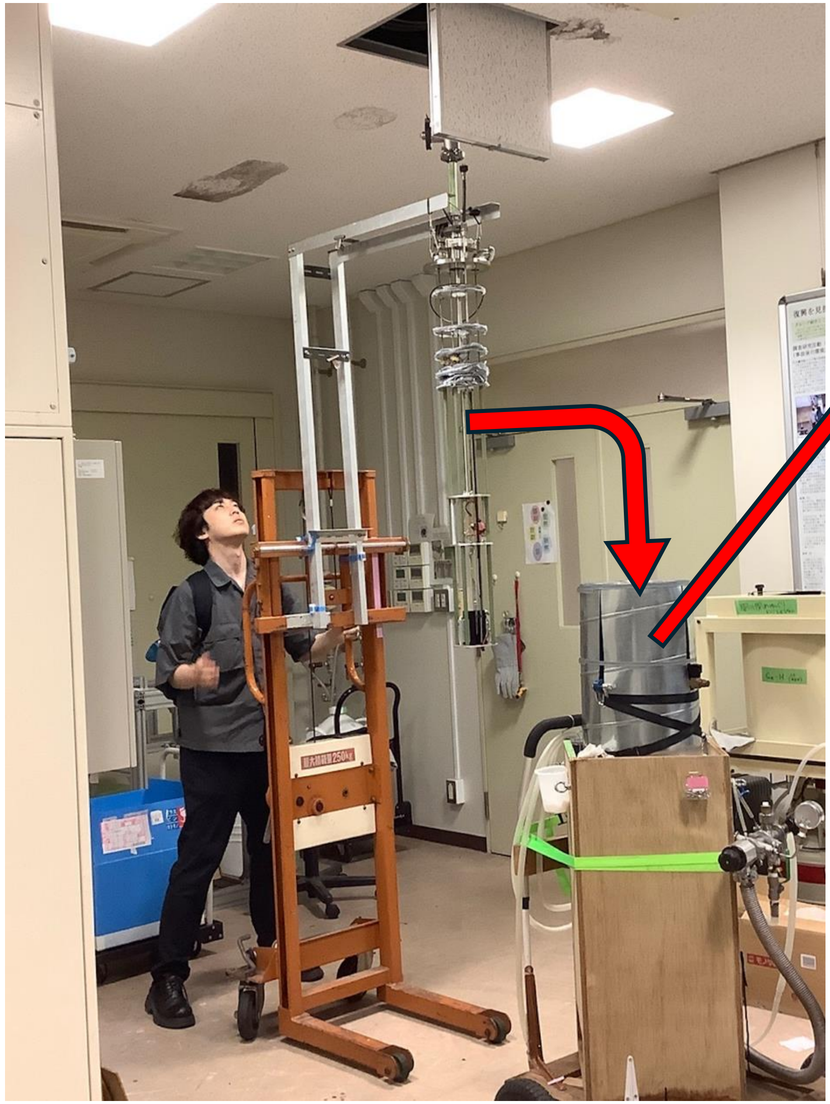




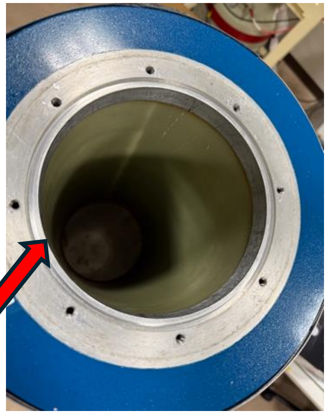
Measurement – Overview of the device

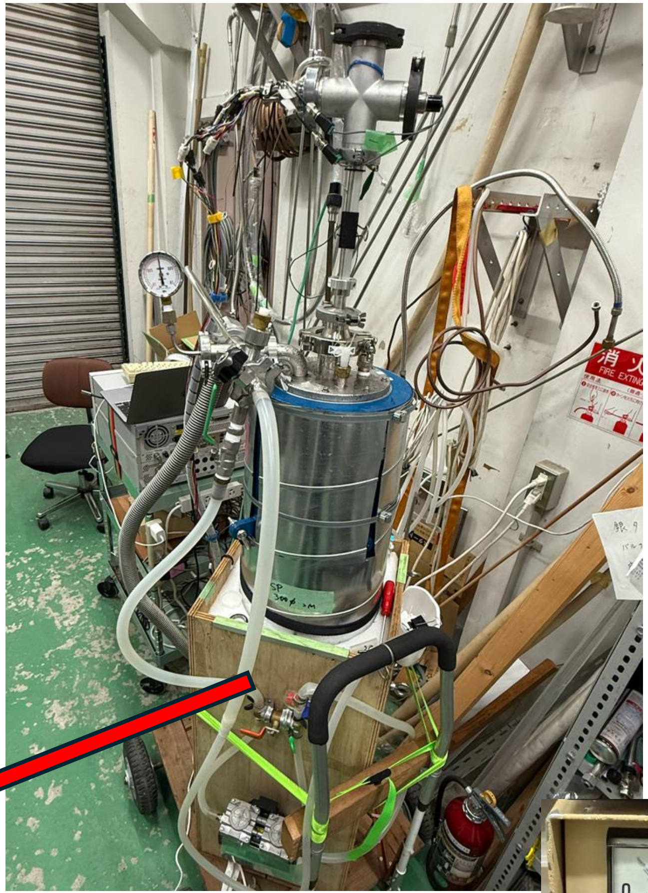


Measurement – Liquid Helium Injection

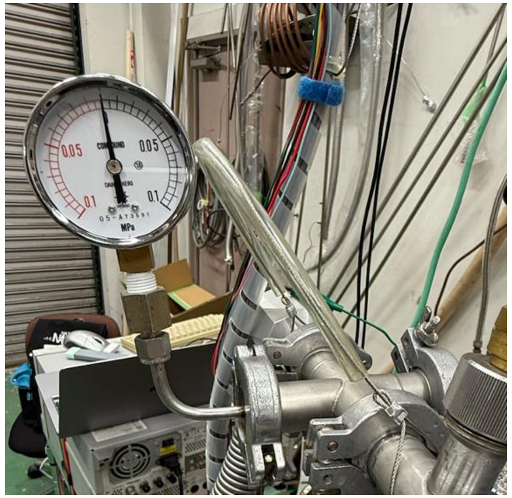


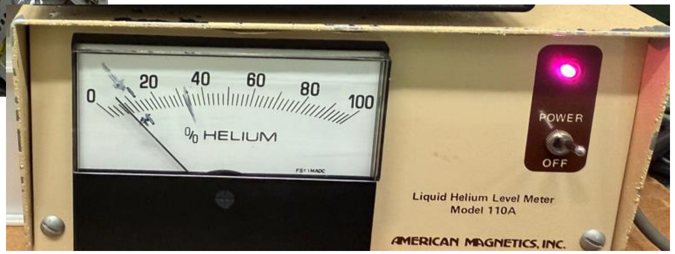
Vacuum Chamber



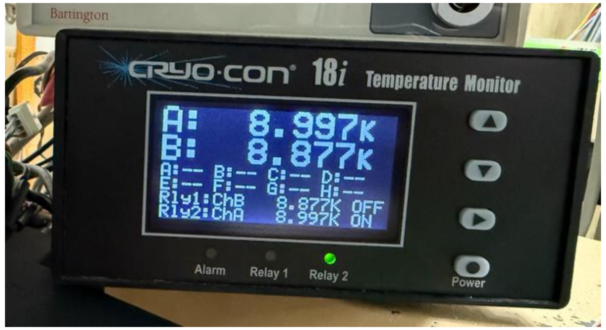


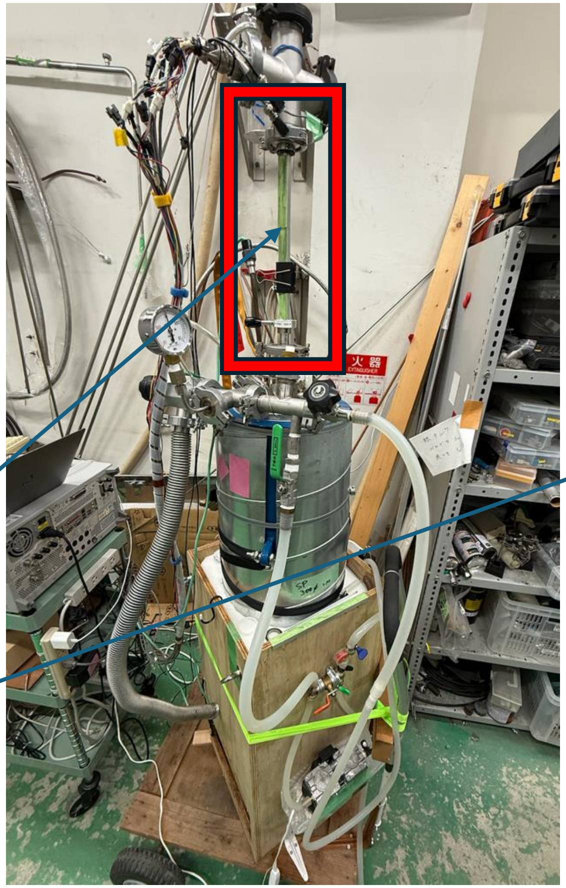
Pressure Gauge

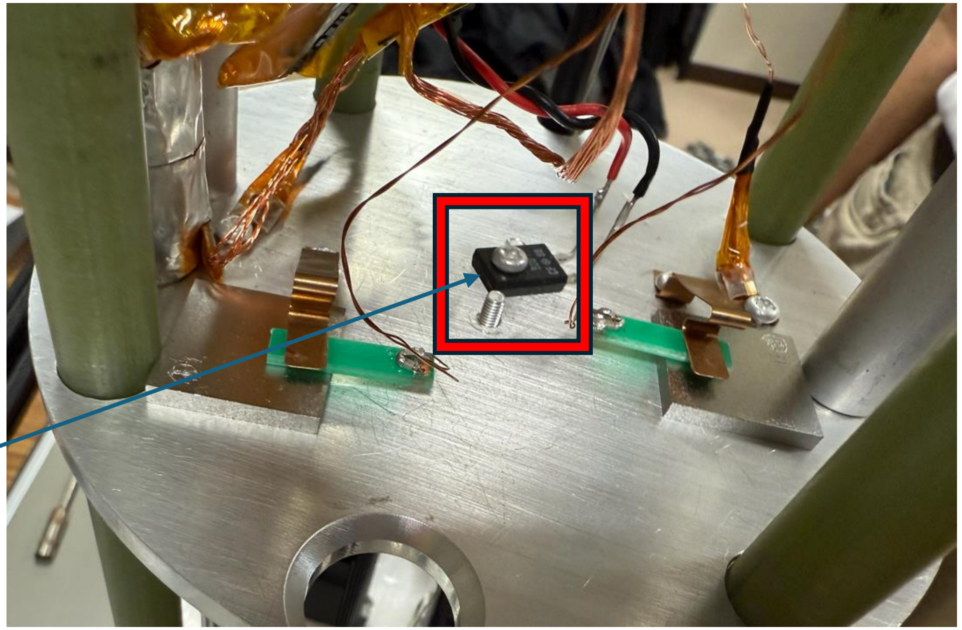


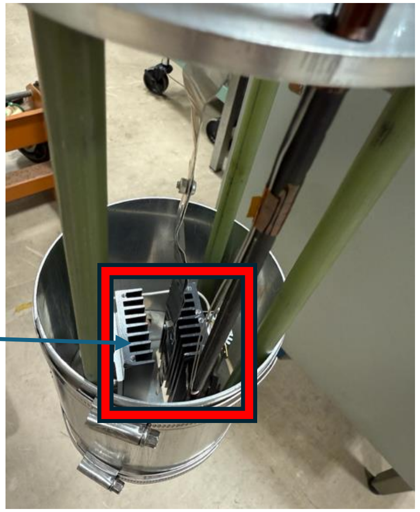


Measurement – Temperature Control







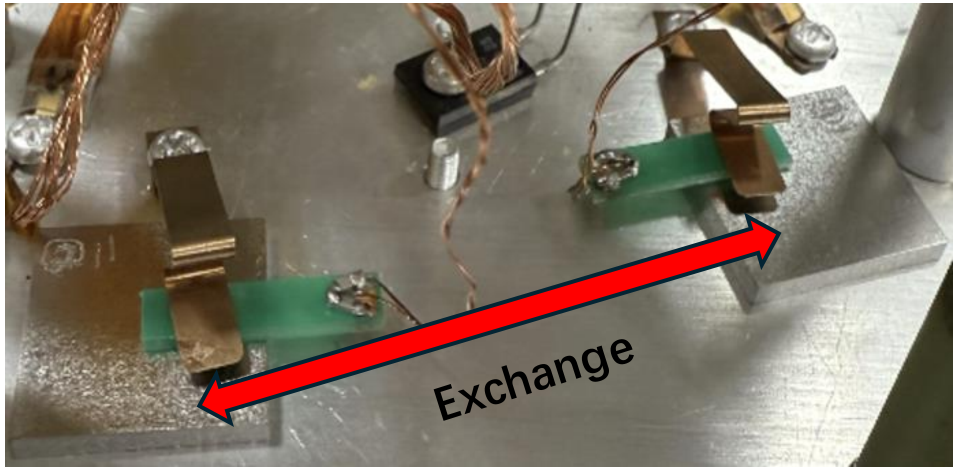


Methods

- Risng/Lowering the coil plane
- Turn on the heater at coil plane
- Turn on the heater at the bottom

Measurement Information

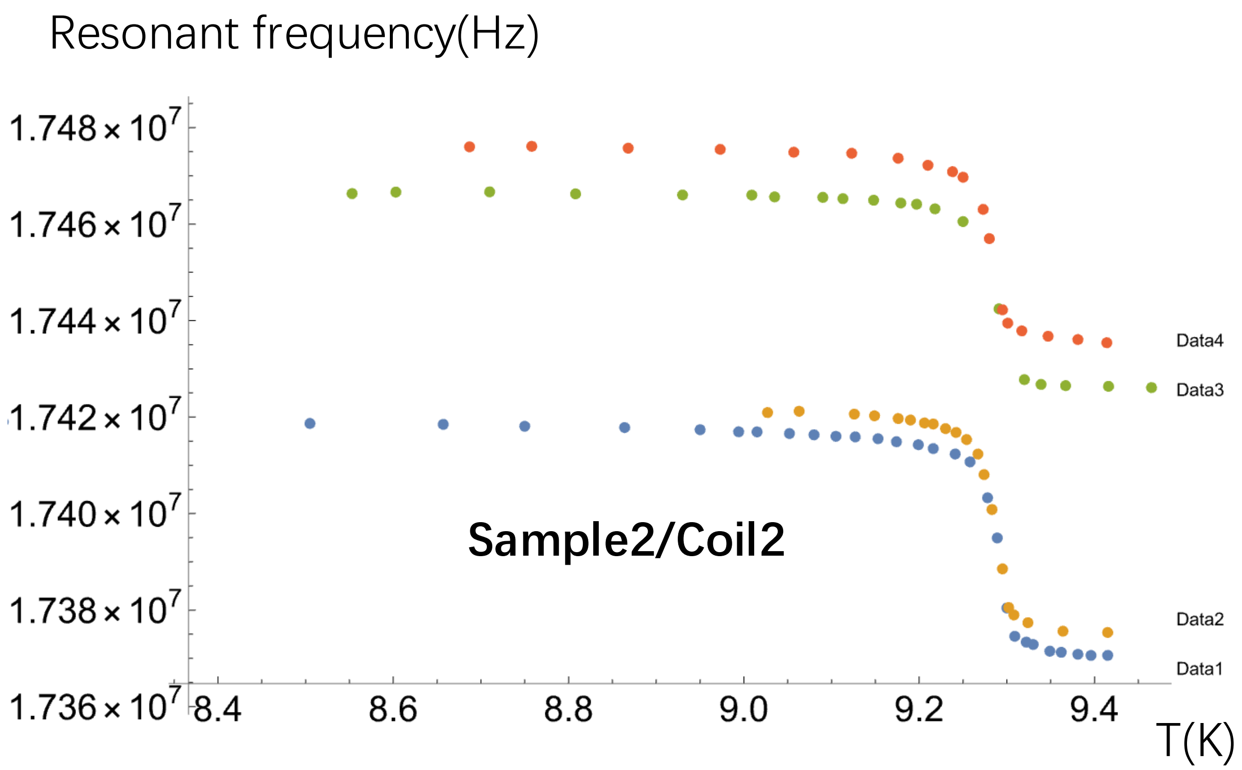
- Sample 1 : Nb, Not annealed
- Sample 2 : Nb, Three hours Annealing
- Totally, 5 measurement results are present.
- Among the 5 measurements, one of the measurement is performed after exchanging the position of two samples.

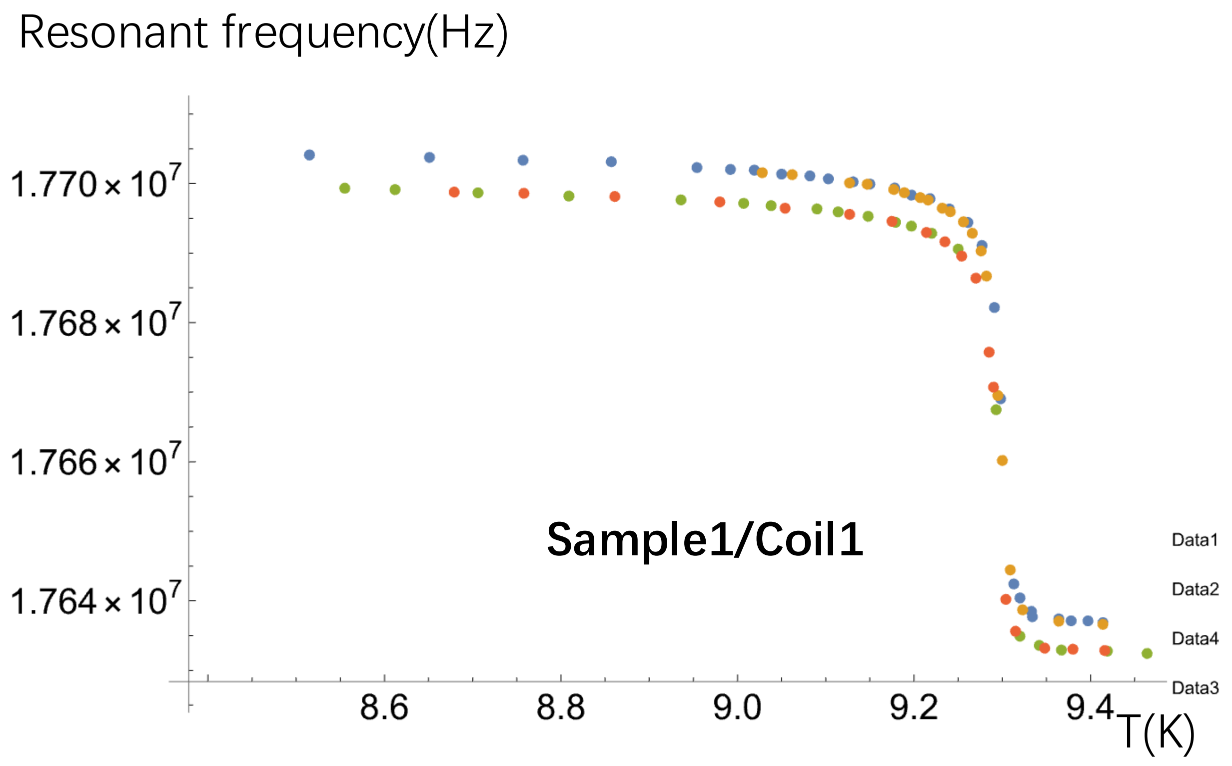


- Table: Information of which sample was measured by which coil during all 5 measurements.

Data #	Sample # Measured by Coil 1	Sample # Measured by Coil 2
1	1	2
2	1	2
3	1	2
4	1	2
5	2	1

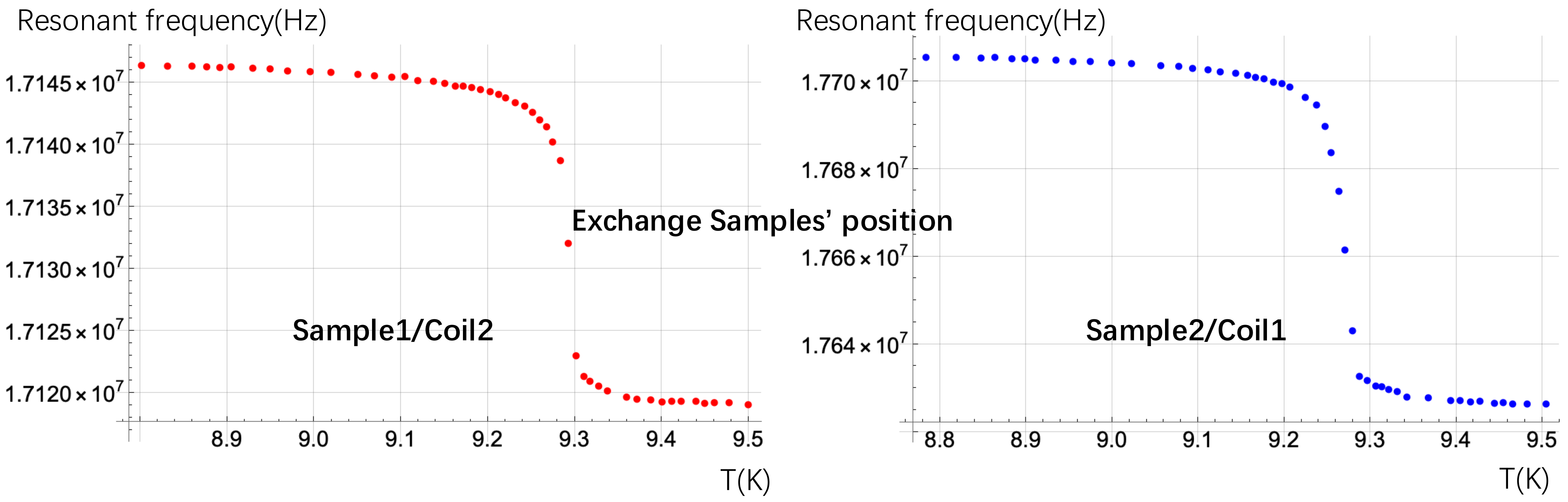
Resonant Frequency Shift (1-4 measurement)





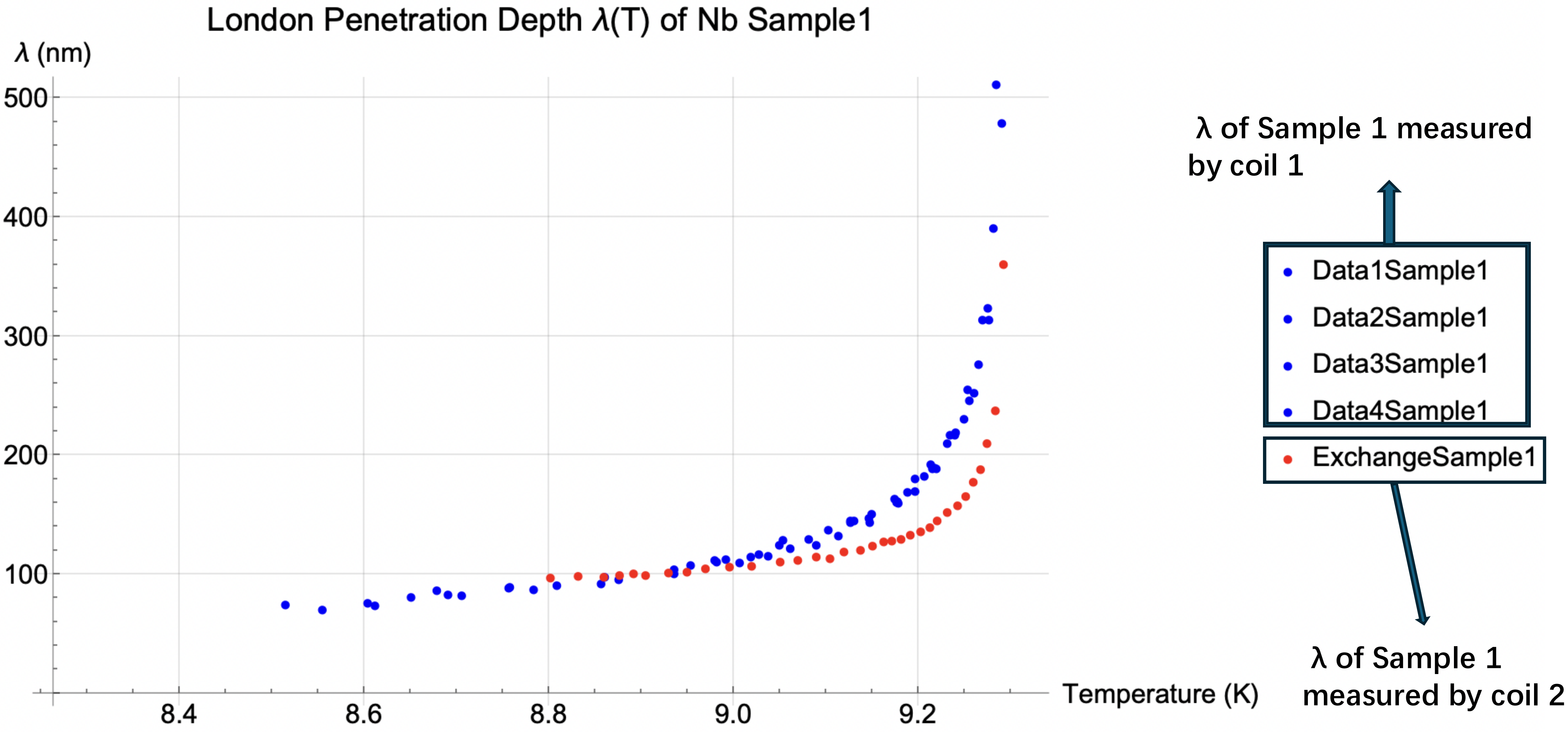
- Data **1**: Blue. Data **2**: Orange. Data **3**: Green. Data **4**: Red.
- Tc appears around 9.3 K.

Resonant Frequency Shift (5th measurement)

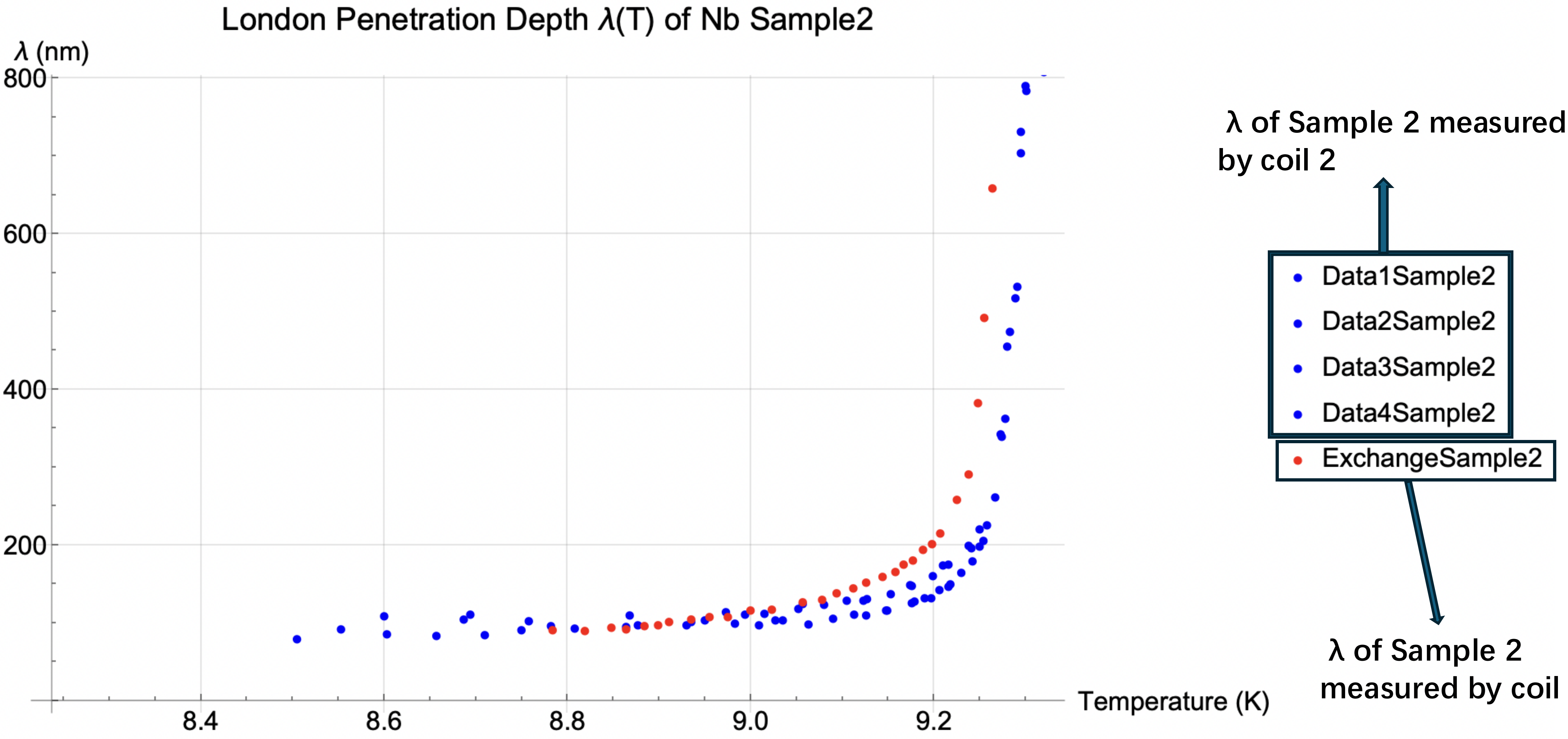


- To evaluate the impact of differences in coils on the results
- Temperature varies very rapid (15 times faster than Data 1~4)

Penetration Depth λ(T) of Sample 1



Penetration Depth λ(T) of Sample 2



λ of Sample 2 measured by coil 2

λ of Sample 2 measured by coil 1

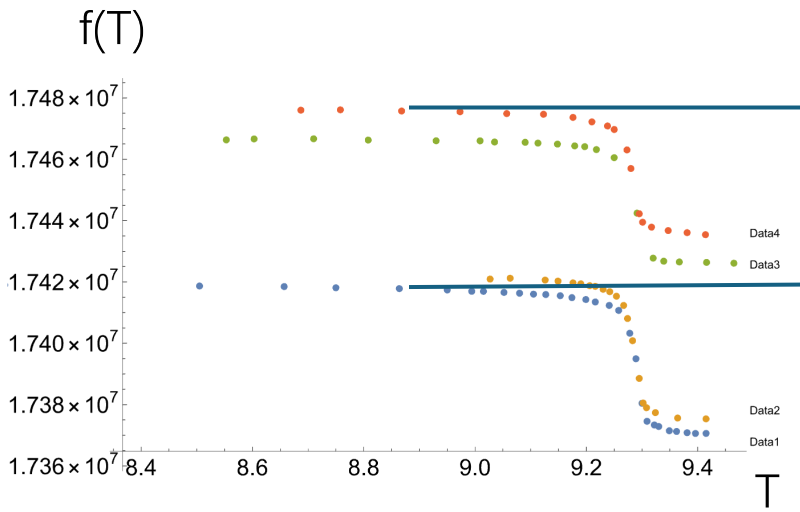
Fitted value of λ(0) and Tc

- **We can fit the result of λ(T) with an empirical formula:**

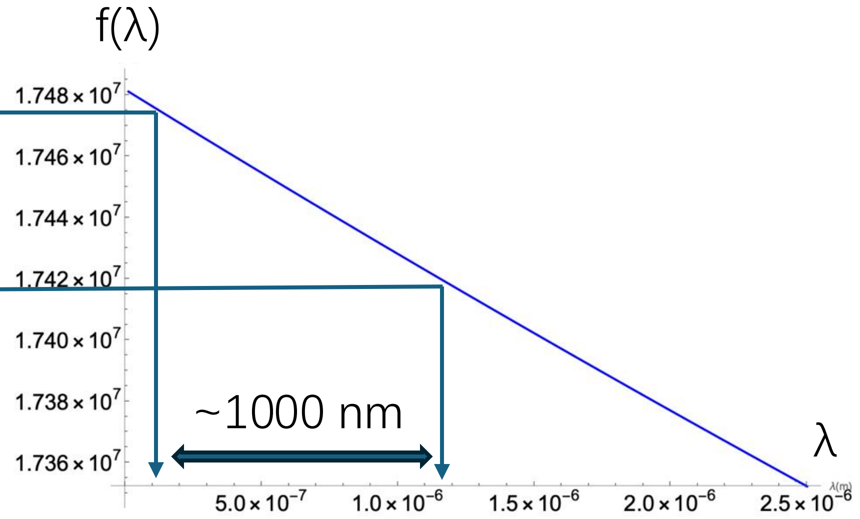
$$\lambda(T) = \lambda(0)\left[1 - \left(\frac{T}{T_c}\right)^4\right]^{-1/2}$$

Fitting parameters:

Tc   λ(0)



- **Why Sample 1 has a large uncertainty in λ(0) :**



Sample / Data Set	Measured by	Tc/K	λ(0)/nm	λ1(0)/λ2(0)
Sample 1 / Data 1 to Data 4	Coil 1	9.303±0.002	37.42±85	1.1353±17.5
Sample 2 / Data 1 to Data 4	Coil 2	9.3±0.002	32.96±502	
Sample 1 / Data 5	Coil 2	9.310±0.03	30.72	0.8268
Sample 2 / Data 5	Coil 1	9.271±0.03	37.1	

Worse Temperature control.

Summary

- **A theoretical formula for Spiral coil which transforms raw frequency data into London penetration depth temperature dependence has been proposed in this study.**
- **All the theoretical expressions derived in this study have been tested by simulations.**
- **The absolute value of λ(0) and its relative ratio exhibit large uncertainties, which may be caused by variations in the distance between the sample and the coil plane during the measurement.**
- **Differences in spiral coils have significant impact on the measurement result.**

Back up

The application of Relative Ratio of λ

$$\mu_0 H_0 = g E_{\mathrm{acc}}$$

- g depends on the shape of cavities

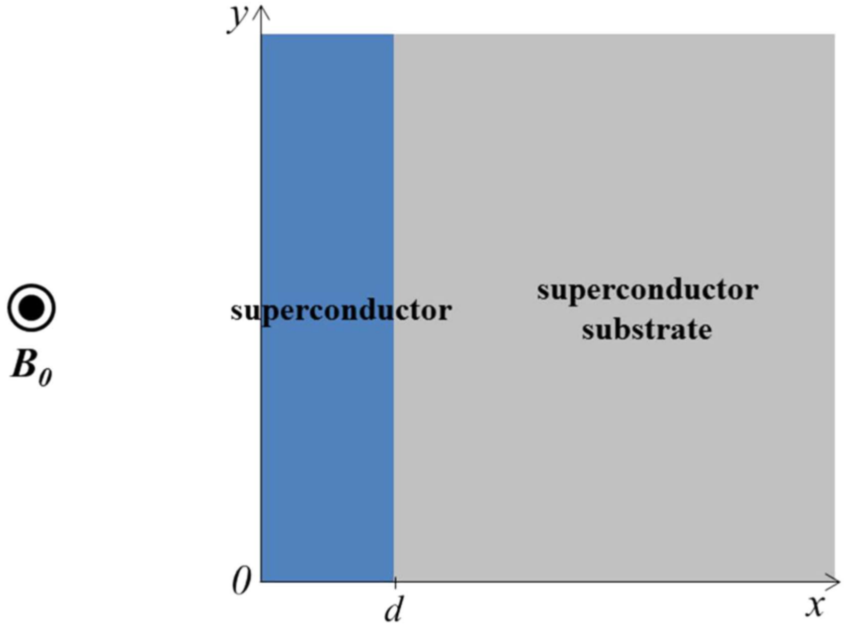
- H_0 is the Amplitude of the oscillating magnetic field at the surface of the superconductor.

- The value of H_0 which makes the Meissner state unstable is called superheating field H_{sh} (~Hc).

$$H_{\mathrm{sh}}(d) = \min\left[c_1(d) H_{\mathrm{sh}}^{(\mathrm{S})}, c_2(d) H_{\mathrm{sh}}^{(\Sigma)}\right],$$

$$c_1(d) = \frac{\cosh[d/\lambda_0^{(\mathrm{S})}] + [\lambda_0^{(\Sigma)}/\lambda_0^{(\mathrm{S})}]\sinh[d/\lambda_0^{(\mathrm{S})}]}{\sinh[d/\lambda_0^{(\mathrm{S})}] + [\lambda_0^{(\Sigma)}/\lambda_0^{(\mathrm{S})}]\cosh[d/\lambda_0^{(\mathrm{S})}]},$$

$$c_2(d) = \cosh[d/\lambda_0^{(\mathrm{S})}] + [\lambda_0^{(\Sigma)}/\lambda_0^{(\mathrm{S})}]\sinh[d/\lambda_0^{(\mathrm{S})}].$$



λ_0 or the relative ratio of λ_0 for different materials is necessary.

Derivation of L(λ)

- **Theoretically, we want to calculate L(λ). We can calculate L(λ) from the result of Z(λ). The impedance Z(λ) along the coil can be derived as follows:**

$$\varepsilon = \oint_C E \cdot d\ell$$

$$E = -\frac{\partial \vec{A}}{\partial t} - \nabla\phi = i\omega\vec{A}$$

$$\varepsilon = i\omega \int \vec{A} \cdot d\ell$$

$$\varepsilon = IZ$$

$$Z = \frac{i\omega}{I} \oint_C \vec{A} \cdot d\ell$$

C : along the coil

$$Z = R + i\omega L - i\frac{1}{\omega C} = Z_0 + i\omega L$$

$$L = \frac{Z - Z_0}{i\omega} = L_0 + \frac{1}{I} \oint_C \vec{A} \cdot dl$$

- **Once we know the magnetic vector potential A(λ), L(λ) is completely solvable**

Equivalent RLC resonant frequency

- **We can take the following approximation of the resonant frequency:**

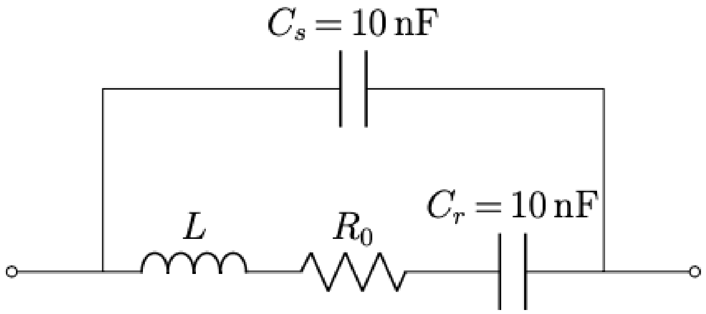
$$f = \frac{1}{2\sqrt{2}\pi}\left[\frac{3}{LC} - \frac{R^2}{L^2} \pm \sqrt{\left(\frac{R^2}{L^2} - \frac{3}{LC}\right)^2 - \frac{8}{L^2C^2}}\right]^{1/2}$$

$$\frac{R^2}{L^2} \ll \frac{1}{LC}$$

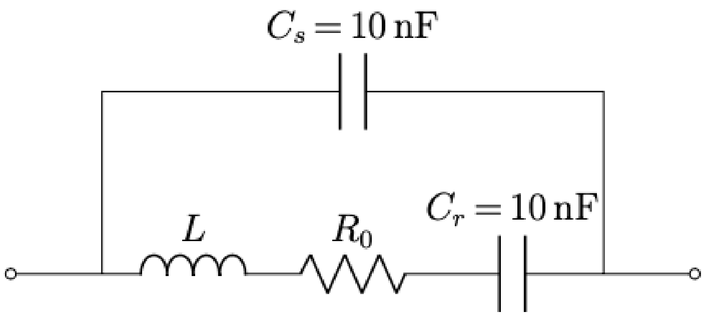
Where R_0 = R

$$f = \frac{1}{2\pi}\frac{1}{\sqrt{LC'}} \quad C' = \frac{C}{2}$$

R： coil resistance



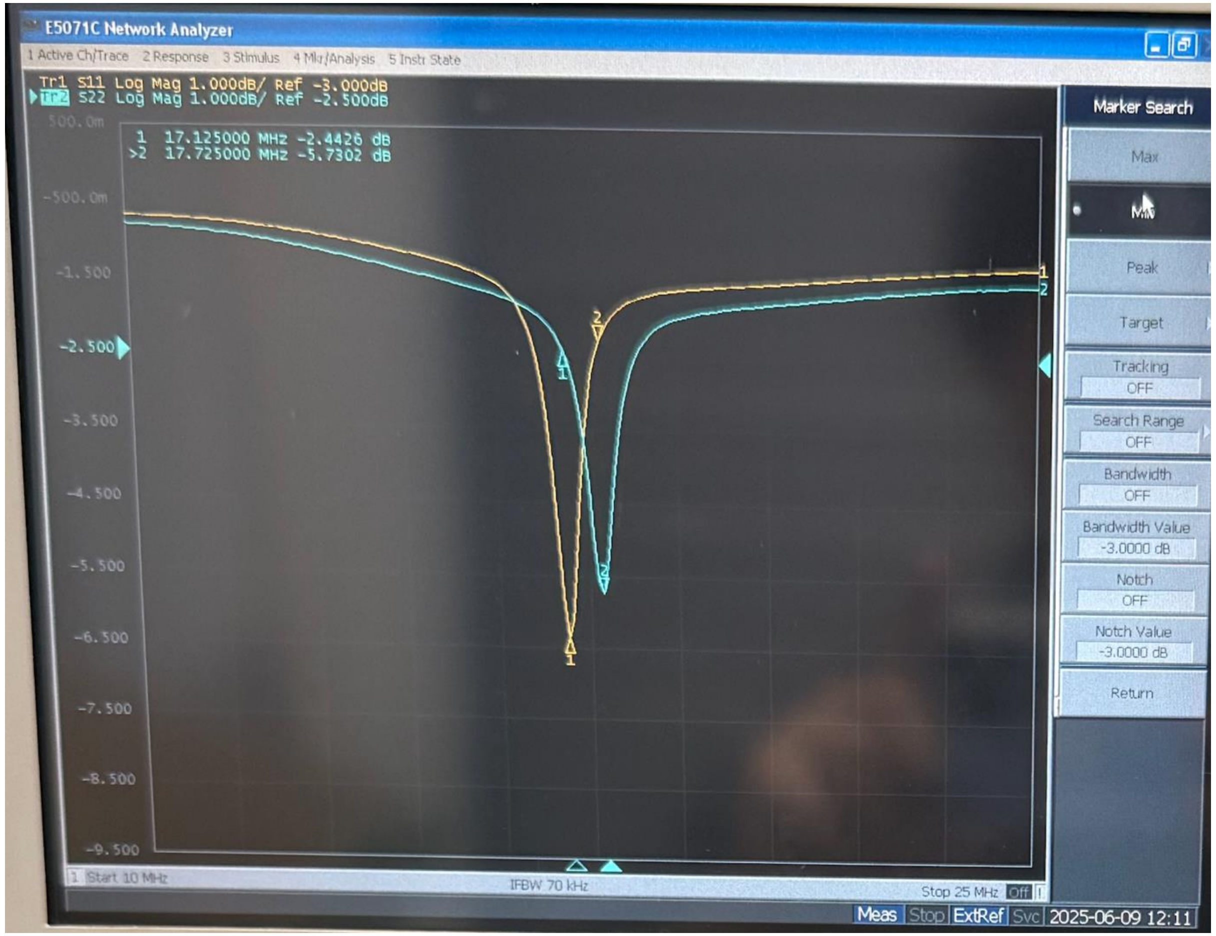
Impedance matching



$$R = \frac{R_0^2 + \left[\omega L - \frac{1}{\omega C}\right]^2}{R_0}$$

Around 150 Ω

C = 10nF
L = 15.6nH
R_0 = 0.002Ω



Inversion Formula – Axial Symmetry

- **If the coil has axial symmetry (like the pancake coil) in three-dimensional space, the only non-zero component of Vector potential A in cylindrical coordinate is A_θ :**

Axial symmetry : $A(r,z) = 0 \cdot \hat{r} + A_\theta(r,z)\hat{\theta} + 0 \cdot \hat{z}$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\iiint \frac{1}{|\vec{r}-\vec{r}'|}\vec{J}_{\text{coil}}(\vec{r}')\,d^3r' \longrightarrow A_\theta(r,z) = \frac{\mu_0}{4\pi}\iiint \frac{r' J_\theta(r',z')\cos\phi}{\left[r^2 + r'^2 + (z-z')^2 - 2rr'\cos\phi\right]^{1/2}}\,d\phi\,dr'\,dz'$$

- **As required by Axial symmetry, the result of A_θ should not depend on θ.**
- **If we recall the solution to Schrodinger equation of Hydrogen atom:**

$$Y_\ell^m(\theta,\phi) = \sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\,e^{im\phi}\,P_\ell^m(\cos\theta)$$

(Spherical Symmetry)

(Associate Legendre Function)

Spherical Symmetry → Axial Symmetry

Can we replace θ integration in $A_\theta(r, z)$ by Legendre Function?

Inversion Formula – Integral Representations

- **The answer is yes!, We can replace the θ integration by using Schlaefli Integral representation of Bessel function:**

$$\int \frac{\cos\psi \, d\psi}{[r^2 + r'^2 + (z-z')^2 - 2rr'\cos\psi]^{1/2}} = \frac{1}{\sqrt{rr'}} Q_{1/2}\left(\frac{r^2 + r'^2 + (z-z')^2}{2rr'}\right)$$ **(Schlaefli Integral)**

- **Furthermore, we can rewrite the result as integral of Bessel function by using:**

$$\int_0^\infty e^{-s|z-z'|} J_1(rs) J_1(r's)\, ds = \frac{1}{\pi\sqrt{rr'}} Q_{1/2}\left(\frac{(z-z')^2 + r'^2 + r^2}{2rr'}\right)$$ **(Lipschitz-Hankel Integral)**

- **Using the two formulas above we have:**

$$A_\theta(r,z) = \frac{\mu_0}{4\pi} \iiint \frac{r' J_\theta(r',z') \cos\phi}{[r^2 + r'^2 + (z-z')^2 - 2rr'\cos\phi]^{1/2}} \, d\phi\, dr'\, dz'$$

$$A_\theta(r,z) = \frac{\mu_0}{2} \iiint r' J_\theta(r,z') e^{-s|z-z'|} J_1(sr) J_1(sr')\, ds\, dr'\, dz'$$

Inversion Formula – Hankel transformation

- **The solution at region 0<z<d and z>d can be solved by Hankel transformation:**

$$\tilde{f}_\nu(k) = \int_0^\infty r J_\nu(kr) f(r)\, dr, \quad f(r) = \int_0^\infty k J_\nu(kr) \tilde{f}_\nu(k)\, dk.$$

$$\int_0^\infty dr\, r J_\nu(kr) \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{\nu^2}{r^2} \right) f(r) = -k^2 \tilde{f}_\nu(k)$$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{\partial^2}{\partial z^2} \right) A(r,z) = -\mu_0 J_\theta \quad (z \le 0),$$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{\ell^2} \right) A(r,z) = 0 \quad (0 \le z \le d),$$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{\partial^2}{\partial z^2} \right) A(r,z) = 0 \quad (d \le z).$$

$$\left(\frac{\partial^2}{\partial z^2} - k^2 \right) \tilde{A}(k,z) = 0 \quad (z \le 0),$$

$$\left(\frac{\partial^2}{\partial z^2} - \beta^2 \right) \tilde{A}(k,z) = 0 \quad (0 \le z \le d),$$

$$\left(\frac{\partial^2}{\partial z^2} - k^2 \right) \tilde{A}(k,z) = 0 \quad (d \le z).$$

Derivation of Supercurrent Js

- **The supercurrent can be derived from London Equations:**

$$\frac{\partial j_s}{\partial t} = \frac{e^2 n_s}{m} E$$

$$\nabla \times j_s = \frac{-e^2 n_s}{m} B$$

$$E = -\frac{\partial \vec{A}}{\partial t} - \nabla\phi = -\frac{\partial \vec{A}}{\partial t}$$

$$\nabla \times \vec{A} = B$$

$$j_s = \frac{-e^2 n_s}{m}\vec{A}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot j_s = 0$$

$$\nabla \cdot j_s = 0$$

(if ρ = 0)

$$\nabla \cdot \vec{A} = 0$$

London penetration depth $\lambda = \sqrt{\frac{m}{u_0 n_s e^2}}$

$$J_{sample} = j_s = \frac{-1}{\mu_0 \lambda^2}\vec{A}$$

With this definition of supercurrent, we can begin to define Maxwell's equations

Spiral coil

- Not Axis-symmetric. • Hankel transformation can not solve it
- We can still solve it by using Green function and the solutions at z<0 are given by:

$$A_r(r,\theta,z) = \frac{\mu_0 I}{4\pi}\sum_{m=0}^{n}\int_0^{2\pi}\left[r^2+(r_m+b\theta')^2+(z+h)^2-2r(r_m+b\theta')\cos(\theta'-\theta)\right]^{-\frac{1}{2}} \times\left[\cos(\theta'-\theta)\sin\alpha-\sin(\theta'-\theta)\cos\alpha\right](r_m+b\theta')\,d\theta' \tag{5.87}$$

$$A_\theta(r,\theta,z) = \frac{\mu_0 I}{4\pi}\sum_{m=0}^{n}\int_0^{2\pi}\left[r^2+(r_m+b\theta')^2+(z+h)^2-2r(r_m+b\theta')\cos(\theta'-\theta)\right]^{-\frac{1}{2}} \times\left[\cos(\theta'-\theta)\cos\alpha+\sin(\theta'-\theta)\sin\alpha\right](r_m+b\theta')\,d\theta' \tag{5.88}$$

- These are solutions in the case that there is no finite thickness superconductor present!
- It is very difficult to obtain the solutions analytically when the finite thickness superconductor is present.

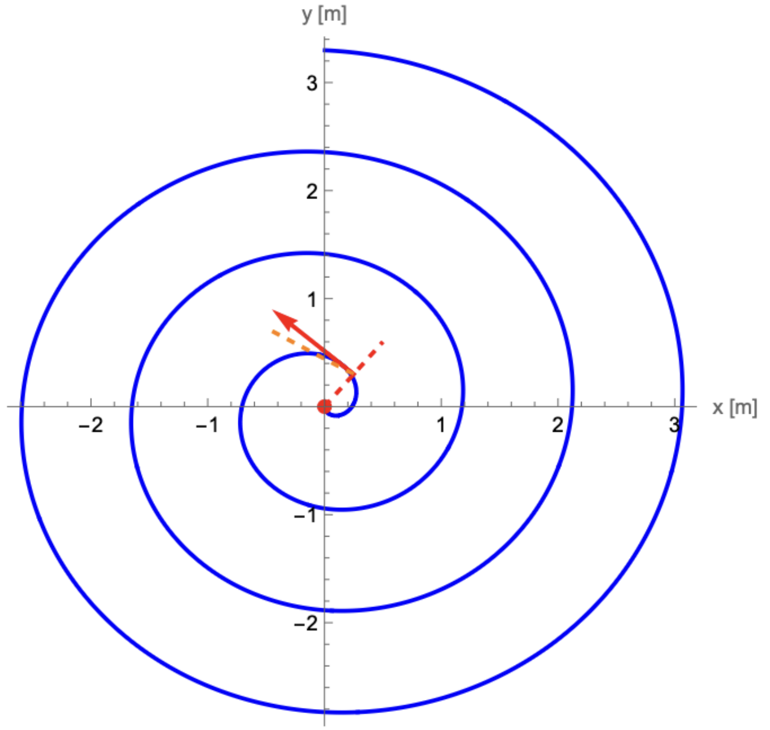
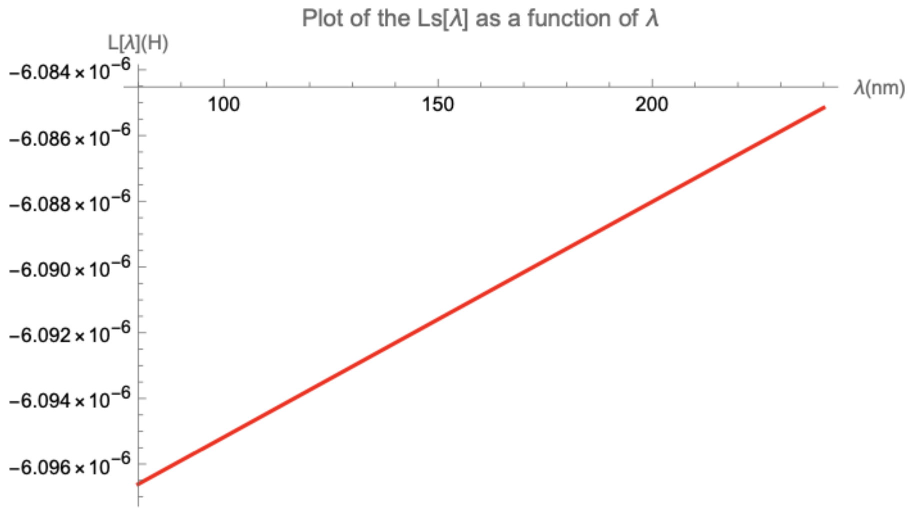
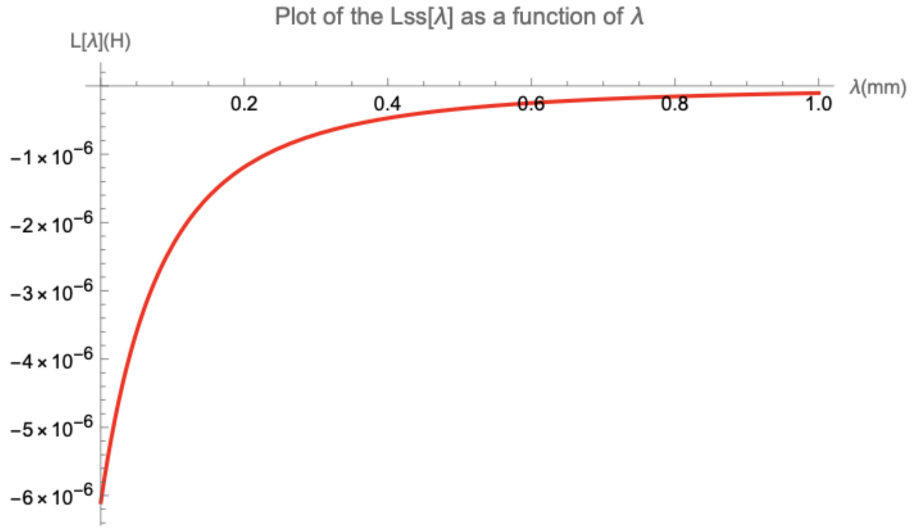
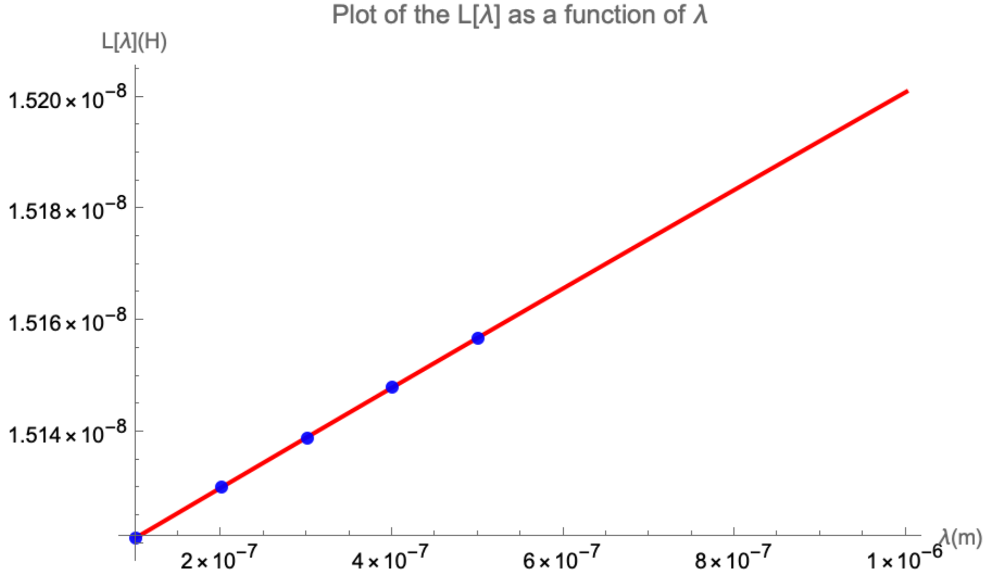


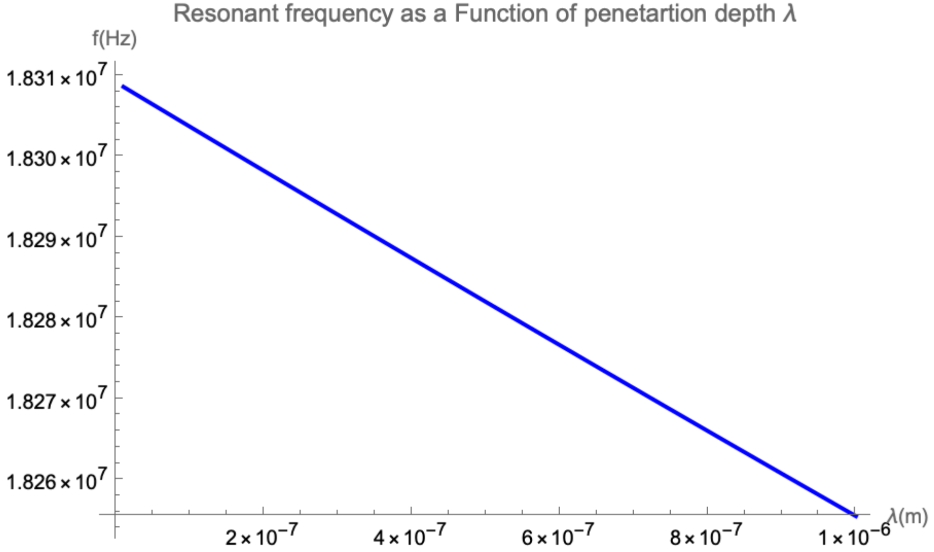
Figure 5.3: The Archimedean spiral.

L (λ) and f(λ) as a function of λ









Comparison: Theoretical result with simulation

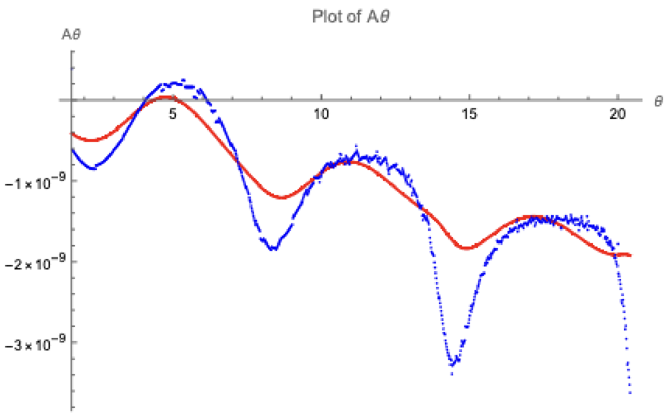


Figure 7.16: Comparison of A_θ between theoretical and COMSOL results.

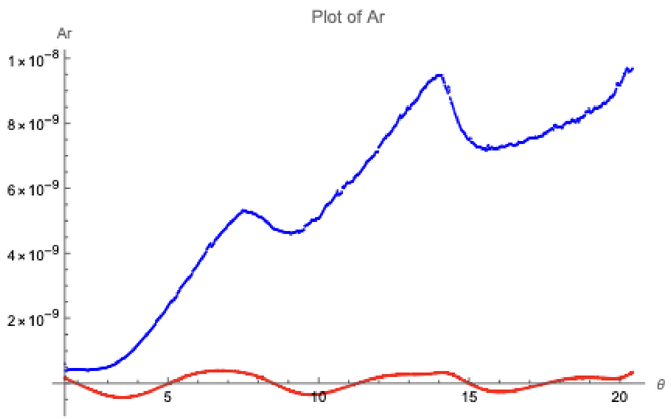
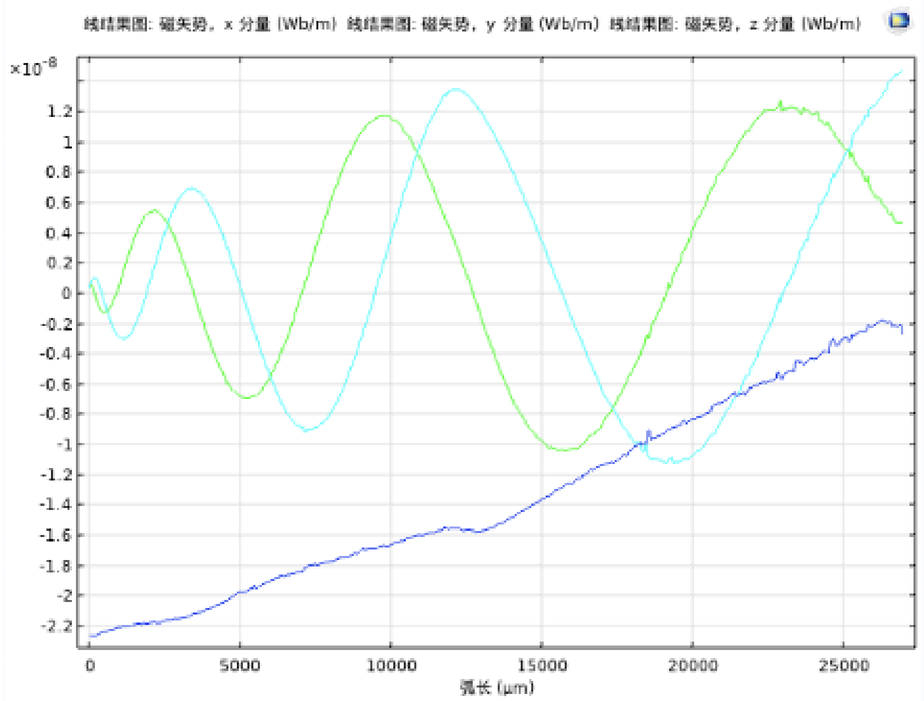
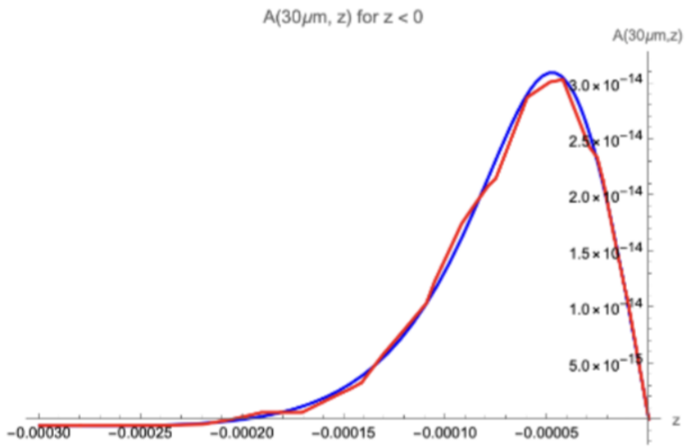


Figure 7.17: Comparison of A_r between theoretical and COMSOL results.





Absolute Calibration

- **Let the value of f(100 nm), calculated from the parametrized inductance, equal the resonant frequency f_i measured in the experiment when λ = 100 nm**

$$L = a \cdot L_0 - c \cdot \pi\mu_0 \int_0^{\infty} \frac{\left\{e^{-sh} \sum_{m=0}^{M-1} (-1)^m J_1(sr_m) \times r_m\right\}^2}{1 + 2s\lambda \coth\left(\frac{d}{\lambda}\right)} ds$$

Find T_{100} : $$100nm = \lambda(0)\left[1 - \left(\frac{T_{100}}{T_c}\right)^4\right]^{-1/2}$$

$$\lambda(T) = \lambda(0)\left[1 - \left(\frac{T}{T_c}\right)^4\right]^{-1/2}$$

Assume λ(0) = 35 nm

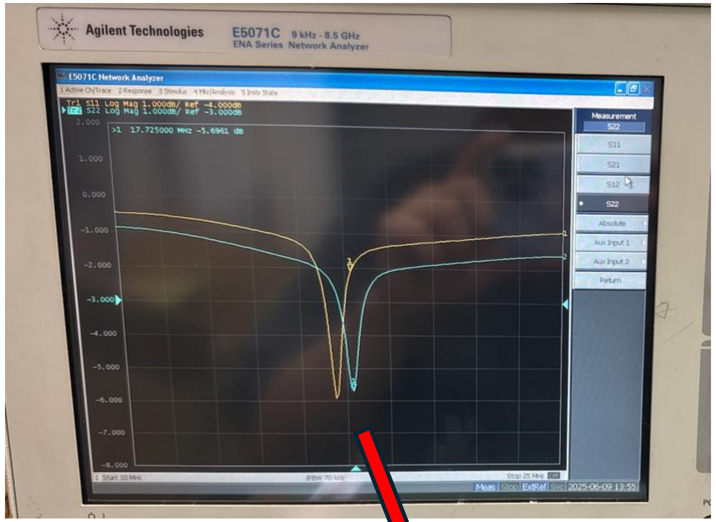
Tc = 9.25 K

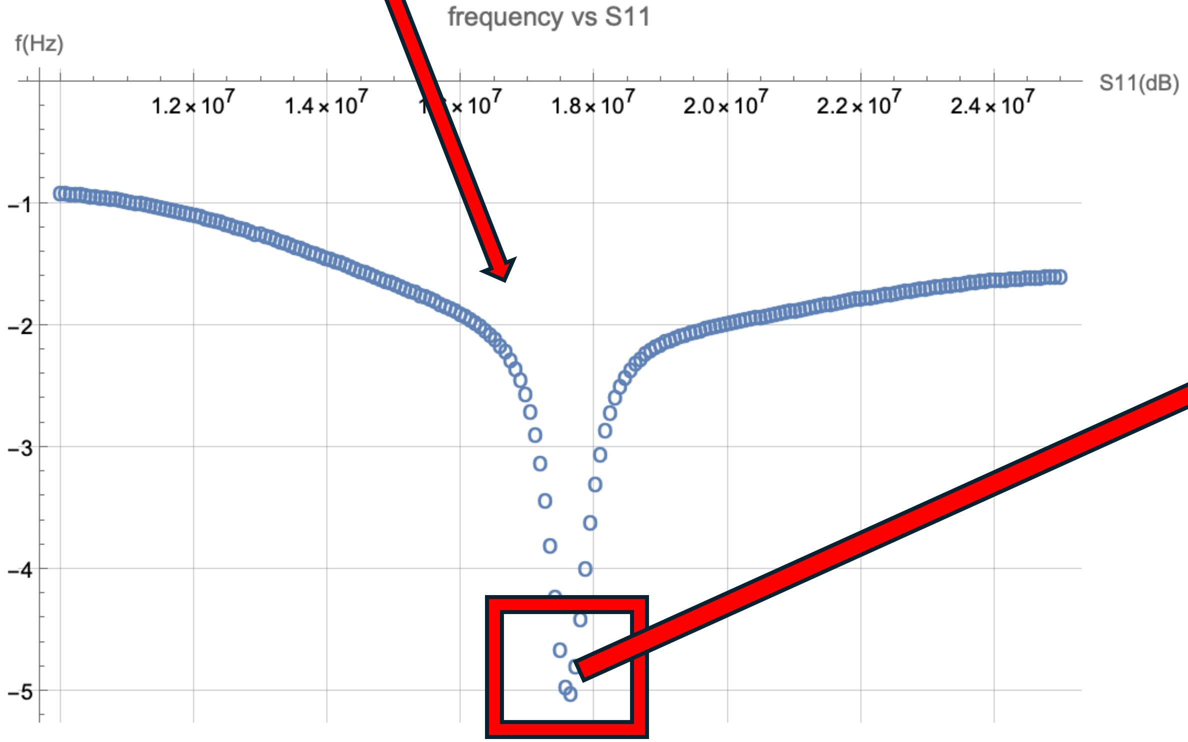
$$\frac{\sum_{i=1}^{n} T_i}{n} = T_{100}$$

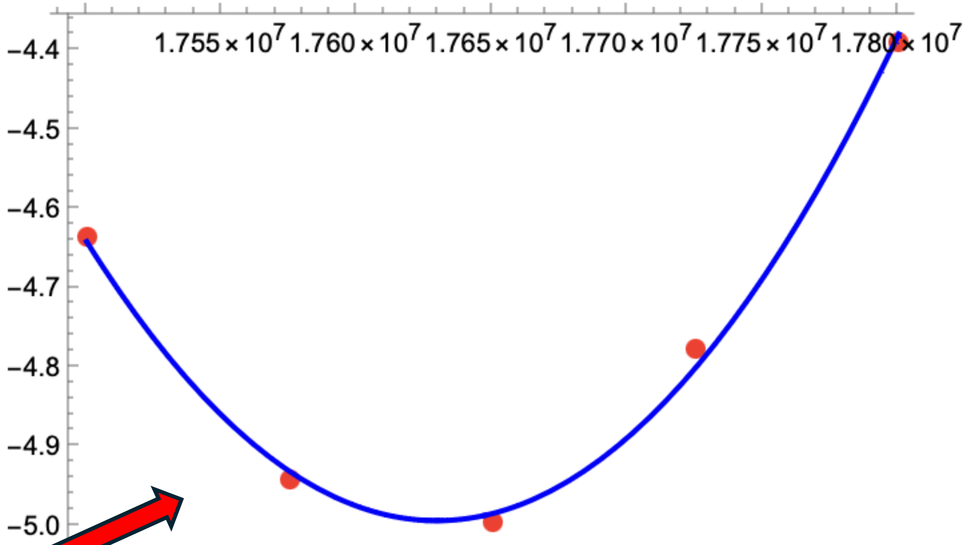
$$f(100nm) = \frac{\sum_{i=1}^{n} f_i}{n}$$

This will make the absolute value of λ Non-physical!

Determination of RF







Fitting function: a*f^2 + b*f + c

F_{min} = -b/2a